PhD Seminars 2004/2005: Prof. T. Porter Semester 2

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1 Seminar 1: 22nd February 2005

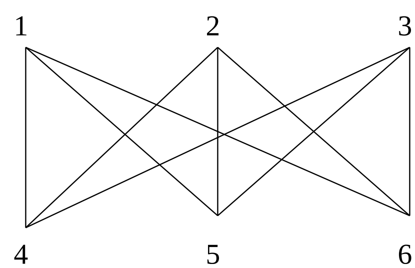
1.1 Groups and Surfaces

Historical note: This seminar was first given by Prof. Porter in 1985!

Let $\Gamma \hookrightarrow S_g$ be a disc embedding (or a map on S) such that $S \setminus \Gamma$ is a disjoint union of discs, where S is an orientable surface of genus g. **Problem:** given Γ, find the minimal g such that $\Gamma \hookrightarrow S_g$ is a disc embedding.

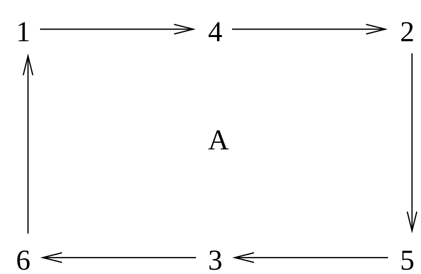
Edmonds' Algorithm. Given Γ, let $V(\Gamma) = \{1, 2, \ldots, n\}$. For each $i \in V(\Gamma)$, let $V(i) = \{k \in V(\Gamma) \mid$ there is an edge from i to k in $\Gamma\}$. Let p_i be a permutation of $V(i)$ of maximal length $n_i = \text{Card}(V(i))$. There is a one-to-one correspondence between disc embeddings in such families of permutations.

Example 1.1 Consider $K_{3,3}$ as shown below, and let $V(1) = V(2) = V(3) = \{4, 5, 6\}$, $V(4) = V(5) = V(6) = \{1, 2, 3\}$, $p_1 = p_2 = p_3 = (4\,5\,6)$ and $p_4 = p_5 = p_6 = (1\,2\,3)$.

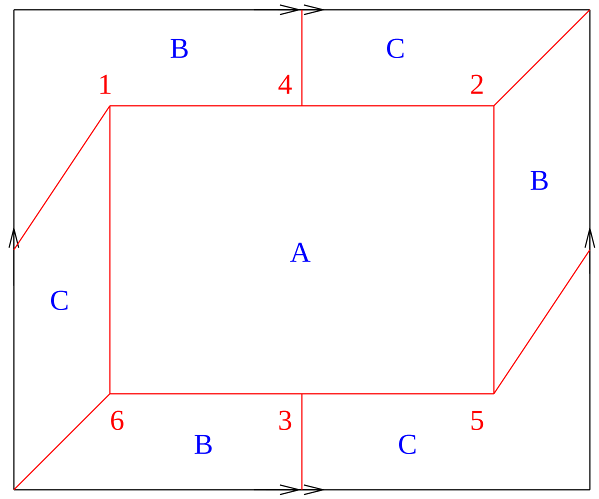


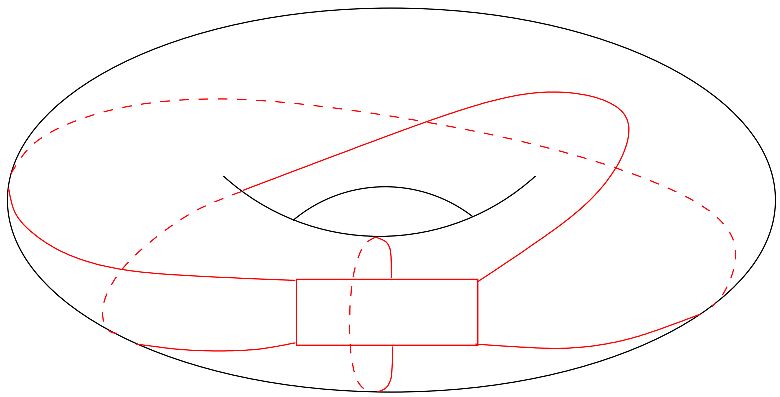
Let $\Omega = \{(a, b) \mid a, b \in V(\Gamma),\ b \in V(a)\}$ be a set of 'darts' $a \to b$, and let $P^* : \Omega \to \Omega$ be defined by $P^*(a, b) = (b, p_b(a))$.

In the example, $(1, 4) \to (4, 2) \to (2, 5) \to (5, 3) \to (3, 6) \to (6, 1) \to (1, 4)$ gives the orbit A shown below. We also have orbits $B = (1, 5) \to (5, 2) \to (2, 6) \to (6, 3) \to (3, 4) \to (4, 1) \to (1, 5)$ and $C = (1, 6) \to (6, 2) \to (2, 4) \to (4, 3) \to (3, 5) \to (5, 1) \to (1, 6)$.



We can now use $v - e + f = 2 - 2g$ to obtain $g = 1$ $(6 - 9 + 3 = 2 - 2g)$, and we can embed on a torus as follows:


B
C
1
4
2
B
A
C
6
3
5
B
C

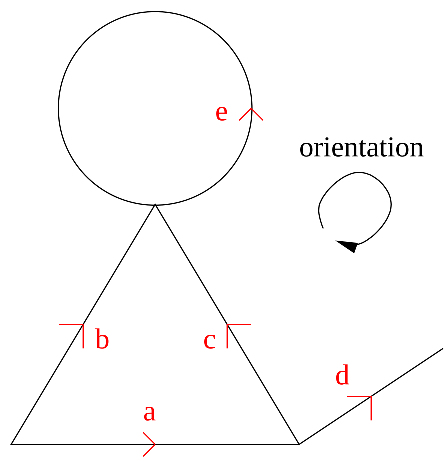


2 Seminar 2: 1st March 2005

2.1 Groups and Surfaces 2

Let M be a map (an embedded graph) on an oriented surface S (with some extra conditions), and let Ω be the set of darts (directed edges) in M. Define r_0 to be the permutation that maps a dart x to the first dart encountered when we move clockwise around the vertex pointed to by x, define r_1 to be the permutation that interchanges the two darts on the same edge, and define $r_2 = (r_0 r_1)^{-1}$.

Consider the following example, which we shall call "The Cat".



Let $\Omega = \{a^+, a^-, b^+, b^-, c^+, c^-, d^+, d^-, e^+, e^-\}$. We note that

$$r_0 = (a^+ c^- d^-)(a^- b^-)(b^+ e^+ e^- c^+)(d^+)$$

and that $r_1(x^+) = x^-$ and $r_1(x^-) = x^+$. As $r_2 = r_1^{-1} r_0^{-1}$, we further note that

$$r_2 = (a^+ b^- c^+)(a^- d^- d^+ c^- e^- b^+)(e^+).$$

Groethendieck's Cartographic Group, $C_2^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_1^2 = \rho_0\rho_1\rho_2 = 1 \rangle$, acts on Ω. In fact C_2^+ has a transitive action on Ω but ρ_1 has no fixed point. If we want to allow ρ_1 to fix a 'dart', we introduce the concept of a 'free edge' where just one dart resides on an edge (the opposite dart isn't there!)

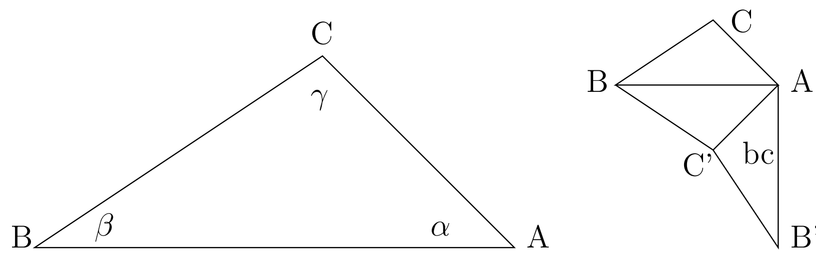
To work the theory in the opposite direction, we define $\Delta = (m, 2, n) = \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^m = \rho_1^2 = \rho_2^n = \rho_0\rho_1\rho_2 = 1 \rangle$, and we let $\hat{M}(m, n)$ be the universal map of type (m, n). Here m regular n-gons will meet at each vertex on a simply connected Riemann surface $U = U(m, n)$. This will be a hyperbolic plane if $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, an Euclidean plane if $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, and a Riemann sphere if $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$. For example, $(\infty, 2, 3)$ has infinitely many triangles, and $\hat{M}(\infty, 3)$ has vertices $\mathbb{Q} \cup \{\infty\}$, where $\frac{a}{b}$ is joined to $\frac{c}{d}$ if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$.

3 Seminar 3: 8th March 2005

3.1 Triangle Groups

This seminar follows Chapter 5 of D.L. Johnson's book "Topics in the Theory of Group Presentations", LMS Lecture Note Series 42.

Consider the triangle shown below.



We want to tessellate 'space' by these triangles. Let a denote the reflection in line BC, b the reflection in CA, and c the reflection in AB. Then, for example, bc leads to a counterclockwise rotation about A through 2α, as shown above (do 'c' first then 'b'). For a tessellation, we need $\alpha = \frac{\pi}{m}$, $\beta = \frac{\pi}{n}$ and $\gamma = \frac{\pi}{\ell}$, where $m, n, \ell \geqslant 2$. It follows that $(ab)^\ell = (bc)^m = (ca)^n = 1$. The triangle group has presentation

$$\Delta(\ell, m, n) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^\ell = (bc)^m = (ca)^n = 1\rangle.$$

In the Euclidean case, where $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 1$, there are three solutions: $(3, 3, 3)$ (tessellation by equilateral triangles, infinite), $(2, 4, 4)$ and $(2, 3, 6)$.

The von Dyck group $D(\ell, m, n)$ has index 2 in $\Delta(\ell, m, n)$, is generated by (ab) and (bc), and has presentation

$$D(\ell, m, n) = \langle x, y \mid x^\ell = y^m = (xy)^n = 1\rangle.$$

We note that $2(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1)^{-1} = |D(\ell, m, n)|$.

In the Elliptic case, where $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$, we have the following cases:

(ℓ, m, n)	$D(\ell, m, n)$	order	figure
$(2, 2, n)$	D_{2n}	$2n$	regular n-gon
$(2, 3, 3)$	A_4	12	tetrahedron
$(2, 3, 4)$	S_4	24	octahedron
$(2, 3, 5)$	A_5	60	icosahedron

Each triangle in the tessellation has area (by Gauss-Bonnet) $(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1)\pi$. But the area of a unit sphere is 4π, so there are $\frac{4}{(\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}-1)}$ $(= |\Delta(\ell, m, n)|)$ triangles in all.

Here is a diagram for the $(2, 3, 3)$ case. For the $(2, 3, 4)$ case, add an equator and two more great circles.