

PhD Seminars 2004/2005: Prof. T. Porter Semester 1

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1 Seminar 1: 5th October 2004

1.1 Homotopical Syzygies

In this series of seminars, we will follow Loday's 2000 paper "Homotopical Syzygies".

Consider a presentation $\mathcal{P} = \langle X \mid R \rangle$ presenting a group G . Let $K(\mathcal{P})$ have a single vertex, a 1-cell for each $x \in X$, and a 2-cell for each $r \in R$. The module of identities is $\pi_2(K(\mathcal{P}))$ which is isomorphic to $\ker(C(\mathcal{P}) \xrightarrow{\partial} F(X))$. Homotopical 2-syzygies: $C(\mathcal{P}) \cong \pi_2(K(\mathcal{P}))_2, K(\mathcal{P})_1, x_0$.

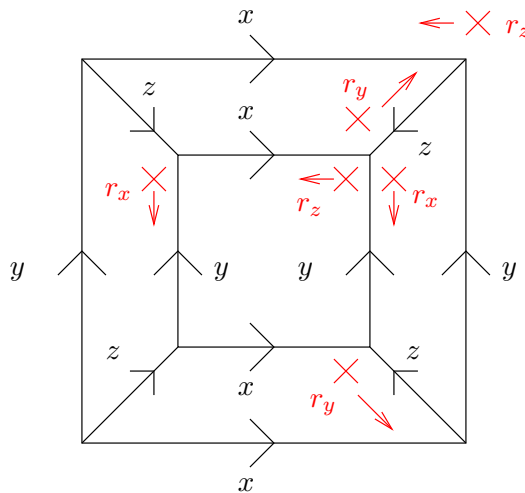
In the homological case, $C_0(G) = \mathbb{Z}G$, $C_1(G) = \mathbb{Z}G^{(X)}$ (e_x or e_x^{-1}), $C_2(G) = \mathbb{Z}G^{(R)}$ (e_r or e_r^2), $d_1(e_x) = \varphi(x) - 1$, and $d_2(e_r) = \sum_{x \in X} \frac{\partial r}{\partial x} e_x$, where $\frac{\partial}{\partial x}$ denotes a Fox derivative. Homological 2-syzygies: $\ker d_2$.

Example 1.1 Let $\mathbb{Z}^3 = \{x, y, z \mid r_x := [y, z], r_y := [z, x], r_z := [x, y]\}$, where $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. The Jacobi-Witt-Hall identity is the expression

$$[x^y, r_x][y^z, r_y][x^z, r_z] = 1.$$

For $K(\mathcal{P})$, take an empty cube, label each dimension 1 edge with e_x , label each dimension 2 edge with e_y , label each dimension 3 edge with e_z , and then glue the opposite faces to obtain $K(\mathcal{P})$.

Note that $S^2 \cong \partial I^3 \rightarrow K(\mathcal{P})$ represents an element in $\pi_2 K(\mathcal{P})$. If we use $S^2 \setminus \{\infty\} \cong \mathbb{R}^2$, the cube will give a planar graph as follows:



Loday claims that the linearisation of the JWH identity is $[r_x](1-x) + [r_y](1-y) + [r_z](1-z)$ (where $[r_x] = e_{r_x}$, etc.).

1.1.1 Igusa Pictures

Definition 1.2 An Igusa Picture is a finite oriented planar graph (up to isotopy) where every edge is labelled with a generator, every vertex has a preferred sector (marked by a *), and reading the labels of the edges (anticlockwise) gives either a relator or the inverse of a relator.

There is more on the above on page 5 of Loday's paper. It was agreed that the choices of whether to read in a clockwise or an anticlockwise manner and whether edges in and out are positive or negative are important choices.

2 Seminar 2: 19th October 2004

2.1 Nerves, etc.

Let G be a group, and let H_1 and H_2 be subgroups of G . Then $h = \{H_1g : g \in G\} \cup \{H_2g : g \in G\}$ is a covering of G , and $\langle H_1g, H_2g' \rangle \in N(h)_1$ if $H_1 \cap H_2g' \neq \emptyset$.

Remark 2.1 Let $X, \mathcal{U} \subset \mathcal{P}(X)$, and let $\bigcup U = X$ be a covering. Then $N(\mathcal{U})$, the nerve of the covering, has vertices $\langle U \rangle$, and $\{U_0, \dots, U_n\}$ is an n -simplex if $\bigcap_{i=0}^n U_i \neq \emptyset$ (written $\langle \dots \rangle$).

In Vietoris' $V(\mathcal{U})$, the vertices are the elements of X ($\langle x \rangle$), and $\{x_0, \dots, x_n\}$ is an n -simplex if $\exists U \in \mathcal{U}$ s.t. $x_i \in U$ for $i = 0, \dots, n$.

Assume that $\#(G) = 20$, $\#(H_1) = 5$ and $\#(H_2) = 4$. We have four 4-simplices and five 3-simplices (plus their faces). From this information it is possible to deduce that $\text{rank}(\pi_1(K_{4,5})) = 12$. Further, $\text{rank}(\pi_1(V(h))) = 12$ as well because of Dowker's theorem, $|N(\mathcal{U})| \simeq |V(\mathcal{U})|$.

The discussion then moved on to show how this work has similarities with Global Actions/Groupoid Atlases, Dana Scott's Equiological Spaces, Artificial Intelligence, Formal Concept Analysis and Theoretical Computer Science (!).

3 Seminar 3: 2nd November 2004

3.1 Igusa Pictures

In this seminar, the topic of Igusa Pictures was discussed, with definitions and examples from two papers analysed, namely “Homotopical Syzygies” by Jean-Louis Loday and “Pictures of $SL_2(\mathbb{Z})$ ” by Emma L. Smith. It was found that the definitions and *moves* on pictures differed slightly from paper to paper, as can be illustrated with the following definitions of an Igusa Picture.

Definition 3.1 (Loday) An Igusa picture for the presented group $G = \langle X \mid R \rangle$ is a finite oriented planar graph (up to isotopy) with the following additional data:

- Every edge is labelled by a generator,
- Every vertex has a preferred sector (marked by a $*$) and reading the labels of the edges (anticlockwise) gives either a relation or the inverse of a relation.

Definition 3.2 (Smith) A picture p of a group G presented by $\langle X \mid R \rangle$ is a planar directed graph with the following properties:

- Each edge is labelled by an element $g \in G$.
- Each vertex has a preferred sector denoted by $*$ with the property that moving around a vertex counterclockwise from the $*$ while recording each edge crossed as the associated element g if the edge is directed into the vertex and g^{-1} if the edge is directed away from the vertex will return a relation $r \in R \cup R^{-1}$.
- Each vertex is labelled $+$ if its relation is an element of R and $-$ otherwise.

During the seminar, it was discussed what happens when a generator acts on a picture and how we can convert from a 2-syzygy to a picture and back again.

4 Seminar 4: 9th November 2004

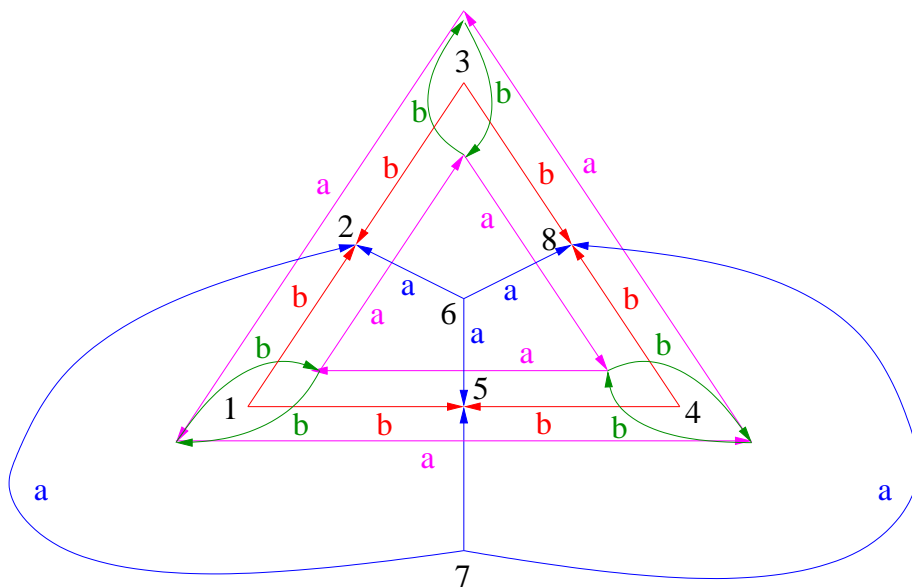
4.1 From Igusa’s pictures to identities among relations

During this seminar, a lively discussion took place trying to decipher the topic of going from Igusa’s pictures to identities among relations (pages 8-11 of Loday’s paper). An example was worked through on the blackboard which greatly enhanced our understanding of the subject.

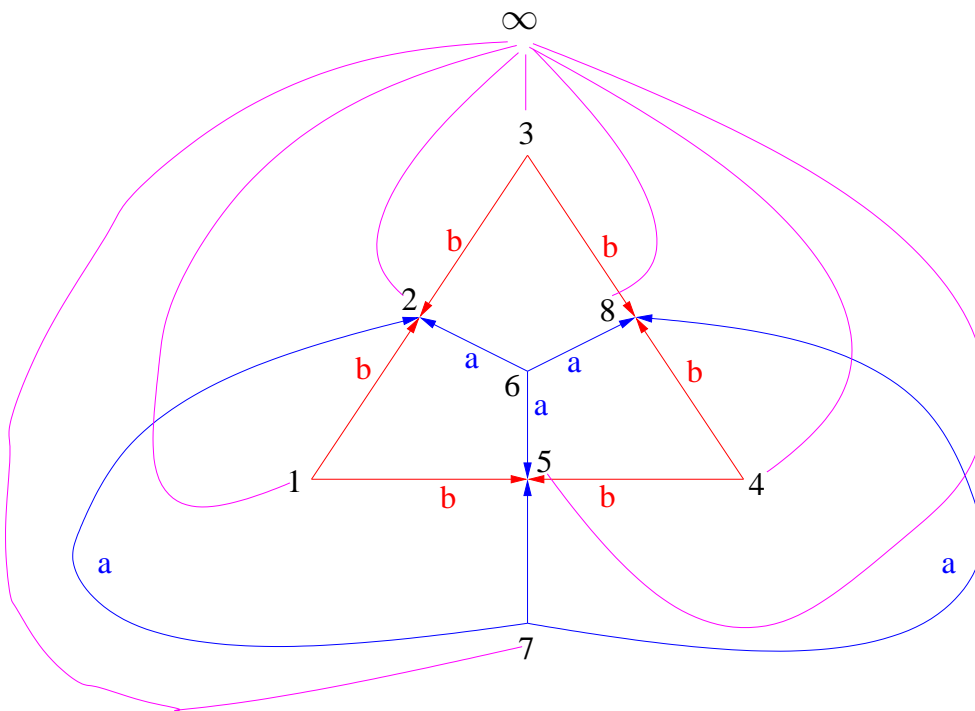
5 Seminar 5: 16th November 2004

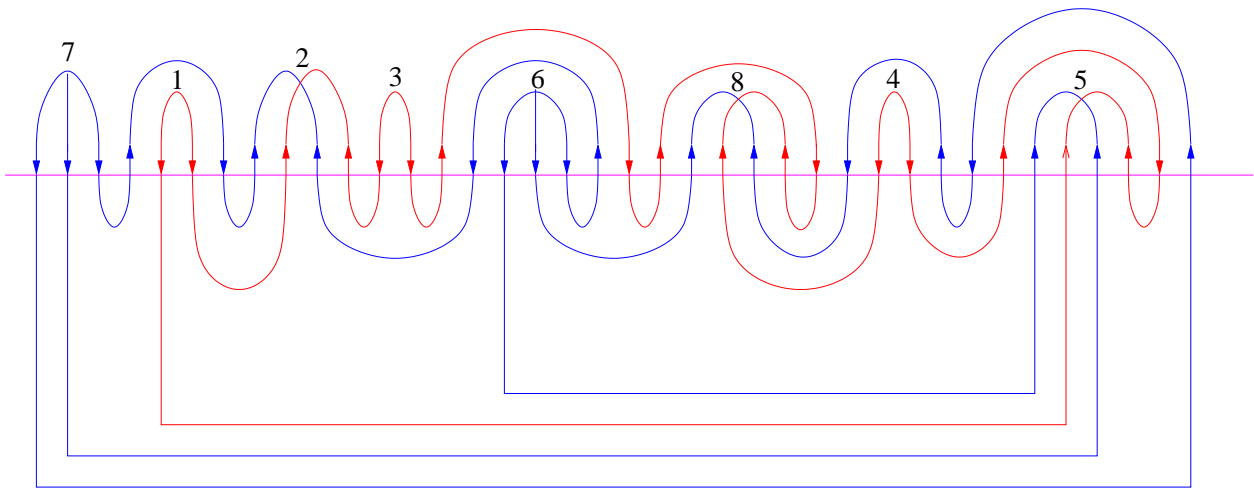
5.1 Example: Cayley Diagram \rightarrow Igusa Picture \rightarrow Identity Amongst Relations

Consider the group $S_3 = \langle a, b \mid r := a^3, s := b^2, t := (ab)^2 \rangle$. Let us transform the Cayley Diagram for S_3 to an Igusa Picture.



Let us now convert the Igusa Picture to an Identity Amongst Relations.





Word: $a^{-1}a^{-1}a^{-1}ab^{-1}b^{-1}a^{-1}ababb^{-1}b^{-1}ba^{-1}a^{-1}a^{-1}a^{-1}ab^{-1}bababb^{-1}a^{-1}b^{-1}b^{-1}aa^{-1}bababb^{-1}a$.

Relation: $r^{-1}(s^{-1})^{a^{-1}}ts^{-1}(r^{-1})^{ab^{-1}}(t)^{b^{-1}}(s^{-1})^a(t)^{b^{-1}a}$.

6 Seminar 6: 7th December 2004

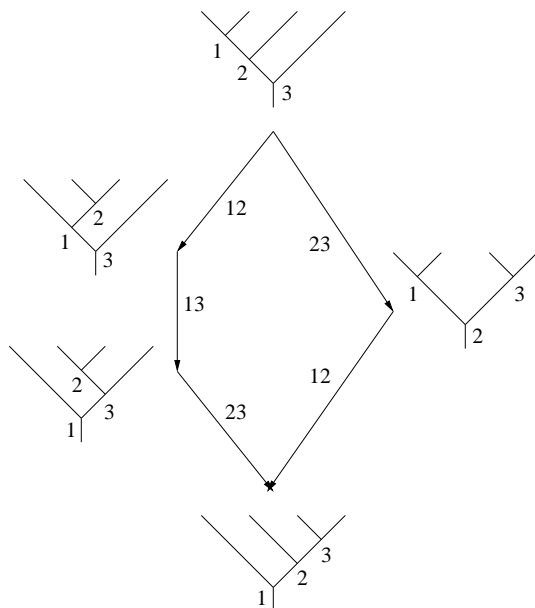
6.1 Homotopical Syzygies: Chapter 3

In this seminar, we worked through Chapter 3 of Loday's paper, looking in particular at the section on the Stasheff group, trying to understand the example and wondering if there are any connections with Kapranov and Saito's work.

The *Stasheff group* Sta_n is the group with generators x_{ij} for $1 \leq i < j \leq n$ and relations $x_{ij}x_{jk} = x_{jk}x_{ik}x_{ij}$ for $i < j < k$ and $x_{ij}x_{kl} = x_{kl}x_{ij}$ for $i < j < k < l$ or $i < k < l < j$. Note that Sta_2 is free on 1 generator and Sta_3 is free on two generators with relation $x_{12}x_{23} = x_{23}x_{13}x_{12}$.

For each n , Loday considers the poset of planar binary rooted trees with n internal vertices. The order is generated by the relation $x < y$ whenever y is obtained from x by moving an internal vertex from left to right. The geometric realisation of this poset is the *Stasheff polytope* K_n (also called the associahedron). For any Stasheff polytope the faces are products of Stasheff polytopes of lower dimensions.

When $n = 3$, Loday writes down the following diagram for K_3 .



We understood that the trees represented bracketings such as $((ab)c)d$, and, assuming that we assign numbers to the vertices of the tree according to a depth first search (working left-to-right), our interpretation of how a tree is changed by moving along an edge was as follows: "Moving along an edge pq moves everything to the right of vertex p (in the source diagram) to the edge immediately to the right of vertex q ".

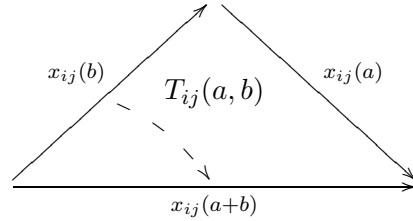
7 Seminar 7: 14th December 2004

7.1 Hieroglyphs

Consider the Steinberg Group with generators $x_{ij}(a)$ for $a \in A$ and $i \neq j$ ($A =$ a ring) and relations $x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$, $x_{ij}(a)x_{k\ell}(b) = x_{k\ell}(b)x_{ij}(a)$ if $i \neq \ell$ or $j \neq k$, and $x_{ij}(a)x_{jk}(b) = x_{jk}(b)x_{ik}(ab)x_{ij}(a)$. Reading the relators right-to-left, we obtain the following diagrams for each relation hieroglyph.

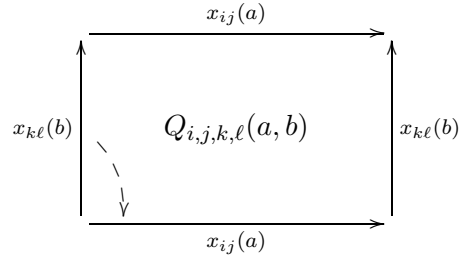
Hieroglyph: $i \xrightarrow{a,b} j$.

Diagram:



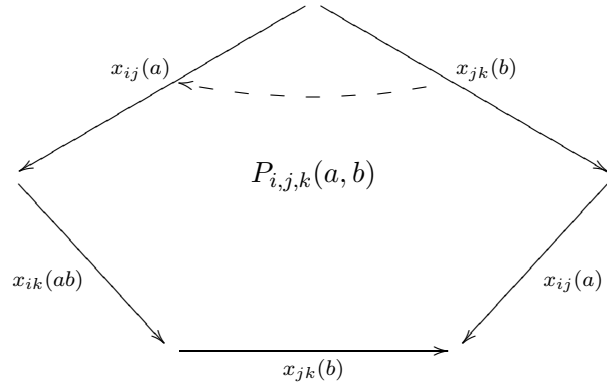
Hieroglyph: $i \xrightarrow{a} j \quad k \xrightarrow{b} \ell$.

Diagram:



Hieroglyph: $i \xrightarrow{a} j \xrightarrow{b} k$.

Diagram:



We can construct more diagrams by considering different hieroglyphs. For example, $i \xrightarrow{a,b,c} j$ gives a tetrahedron, $i \xrightarrow{a} j \quad k \xrightarrow{b} \ell \quad m \xrightarrow{c} n$ gives a cube, $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} \ell$ gives a permutrahedron, and the hieroglyph shown below gives a ‘Chicago building’!

