PhD Seminars 2003/2004: Prof. T. Porter Semester 3

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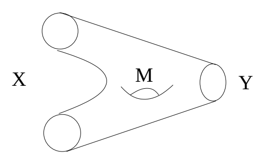
Contents

1 Seminar 1: 20th April 2004

1.1 Homotopy Quantum Field Theories and Stacks

Consider the category of vector spaces with linear maps. We have a monoidal structure $\otimes :$ Vect $\times$ Vect $\to$ Vect (a functor), with associator $a : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$, left unit $\ell : I \otimes A \cong A$, right unit $r : A \otimes I \cong A$, coherence rules, and duals A^*.

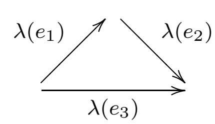
Example 1.1 Cobordism categories (d-cobord) have objects that are d-dimensional smooth closed oriented manifolds, and a typical morphism is of the form $M : X \to Y$, where M is a compact $d+1$ differential manifold with an orientation preserving diffeomorphism $f : X \sqcup -Y \to \partial M$.



The identity map is the cylinder $X \times [0,1]$ and composition is gluing. Further, the monoidal structure is $X \otimes Y = X \sqcup Y$, with $I = \emptyset$, the empty manifold, and $-X =$ a dual manifold (X with opposite orientation).

Definition 1.2 A $(d+1)$-dimensional TQFT assigns a vector space $Z(X)$ to each X in d-cobord, and a linear transformation $Z(M) : Z(X) \to Z(Y)$ to a cobordism $M : X \to Y$ such that $Z(X \sqcup Y) \cong Z(X) \otimes Z(Y)$ and $Z(-X) \cong Z(X)^*$, i.e. Z is a monoidal functor from d-cobord to Hilb.

One construction of a TQFT was described in 1992 by Yetter. Assume that the manifolds are triangulated and fix a finite group G, a manifold X and a triangulation T. A G-colouring of T is a map $\lambda : T_1 \to G$ such that if $\sigma \in T_2$ (and $\partial\sigma = e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3}$) then $\lambda(e_1)^{\varepsilon_1}\lambda(e_2)^{\varepsilon_2}\lambda(e_3)^{\varepsilon_3} = 1$. For example, the diagram



gives $\lambda(e_1)\lambda(e_2)\lambda(e_3)^{-1} = 1$ (integrating the colouring around a triangle gives the identity).

Let $\Lambda_G(T)$ be the set of G-colourings of T, and let $Z_G(X,T)$ be the vector space with $\Lambda_G(T)$ as basis. Passing to refined triangulations, we obtain a diagram of vector spaces and we define $Z_G(X)$ to be the colimit.

- $Z_G(X)$ is a $1+1$ TQFT (i.e. for $n = 2$).
- $Z_G(X)$ is isomorphic to the vector space whose basis is the set of conjugacy classes of morphisms from $\pi_1 X$ to G.

Yetter's construction is equivalent to fixing a space $B = BG$ (the classifying space) and computing a weighted sum over homotopy classes of maps $X \to B$. In 1993, Yetter generalised this to allow B to be a 2-type $B = BM$, a crossed module M. Porter then (in 1996) described how a G-colouring is equivalent to $\lambda : G(T) \to G$, where $G(T)$ is the Dwyer-Kan loop groupoid. By 1999, Turaeu had defined Homotopy QFT, shifting the role of B and finding extra structure on the manifolds given by the 'characteristic map' $X \to B$. A similar construction was adopted by Brightwell & Turner (2000) who classified $(1 + 1)$ HQFTs with B simply connected. Finally Porter & Turner (2002) described HQFTs in terms of 'formal maps', and a link with stacks was formed.

2 Seminar 2: 22nd April 2004

2.1 Homotopy Quantum Field Theories and Stacks, Part 2

Let B be any path connected pointed space, and consider HCobord(n, B). A B-manifold M is a closed n-manifold with base points m_i for each component M_i of M, with $g : M \to B$ such that $g(m_i) = *$. We can think of B as the 'background space' or the 'target space', giving some 'structure' to M.

A B-cobordism is of the form $(W, F) : (M, g) \to (N, h)$, where $W : M \to N$ is a cobordism (dimension $n+1$) and $F : W \to B$ is the homotopy class of maps rel ∂W with $A_M = g$ and $A_N = h$. The monoidal structure is $\otimes = \sqcup$.

Theorem 2.1 (Rodrigues, 2001) *HCobord(n, B) is a symmetric monoidal category.*

Definition 2.2 An $(n + 1)$ HQFT assigns the following:

- $\tau(M, g)$, a vector space to each B-manifold;
- $\tau\psi : \tau(M, g) \xrightarrow{\cong} \tau(N, h)$ for $\psi : (M, g) \to (N, h)$ over B;
- $\tau(W, F) : \tau(M, g) \to \tau(N, h)$ for $(W, F) : (M, g) \to (N, h)$, a B-cobordism;
- axioms.

An important change (from V.T. to G.R.) is the following: If $(I \times M, 1_g)$ is the identity B-cobordism on (M, g) (i.e. a cylinder), let $\tau(I \times M, 1_g) = id_{\tau(M,g)}$. (V.T. asks for $\tau(I \times M, F) = id_{\tau(M,g)}$ which is too strong — G.R. adjusted the axioms).

Theorem 2.3 (V.T., amended by G.R.) *An $(n + 1)$-dimensional HQFT τ with background B depends (up to natural isomorphism) only on the homotopy $(n + 1)$-type of B.*

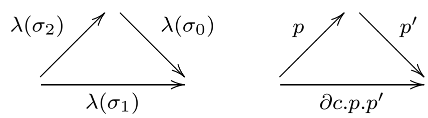
The above theorem suggests a project — given an algebraic model for the n-type of B, classify all HQFTs with that particular B as background.

- $n = 1$: 1-type modelled by a group G with $B \simeq k(G, 1)$ and $\pi_1(B) \cong G$. This gives a classification in terms of G-algebras (Turaeu).
- $n = 2$: With a simply connected B and an abelian group G, we get $B \simeq k(G, 2)$ and $\pi_2(B) \cong G$. This gives a classification (Brightwell & Turner (2000), Rodrigues (2001)) HQFT$(1, k(G, 2)) \simeq$ the category of G-Frobenius algebras (= Frobenius objects in G-vector spaces).
- $n > 2$: The easiest model (for us) is a simplicial group G with $NG_k = 1$ for $k > n$ so that we see that the classifying space of G is equal to $|\bar{W}G|$.

2.2 Formal Maps

Let M be a crossed module, let X be a manifold, and let T be a triangulation of X. A simplicial formal map on T is (1) an ordering $\leqslant$ on T_0 so that each simplex is totally ordered; (2) a simplicial map $\partial :\ T \to \bar{W}G(M) =: \mathrm{Ner}\ M$.

Taking the definition apart, if v is a vertex of T, then $\lambda(v) =$ a single vertex of $\bar{W}G(M)$; if $\sigma \in T_1$, then $\lambda(\sigma) \in P$; and if $\sigma \in T_2$, then $\lambda(\sigma) = C \rtimes P$, illustrated by the left hand side of the following diagram (or better illustrated by the second half of the diagram, where $\lambda(\sigma) = ((c, p), p')))$.



Now $\lambda :\ T \to \mathrm{Ner}(M)$ gives

$$\begin{array}{rcl} |\lambda| : & |T| & \longrightarrow BM\ , \\ & \big\| \cong & \\ & X & \end{array}$$

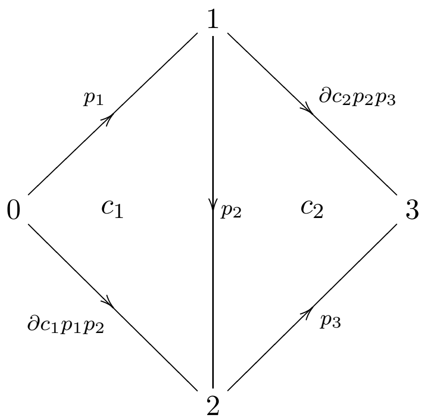
i.e. a characteristic map for a B-manifold $(X, |\lambda|)$ with $B = BM$ — and these are enough to specify the B-manifold after studying the effect of reordering and subdivision (up to a natural notion of equivalence of formal maps).

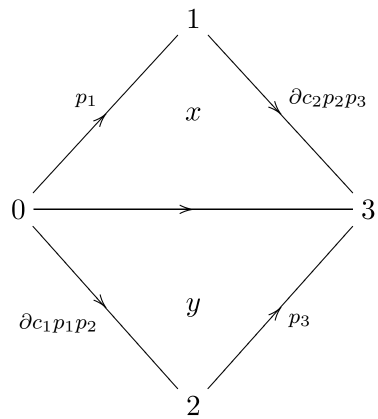
Remark 2.4 Combining simplicies gives cellular formal maps, allowing for a complete classification of 1+1 HQFTs.

3 Seminar 3: 30th April 2004

3.1 Homotopy Quantum Field Theories and Stacks, Part 3

Consider the two diagrams shown below.

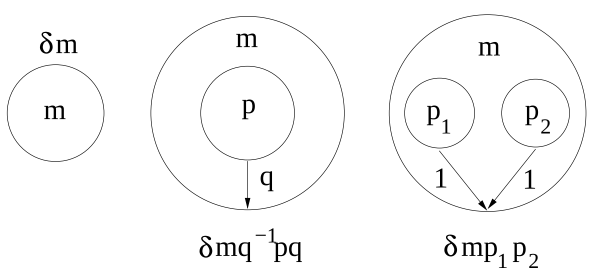




It is apparent that $\partial x p_1 \partial c_2 p_2 p_3 = \partial y \partial c_1 p_1 p_2 p_3$, and so $\partial x p_1 \partial c_2 p_1^{-1} \partial c_1^{-1} = \partial y$. If we assume that $x = 1_c$, then $y = {}^{p_1} c_2 c_1^{-1}$.

Remark 3.1 It is useful to think of these transitions between triangles as rewriting.

For $n = 1$, the key generating formal maps are shown below[1]. These allow for a complete classification of $1 + 1$ HQFTs.

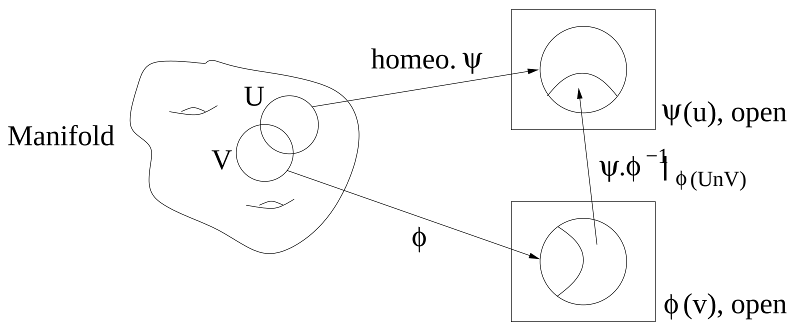


[1] In the diagram, the symbol δ is used as a substitute for the symbol ∂ (couldn't find ∂ in Xfig!)

4 Seminar 4: 9th May 2004

4.1 Homotopy Quantum Field Theories and Stacks, Part 4

4.1.1 Vector Bundles



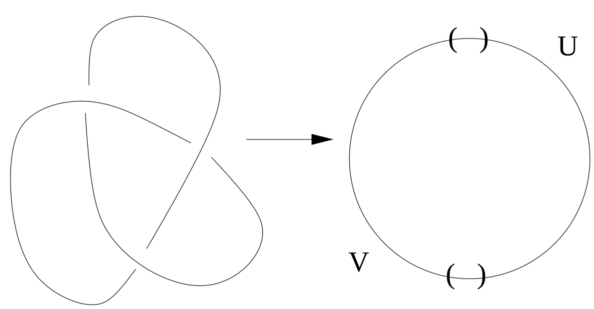
For the above diagram, a change of coordinates gives a map

$$\begin{array}{ccc} (U \cap V) \times \mathbb{R}^n & \xrightarrow{\quad \xi \quad} & (U \cap V) \times \mathbb{R}^n \\ & \searrow \quad \swarrow_{\xi_x \in \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{G}\ell_n(\mathbb{R})} & \\ & U \cap V & \end{array}$$

4.1.2 Fibre Bundles

Replacing $\mathbb{R}^n$ by some fibre F and $\mathrm{G}\ell_n(\mathbb{R})$ by some group G acting on F, we obtain a map p from a total space E to a base X, $F: \ E \xrightarrow{p} X$.

Example 4.1 In the diagram below, the trefoil (which is isomorphic to S^1) maps to a partitioned circle, the map being $e^{i\theta} \to e^{2i\theta}$. For the bundle, $F = \{+, -\}$, $G = C_2 = \{e, a\}$, and $U \cap V = 2$ pieces N, S (respectively e, a).



Other examples include the consideration of Möbius bands, cylinders with no twisting, and triangles (see the RPA CD-ROM and web site). In these examples, loops in X $(= S^1)$ correspond to transformations in G (i.e. a homomorphism $\pi_1(X) \to G$).

Now consider an open cover $\mathcal{U}$ of X. If we assume that $U_0 \leqslant U_1 \leqslant \ldots \leqslant U_n$, then

$$\mathrm{Ner}(\mathcal{U})_n = \{\langle U_0, \ldots, U_n \rangle \mid U_i \in \mathcal{U},\ \cap U_i \neq \emptyset\}.$$

If X is a manifold and $\mathcal{U}$ is fine enough, then $|\mathrm{Ner}(\mathcal{U})| \cong X$.

4.1.3 Link with stacks

Consider the diagram

$$\begin{array}{ccc} E_\lambda & \longrightarrow & WG\ , \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \lambda\ } & \bar{W}G \end{array}$$

where $G = G(\mathcal{M})$ and T is a simplicial fibre bundle (classical construction). What is this topologically? We know that $X \cong |T| \to BG = |\bar{W}G|$ gives X 'extra structure', but what extra structure is it?

A triangulation of X is equivalent to an open covering of X. For an open covering of X, $N(X,\mathcal{U})$ is a Čech nerve, and a formal map $N(X,\mathcal{U}) \to \bar{W}G$ interprets as non-abelian descent data. Further, for $U \in \mathcal{U}$, we get a trivial bundle $U \times \Xi(\mathcal{M})$, and the fibre is a (discrete) category (the underlying Cat^1-group of $\mathcal{M}$).

Remark 4.2 A construction on $\langle U_0, U_1, U_2 \rangle$ with $\lambda_{0,1,2} \in C \rtimes P \rtimes P$ yields a stack of groupoids on X, and this is work related to M.F.B.'s Ph.D. thesis.