PhD Seminars 2004/2005: Dr. C. D. Wensley Semester 2

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1 Seminar 1: 1st February 2005

1.1 Matroids and Coxeter Groups

Definition 1.1 A matroid M is a pair $(E, \mathcal{B})$ where E is a non-empty finite set and $\mathcal{B}$ is a non-empty collection of subsets of E, called bases, which satisfy the following conditions:

- No base properly contains another base: $B_1 \subseteq B_2 \Rightarrow B_1 = B_2$.
- If B_1, B_2 are bases, and $e \in B_1$, then there exists $f \in B_2$ such that $(B_1 \setminus \{e\}) \cup \{f\}$ is a basis.

The second axiom is an example of a *replacement property.*

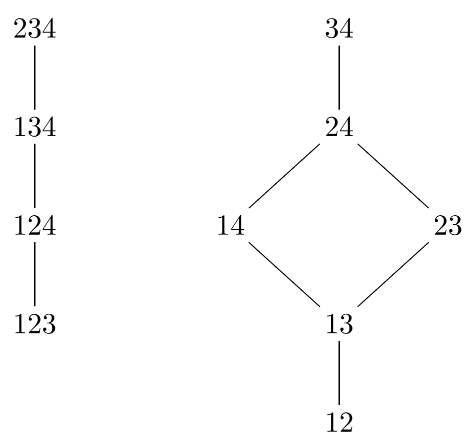
It is also possible to define a matroid in terms of independent sets and in terms of cycles. Given a finite subset E of $\mathbb{R}^n$, Whitney (1936) noticed a replacement property for linearly independent subsets of E: if $\mathcal{B}$ is a set of maximal linearly independent subsets of E, then $A, B \in \mathcal{B}$ and $a \in A \setminus B \Rightarrow \exists\, b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

1.1.1 The Maximality Property (apply a greedy algorithm)

In any total ordering of E there is a unique basis given by choosing the first elements you can (and we can identify total orderings with permutations). Let $[n] := \{1, 2, \dots, n\}$ and $\mathcal{P}_{n,k} :=$ the $\binom{n}{k}$ k-element subsets of $[n]$. For $A = \{a_1 < \cdots < a_k\}$ and $B = \{b_1 < \cdots < b_k\}$ in $\mathcal{P}_{n,k}$, define a partial order on $\mathcal{P}_{n,k}$ by

$$A \leqslant B \Leftrightarrow a_1 \leqslant b_1, \cdots, a_k \leqslant b_k.$$

Example 1.2 When $n = 4$ and $k = 3$ or 2, we have the following diagrams:



Definition 1.3 Let $W = S_n =$ all permutations of $[n]$. The *Gale ordering* on $\mathcal{P}_{n,k}$ induced by $w \in W$ is given by $A \leqslant^w B \Leftrightarrow Aw \leqslant Bw$.

Example 1.4 If $w = \binom{1\,2\,3\,4\,5}{4\,5\,1\,3\,2} = \binom{i}{iw}$, then $4 <^w 5 <^w 1 <^w 3 <^w 2$.

Theorem 1.5 (Gale, 1968) *Let $\mathcal{B} \subseteq \mathcal{P}_{n,k}$. Then $\mathcal{B}$ is (the collection of bases of) a matroid iff $\mathcal{B}$ satisfies the maximality property "for every $w \in S_n$ there is a unique $A \in \mathcal{B}$ which is maximal in $\mathcal{B}$ with respect to $\leqslant^w$".*

Let $n = 4$ and $k = 2$ so that $E = \{1, 2, 3, 4\}$. Then $\mathcal{B} = \mathcal{P}_{4,2}$ is a matroid, $\{\{1, 2\}\}$ etc. are all matroids, $\{\{1, 2\}, \{1, 3\}\}$ is a matroid, $\{\{1, 2\}, \{3, 4\}\}$ is not a matroid (as $\{\{1, 2\}, \{3, 4\}\}^{(2,4)} = \{\{1, 4\}, \{2, 3\}\}$), $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ is a matroid, $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is a matroid, and $\{\{1, 2\}, \{1, 3\}, \{3, 4\}\}$ is not a matroid.

Definition 1.6 A *Coxeter group* is a group W with generating set S and relations $s_i^2 = 1$ and $(sr)^{m_{s,r}} = 1$ for some $m_{s,r} \in \mathbb{Z}^+$, $r \neq s \in S$. Example of a Coxeter group: $S_n = \langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1,\ (s_i\, s_{i+1})^3 = 1,\ [s_i, s_j] = 1 \text{ when } |i - j| \geqslant 2\rangle$.

Definition 1.7 A *reduced expression* for $w \in S_n$ has the form $w = s_1 s_2 \ldots s_\ell$ of minimal length.

Definition 1.8 The Bruhat order $\leqslant$ on W is defined by: v covers $u \Leftrightarrow u = s_1 s_2 \ldots s_\ell$ and $v = s_0 s_1 \ldots s_\ell$ or $v = s_1 s_2 \ldots s_\ell s_{\ell+1}$ and these are reduced expressions.

For $R \subset S$, $P = \langle R\rangle \leqslant W$ is called a standard parabolic subgroup of W. Every coset Pw has a maximal element $\max Pw$, and the Bruhat order on W/P is $Pu \leqslant Pv \Leftrightarrow \max Pu \leqslant \max Pv$.

Example 1.9 Let $P = \langle r, s\rangle$ in S_4 (where $r = (1\,2)$, $s = (2\,3)$ and $t = (3\,4)$). Then $P \cong S_3$, and W/P is a straight line. However if $P = \langle r, t\rangle \cong C_2 \times C_2$, then W/P has a diamond in the middle of the straight line. In S_n, if $P = \langle s_1, \ldots, s_{k-1}, s_{k+1}, \ldots s_{n-1}\rangle = \mathrm{stab}\{1, \ldots, k\}$, then $W/P \cong \mathcal{P}_{n,k}$.

Definition 1.10 A *Coxeter matroid* for W and P is a subset $\mathcal{B} \subseteq W/P$ which satisfies the maximality property "for $w \in W$ there is a unique $A \in \mathcal{B}$ such that for all $B \in \mathcal{B}$, $B \leqslant^w A$".

The matroids of rank k on $[n]$ are precisely the Coxeter matroids for $W = S_n$ and the maximal parabolic subgroup $\langle s_1, \ldots, s_{k-1}\rangle$.