# PhD Seminars 2003/2004: Prof. T. Porter Semester 2

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## Contents

# 1 Seminar 1: 13th January 2004

## 1.1 Rings with Several Objects

### 1.1.1 Basic Definitions

**Definition 1.1** In a (pre) additive category $\mathcal{C}$, (i) each $\mathcal{C}(p,q)$ is an abelian group, and (ii) the composition $\mathcal{C}(p,q) \times \mathcal{C}(q,r) \to \mathcal{C}(p,r)$ is bilinear (so that if $a_1,\ a_2 :\ p \to q$ and $b :\ q \to r$, then $(a_1 + a_2)b = a_1 b + a_2 b$, etc.).

**Remark 1.2** A ring is equivalent to a preadditive category with a single object. Alternatively, if $\mathcal{C}$ is preadditive, then $\mathcal{C}(p,p)$ is a ring (with 1).

**Example 1.3** Let $\pi$ be a small category. Then $\mathbb{Z}\pi$ is a preadditive category with $\mathbb{Z}\pi(p,q)$ being the free abelian group on $\pi(p,q)$. This generalises the monoid ring construction.

**Remark 1.4** Later on we will have $k$, a commutative ring or field, and we will consider $k$-additive categories where $\mathcal{C}(p,q)$ is a $k$-module and composition is $k$-bilinear. This example generalises to $k\pi$.

**Definition 1.5** For any small preadditive $\mathcal{C}$ we can form a ring with the arrows of $\mathcal{C}$ as elements and with $\alpha \cdot \beta = 0$ if $\alpha \cdot \beta$ is not defined, and $\alpha \cdot \beta = \alpha \cdot \beta$ if $\alpha \cdot \beta$ is defined. Note that this 'destroys structure'.

**Definition 1.6** Let $\pi$ be a category and let $\alpha :\ p \to q$, $\beta :\ q \to p$ in $\pi$ be such that $\alpha\beta = 1_p$. We say that $\beta$ is a retraction or a split epimorphism, that $\alpha$ is a coretraction or a split monomorphism, and that $p$ is a retract of $q$. Also $\theta = \beta\alpha$ is an idempotent, $\theta^2 = \theta$, and further it is a split idempotent.

**Definition 1.7** A category $\pi$ is idempotent complete if all idempotents split, i.e. given any object $q$ and a $\theta \in \pi(q,q)$ with $\theta^2 = \theta$, then there is a $p$, an $\alpha :\ p \to q$ and a $\beta :\ q \to p$ with $\theta = \beta\alpha$ and $\alpha\beta = 1_p$.

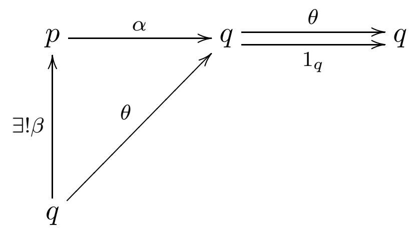
**Lemma 1.8** *If $\pi$ has equalisers, then it is idempotent complete.*

**Proof:** Given $\theta :\ q \to q$ with $\theta^2 = \theta$, let $p = \mathrm{Eq}(\theta, 1_q)$, i.e.

$$p \xrightarrow{\ \alpha\ } q \underset{1_q}{\overset{\theta}{\rightrightarrows}} q$$

is an equaliser diagram ($\alpha\theta = \alpha$ and $\alpha$ is universal with this property). **Claim:** $p$ is a retract of $q$.

Consider the following diagram (from which we get a uniquely defined $\beta : q \to p$ such that $\theta = \beta\alpha$).



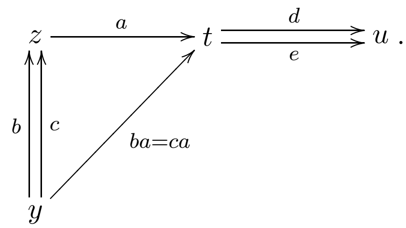

We now need to show that $\alpha\beta = 1_p$.

**Sublemma:** Any equaliser is a monomorphism (and hence $\alpha$ is one).

**Proof:** Recall that $a : z \to t$ is a monomorphism if given any $b, c : y \longrightarrow z$ such that $ba = ca$ then $b = c$. Suppose that

$$z \xrightarrow{\ a\ } t \underset{e}{\overset{d}{\rightrightarrows}} u$$

is an equaliser diagram (so that $ad = ae$ and we have a universal property). Let us look at




Suppose that $ba = ca$ so that $bad = bae$ and therefore $cad = cae$. Then there exists an unique $f : y \to z$ so that $b = c$ as required. □

*(Back to the proof of the first lemma)* Now $\alpha\beta\alpha = \alpha\theta = \alpha$ (by construction) $= 1_p\alpha$. But $\alpha$ is monic so that $\alpha\beta = 1_p$ as required. □

**Definition 1.9** A full subcategory $\pi \subset \bar{\pi}$ will be called a cover for $\bar{\pi}$ if every object in $\bar{\pi}$ is a retract of one in $\pi$.

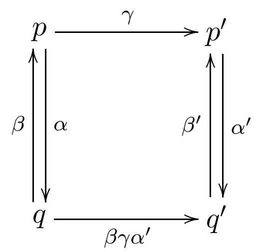
**Definition 1.10** If $\pi$ is a cover for $\bar{\pi}$ and $\mathbf{a}$ is idempotent complete, then given $F : \pi \to \mathbf{a}$ we can extend it to $\bar{F} : \bar{\pi} \to \mathbf{a}$.

**(Aside)** If $p \in \bar{\pi}$, then there is a $q \in \pi$, a monomorphism $\alpha : p \to q$, an epimorphism $\beta : q \to p$, and $\alpha\beta = 1_p$. Now $F(\beta\alpha)$ is an idempotent in $\mathbf{a}$ so there exists a factorisation

$$F(q) \xrightarrow{\ \ b\ \ } \bar{F}(p) \xrightarrow{\ \ a\ \ } F(q)$$

defining $\bar{F}(p)$ (where $\bar{F}(p)$ is chosen).

Now take $\bar{F}(\alpha) = a$ and $\bar{F}(\beta) = b$ (for each $q$), and suppose that $\gamma : p \to p'$ in $\bar{\pi}$. Then we have the diagram




and thus the sequence

$$\bar{F}(p) \xrightarrow{F(\alpha)} F(q) \xrightarrow{F(\beta\gamma\alpha')} F(q') \xrightarrow{F(\beta')} \bar{F}(p')\ .$$

Take this to be $\bar{F}(\gamma)$ and (hopefully!) it works.

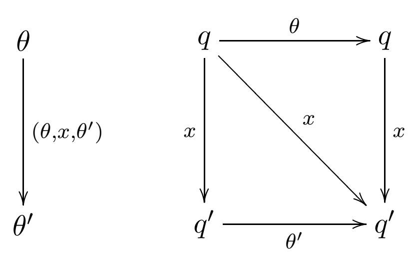
Similarly, we can extend $\eta : F \to G$ to $\bar{\eta} : \bar{F} \to \bar{G}$. **(End of Aside)**.

**Proposition 1.11** *If $\pi$ is a cover for $\bar{\pi}$ and $\bar{\pi}$ is small, then the inclusion $\pi \to \bar{\pi}$ induces an equivalence of categories $\mathbf{a}^{\bar{\pi}} \simeq \mathbf{a}^{\pi}$ for any idempotent complete $\mathbf{a}$.*

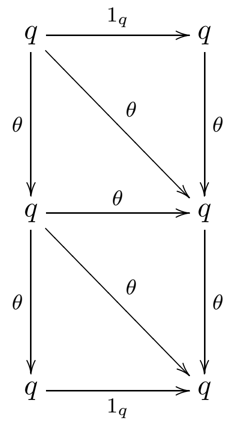
**Definition 1.12** If $\pi$ is a cover for $\bar{\pi}$ and $\bar{\pi}$ is idempotent complete, then $\bar{\pi}$ is called an idempotent completion of $\pi$ (such things are unique up to equivalence).

### 1.1.2 Idempotent Completions Exist (Freyd)

Given $\pi$, let $\hat{\pi}$ be such that its objects are idempotents of $\pi$. In the following diagram, we have $x : q \to q'$ such that $x\theta' = x = \theta x$ and composition is given by $(\theta, x, \theta')(\theta', y, \theta'') = (\theta, xy, \theta'')$.



The functor $\pi \to \hat{\pi}$ sends $q$ to $1_q : q \to q$. Suppose $q \in \pi$ and $\theta^2 = \theta : q \to q$. Then we have the diagram



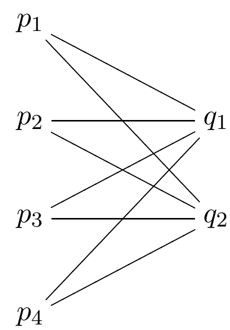

with $(1_q, \theta, \theta) : 1_q \to \theta$ as the '$\beta$' and $(\theta, \theta, 1_q)$ as the '$\alpha$'.

## 2 Seminar 2: 15th January 2004

### 2.1 Rings with Several Objects, Part 2

#### 2.1.1 Mat($\mathcal{C}$)

Given $\mathcal{C}$, we can form Mat($\mathcal{C}$) with objects $(p_1, \ldots, p_m)$ (with $p_i \in \mid \mathcal{C} \mid$) and arrows $[\gamma_{ij}] : (p_1, \ldots, p_m) \to (q_1, \ldots, q_n)$ (with $\gamma_{ij} : p_i \to q_j$ in $\mathcal{C}$). As an example, we have the following diagram (which is analogous to $p_1 \oplus p_2 \oplus p_3 \oplus p_4 \to q_1 \oplus q_2$ $((0,0,0,0)[\gamma_{ij}] = (\gamma_{11}a, \gamma_{12}a)$; 'partitioned matrices')).




Let $\mathbf{a}$ be additive with finite products. Then we have $\mathbf{a}^{\mathrm{Mat}(\mathcal{C})} \xrightarrow{\simeq} \mathbf{a}^{\mathcal{C}}$ which is induced by $\mathcal{C} \longrightarrow \mathrm{Mat}(\mathcal{C})$. An important point is that Mat($\mathcal{C}$) has finite products: if $\underline{p}$, $\underline{q} \in \mathrm{Mat}(\mathcal{C})$ then $(\underline{p}, \underline{q})$ is their product. Further, $(\underline{q}, \underline{p}) \cong (\underline{p}, \underline{q})$ with $\begin{pmatrix} 0 & I_P \\ I_Q & 0 \end{pmatrix}$ as the isomorphism; we also have $\mathbf{a}^{\mathcal{C}} \xrightarrow{\simeq} \mathbf{a}^{\mathrm{Mat}(\mathcal{C})} \xrightarrow{\simeq} \mathbf{a}^{\widehat{\mathrm{Mat}(\mathcal{C})}}$ (where $\widehat{\mathrm{Mat}(\mathcal{C})}$ is 'amenable').

For $F : \mathrm{Mat}(\mathcal{C}) \to \mathbf{a}$, we have $F(p, q) = F(p) \oplus F(q)$ and so it's determined by its operation on 1-by-1 matrices and length one strings.

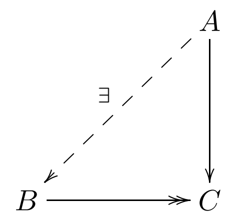
**Proposition 2.1** *Let $\tau$ be a complete lattice (as a category). Suppose $\tau \subset \pi$ (for some category $\pi$) so that if $\tau(p, q) = \phi$ then $\pi(p, q) = \phi$. Then $\tau$ is a retract of $\pi$.* ***Proof:*** *We can assume that $\pi$ is a poset. Let $u : \tau \to \pi$ so that $\tau$ is just a full subcategory of $\pi$. Let us construct $r : \pi \to \tau$ as follows: $p \in \pi$, $r(p) = \inf\{t \in \tau \mid t \geqslant p\} \in \tau$. Then $ru = \mathrm{Id}_\tau$.* □

#### 2.1.2 Functor Categories

Let $\mathbf{a}$ be a preadditive category and let $A$ be an object of $\mathbf{a}$. Then $\mathbf{a}_A = \mathbf{a}(A, \_) : \mathbf{a} \to \mathrm{Ab}$ so that $\mathbf{a}_A(B) = \mathbf{a}(A, B)$ is a representable functor. Similarly, $\mathbf{a}^*_A = \mathbf{a}(\_, A) : \mathbf{a}^{\mathrm{op}} \to \mathrm{Ab}$ and (ugh!) $\mathbf{a} : \mathbf{a}^{\mathrm{op}} \times \mathbf{a} \to \mathrm{Ab}$, $\mathbf{a}(A, B) = \mathbf{a}(A, B)$!

**Definition 2.2** An object $A$ in $\mathbf{a}$ is called projective if $\mathbf{a}_A$ preserves epimorphisms. It follows that

$B \twoheadrightarrow C$ gives $\mathbf{a}(A, B) \twoheadrightarrow \mathbf{a}(A, C)$ and we have the diagram




**Definition 2.3** $A$ is small if $\mathbf{a}_A$ preserves coproducts.

**Definition 2.4** $A$ is faithful if $\mathbf{a}_A$ is a faithful functor (if $f, g : B \to C$ and $\mathbf{a}(A, f) = \mathbf{a}(A, g)$, then $f = g$ and it reflects isomorphisms (if $\mathbf{a}(A, f)$ is an isomorphism, then so was $f$)).

**Definition 2.5** A family $\mathcal{F}$ of objects is faithful if $\times_{A \in \mathcal{F}} \mathbf{a}_A$ is faithful.

**Lemma 2.6 (The Additive Yoneda Lemma)** *If $\mathcal{C}$ is a preadditive category, then $\mathrm{Ab}^{\mathcal{C}}(\mathcal{C}, F) \simeq F$ (meaning $\mathrm{Ab}^{\mathcal{C}}(\mathcal{C}(*, \_), F(\_)) \simeq F(*)$). In particular, $\mathrm{Ab}^{\mathcal{C}}(\mathcal{C}_A, F) \cong F(A)$.* ***Proof:*** *If $\eta : \mathcal{C}_A \to F$ is a natural transformation, then $\eta_A : \mathcal{C}(A, A) \to F(A)$ and the isomorphism is $\eta \to \eta_A(\mathrm{id}_A)$.* □

**Theorem 2.7 (Freyd)** *A category $\mathbf{a}$ is equivalent to one of the form $\mathrm{Ab}^{\mathcal{C}}$ ($\mathcal{C}$ a preadditive category) iff $\mathbf{a}$ is abelian with coproducts and a faithful set of small projectives.*

(Next time we will prove the above theorem).

# 3 Seminar 3: 20th January 2004

## 3.1 Rings with Several Objects, Part 3

Before we prove the theorem that was stated at the end of the previous seminar, let us look at $\mathrm{Ab}^R$ for a ring $R$. Notice that this is just $R-\mathrm{Mod}$ and that the faithful set of small projectives is equal to $\{R\}$.

**Remark 3.1** In Ab itself, $\mathbb{Z}/2\mathbb{Z}$ is not faithful, e.g. for

$$B = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = C$$

with $f(n,1) = (0,1)$ and $g(n,1) = (n,1)$,

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{any}} B \overset{f}{\underset{g}{\rightrightarrows}} C$$

are always equal.

### 3.1.1 The Proof of Freyd's Theorem

**First Direction:** If $\mathbf{a} \simeq \mathrm{Ab}^{\mathcal{C}}$, then $\mathcal{C}_p = \{\mathcal{C}(p, \_) \ : \ \mathcal{C} \to \mathrm{Ab} \mid p \in \mathrm{Ob}(\mathcal{C})\}$ is a faithful set of small projectives 'in' $\mathbf{a}$.

- **Small.** Let $A$ correspond to $\mathcal{C}_p$. Does $\mathbf{a}(A, \_)$ preserve coproducts? If $M \in \mathbf{a}$ (fudging the distinction between $\mathbf{a}$ and $\mathrm{Ab}^{\mathcal{C}}$), i.e. $M \in \mathrm{Ab}^{\mathcal{C}}$ and $a = \mathrm{Ab}^{\mathcal{C}}$, then $\mathbf{a}(\mathcal{C}_p, M) \cong M(p)$ by the Yoneda lemma. Therefore $\mathbf{a}(\mathcal{C}_p, M \oplus N) \cong M(p) \oplus N(p)$ and so $\mathbf{a}(\mathcal{C}_p, \_)$ preserves coproducts.

- **Faithful.** Suppose $f_1, \ f_2 : \ M \to N$ in $\mathbf{a}$. Consider the diagram

$$\begin{array}{c} \prod_p \mathbf{a}(\mathcal{C}_p, M) \cong \prod_p M(p) \ . \\ {\scriptstyle \pi f_1(p)} \Big\downarrow\Big\downarrow {\scriptstyle \pi f_2(p)} \\ \prod_p N(p) \end{array}$$

  If $\pi f_1(p) = \pi f_2(p)$ then we must have $f_1 = f_2$.

- **Projective.** Suppose $M \to N$ is an epimorphism. Claim: for each $p$, $M(p) \to N(p)$ is an epimorphism of $\mathcal{C}(p,p)$ modules. (Proof of Claim ...) It now follows that $\mathbf{a}(\mathcal{C}_p, M) \xrightarrow{\mathbf{a}(\mathcal{C}_p, \pi)} \mathbf{a}(\mathcal{C}_p, N)$ is an epimorphism.

**Second Direction.** Suppose that $\mathbf{a}$ is abelian with coproducts and a faithful set of small projectives, and let us define $T : \mathbf{a} \to \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}$ as $T(A)(p) = \mathbf{a}(p, A)$. Claim: $T$ is an equivalence.

As the $p$'s are projective and small and as $\mathcal{P}$ is faithful, then $T$ is exact, coproduct preserving and faithful. For a fixed $B$ in $\mathbf{a}$, consider $\mathbf{a}(A, B) \xrightarrow{\theta_{A,B}} \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(TA, TB)$ which is induced by $T$. When $A \in \mathcal{P}$ and $A = p$, then $\mathbf{a}(p, B) = \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(\mathcal{P}_p^{\mathrm{op}}, TB) = \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(TA, TB) = TB(p)$, i.e. $\theta_{A,B}$ is the identity! Therefore if $A$ is a coproduct of $p$'s, then $\theta$ is an isomorphism because $T$ preserves coproducts.

Now $\mathcal{P}$ is faithful implies that $\prod_{p\in\mathcal{P}} \mathbf{a}(p, \_)$ is a faithful functor. But coproducts ($\coprod$) exist so that $\mathbf{a}(\coprod_p p, \_)$ is a faithful functor. Mitchell claims that this implies that for any $A$ there is a (right) exact sequence

$$\oplus_{j\in J} p_j \to \oplus_{i\in I} p_i \to A \to 0$$

(i.e. a presentation of $A$). This will do the trick as applying $\theta$ to this sequence we get the following diagram (where the leftmost $\theta$ will also be an isomorphism).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{a}(A,B) & \longrightarrow & \mathbf{a}(\oplus p_i, B) & \longrightarrow & \mathbf{a}(\oplus p_j, B) \\
 & & \downarrow \theta & & \cong \downarrow \theta & & \cong \downarrow \theta \\
0 & \longrightarrow & \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(TA, TB) & \longrightarrow & \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(T(\oplus p_i), TB) & \longrightarrow & \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}(T(\oplus p_j), TB)
\end{array}$$

It is sufficient to prove that given $A$ there exists an epimorphism $\oplus p_i \to A$. This will show that $T$ is an embedding, but we will also need to show that $T$ is essentially epimorphic. Suppose $M \in \mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}$. The representables form a faithful set in $\mathrm{Ab}^{\mathcal{P}^{\mathrm{op}}}$ so that (using the assumptions above!) there is a sequence

$$\oplus_{j\in J} \mathcal{P}_{p_j}^{\mathrm{op}} \xrightarrow{f} \oplus_{i\in I} \mathcal{P}_{p_i}^{\mathrm{op}} \to M \to 0.$$

This is the sequence $T(\oplus p_j) \xrightarrow{f} T(\oplus p_i) \to M \to 0$. But $T$ is full so there exists an $\alpha$ such that $T\alpha = f$ $(\oplus_j p_j \xrightarrow{\alpha} \oplus_i p_i)$. Let $A = \mathrm{coker}\,(\alpha)$ so that $TA \cong M$.

We will continue the proof in the next seminar…

# 4 Seminar 4: 10th February 2004

## 4.1 Homological Dimension

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Given objects $A$, $C$ in $\mathcal{A}$, The Ext-Functor $\mathrm{Ext}^n_{\mathcal{A}}(A, C)$ is the set of exact sequences $\mathbb{E}: 0 \to C \to B_{n-1} \to \ldots \to B_0 \to A \to 0$ modulo the equivalence $\sim$ generated by

$$\begin{array}{cccccccccccccc}
\mathbb{E} & 0 & \longrightarrow & C & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_0 & \longrightarrow & A & \longrightarrow & 0\,. \\
 & & & \downarrow{=} & & \downarrow{f_{n-1}} & & \cdots & & \downarrow{f_0} & & \downarrow{=} & & \\
\mathbb{E}' & 0 & \longrightarrow & C & \longrightarrow & B'_{n-1} & \longrightarrow & \cdots & \longrightarrow & B'_0 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

For $n = 0$, we have $\mathrm{Ext}^0_{\mathcal{A}}(A, C) := \mathrm{Hom}(A, C)$, and $\mathrm{Ext}^1_{\mathcal{A}}(A, C)$ is made up of equivalence classes of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & B_0 & \longrightarrow & A & \longrightarrow & 0\,. \\
 & & \| & & \downarrow{f_0} & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & B'_0 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Note that in the above diagram, $f_0$ is an isomorphism, but $f_0$ will not be an isomorphism in higher dimensions — for $n = 2$, we will have

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & A & \longrightarrow & 0\,. \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & B_1 \oplus M & \longrightarrow & B_0 \oplus M & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Now if $0 \to C \to B \to A \to 0$ is a short exact sequence and if $X$ is any old object of $\mathcal{A}$, then there is a long exact sequence $\to \mathrm{Ext}^{n+1}(C, X) \to \mathrm{Ext}^n(A, X) \to \mathrm{Ext}^n(B, X) \to \mathrm{Ext}^n(C, X) \to \ldots$.

**Definition 4.1** The homological dimension of $A \in \mathcal{A}$ is defined to be $\mathrm{hd}\, A = \sup\{n \mid \mathrm{Ext}^n(A, \_) \neq 0\}$. The global dimension of $\mathcal{A} = \sup\{\mathrm{hd}\, A \mid A \in \mathcal{A}\}$.

**Definition 4.2** If we can find a projective resolution of $A$, say $X_\bullet \to A$, then

$$\mathrm{Ext}^n(A, C) \cong H^n(\mathrm{hom}(X_\bullet, C)).$$

Some quick properties:

- Each $X_n$ is projective.
- $\to X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_{n-2} \to \ldots$ all have $\ker\partial = \operatorname{im}\partial$.
- $X_1 \to X_0 \to A \to 0$ is exact.

Now $\partial^* F = F\partial$ so that $f \in \ker\partial^*$ $((f\partial : X_{n+1} \to C) = 0)$; and if $f \in \operatorname{im}\partial^*$ then there exists a $g : X_{n+1} \to C$, $f = g\partial$. Further, if $f \in \ker\partial^*$, we can form

$$\begin{array}{lccccccccccc}
 & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & A \\
 & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & & & \| & & \| \\
\mathbb{E}_f : & 0 & \longrightarrow & C & \longrightarrow & \text{pushout} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & A
\end{array}$$

and can show that if $f, f' \in \ker\partial^*$ and if $f - f' = \partial^* g$, then $\mathbb{E} \sim \mathbb{E}'_f$.

### 4.2 Dimension of $k$-categories

Let $k$ be a commutative ring, let $\mathcal{C}$ be a small $k$-category (each $\mathcal{C}(p,q)$ is a $k$-mod), and let $\mathcal{C}^{\mathcal{C}} = \mathcal{C}^{\text{op}} \underset{k}{\otimes} \mathcal{C}$. The objects are pairs $(p_1, p_2)$ of objects in $\mathcal{C}$ with $\mathcal{C}^{\mathcal{C}}((p_1,p_2),(q_1,q_2)) = \mathcal{C}^{\text{op}}(p_1,q_1) \underset{k}{\otimes} \mathcal{C}(p_2,q_2) = \mathcal{C}(q_1,p_1) \underset{k}{\otimes} \mathcal{C}(p_2,q_2)$. Note that we have the 'horrible' bivariant hom functor $\mathcal{C} : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \to k - \text{mod}$ (or Ab), i.e. $\mathcal{C} \in \text{Ab}^{\mathcal{C}^{\mathcal{C}}}$, $\mathcal{C}(p_1,p_2) = \mathcal{C}(p_1,p_2)$.

If $f$ is another object in $\text{Ab}^{\mathcal{C}^{\mathcal{C}}}$, the $n^{\text{th}}$ Hochschild-Mitchell cohomology of $\mathcal{C}$ with coefficients in $F$ is $H^n(\mathcal{C}, F) = \text{Ext}^n_{\mathcal{C}^{\mathcal{C}}}(\mathcal{C}, F)$. Therefore $\dim_k(\mathcal{C}) = \sup\{n \mid H^n(\mathcal{C}, \_) \neq 0\} = \text{hd}_{\mathcal{C}^{\mathcal{C}}}\mathcal{C}$. Note: if $\mathcal{C} = k\pi$, write $\dim_k\pi$, etc. Questions: (i) What does $\dim_k\pi$ tell us about $\pi$?; (ii) How might one calculate $\dim_k\pi$?

### 4.3 The Standard Complex

Let $n \geqslant -1$ and consider $\bar{S}_n(\mathcal{C})$ which is in $\text{Ab}^{\mathcal{C}^{\mathcal{C}}}$.

- $\bar{S}_{-1}(\mathcal{C}) = \mathcal{C}(\_,\_) : \mathcal{C}^e \to k - \text{mod}$.
- $\bar{S}_n(\mathcal{C}) = \underset{(p_0,\ldots,p_n)}{\oplus} \mathcal{C}(\_,p_0) \underset{k}{\otimes} \mathcal{C}(p_0,p_1) \underset{k}{\otimes} \ldots \underset{k}{\otimes} \mathcal{C}(p_{n-1},p_n) \underset{k}{\otimes} \mathcal{C}(p_n,\_)$.

At 0, given $(p,q)$ we get an element

$$p \xrightarrow{\alpha_0} p_0 \xrightarrow{\alpha_1} p_1 \to \ldots \xrightarrow{\alpha_n} p_n \xrightarrow{\alpha_{n+1}} q.$$

Define $\partial : \bar{S}_n(\mathcal{C}) \to \bar{S}_{n-1}(\mathcal{C})$ so that

$$\partial(\alpha_0 \otimes \alpha_1 \otimes \ldots \otimes \alpha_n \otimes \alpha_{n+1}) = \sum_{i=0}^{n} (-1)^i \alpha_0 \otimes \ldots \otimes \alpha_i\alpha_{i+1} \otimes \ldots \otimes \alpha_{n+1}$$

(this is an alternating sum of simplicial face maps $d_i(\alpha_0 \otimes \ldots \otimes \alpha_{n+1}) = \alpha_0 \otimes \ldots \otimes \alpha_i \alpha_{i+1} \otimes \ldots \otimes \alpha_{n+1}$).

We define $s_n : \bar{S}_n(\mathcal{C}) \to \bar{S}_{n+1}(\mathcal{C})$ as $s_n(\underline{\alpha}) = 1 \otimes \underline{\alpha}$ which is natural in $\mathcal{C}$ but not in $\mathcal{C}^{\mathrm{op}}$. Therefore we write $s_\bullet(\mathcal{C})$ for $\to \ldots \to \bar{S}_2(\mathcal{C}) \to \bar{S}_1(\mathcal{C}) \to \bar{S}_0(\mathcal{C}) \to 0$ which is an acyclic complex over $\mathcal{C}$, and if $\mathcal{C}$ is $k$-projective (i.e. each $\mathcal{C}(p,q)$ is projective) then each $s_n(\mathcal{C})$ is a projective module and $s_\bullet(\mathcal{C})$ is a projective resolution of $\mathcal{C}(\_,\_)$ known as the standard resolution.

## 5 Seminar 5: 24th February 2004

### 5.1 The Standard Complex (continued)

Let $\mathcal{D}$ be another $k$-category, let $G \in \mathrm{Ab}^{\mathcal{C}^{\mathrm{op}} \overset{\otimes}{k} \mathcal{D}}$ (a left-$\mathcal{C}$-module-right-$\mathcal{D}$-module), and let

$$S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G = \int_p S_\bullet(\mathcal{C})(\_, p) \overset{\otimes}{k} G(p, \_).$$

**Remark 5.1** If $a = S_\bullet(\mathcal{C})(x, p)$, $r = \mathcal{C}(p, p')$ and $b = G(p', q)$, then in $a \otimes r \otimes b$, $a \otimes r = S_\bullet(\mathcal{C})(x, p') = a.r$ and $r \otimes b = G(p, q) = rb$ so that $a.r \otimes b \equiv a \otimes rb$.

$S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G$ gives an acyclic left complex over $C \overset{\otimes}{\mathcal{C}} G \cong G$ in $\mathrm{Ab}^{\mathcal{C}^{\mathrm{op}} \overset{\otimes}{k} \mathcal{D}}$. If $G$ is $\mathcal{D}$-projective, this is a projective resolution of $G$.

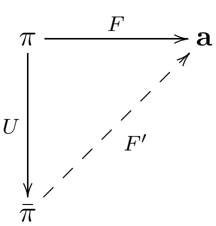
**Example 5.2** If $U : \mathcal{C} \to \mathcal{D}$, then $\mathcal{D}(U\_, \_) : \mathcal{C}^{\mathrm{op}} \overset{\otimes}{k} \mathcal{D} \to \mathrm{Ab}$ or $k - \mathrm{Mods}$ is $\mathcal{D}$-projective.

Assume that $\mathcal{C}$ is $k$-projective and that $G \in \mathrm{Ab}^{\mathcal{C}^{\mathrm{op}} \overset{\otimes}{k} \mathcal{D}}$ is also $k$-projective. Then $S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G$ is $\mathcal{C}^{\mathrm{op}}$-projective and $S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G$ is a $\mathcal{C}^{\mathrm{op}}$-projective resolution of $G$. If $F \in \mathbf{a}^{\mathcal{C}}$, $\mathbf{a}$ abelian with coproducts, then $F \overset{\otimes}{\mathcal{C}} S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G \longrightarrow F \overset{\otimes}{\mathcal{C}} G$ has homology $H_n(F \overset{\otimes}{\mathcal{C}} S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G) = \mathrm{Tor}_n^{\mathcal{C}}(F, G)$. And if $F \in \mathbf{a}^{\mathcal{C}^{\mathrm{op}}}$, then $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G, F)$ has cohomology $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(G, F)$.

**Remark 5.3** $F \overset{\otimes}{\mathcal{C}} S_\bullet(\mathcal{C}) \overset{\otimes}{\mathcal{C}} G$ is the two-sided bar construction beloved of homotopy theorists!

#### 5.1.1 A Special Case

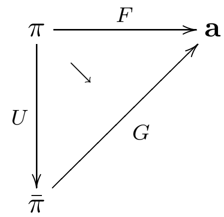
Let $\pi \overset{U}{\longrightarrow} \bar{\pi}$ be an inclusion of small categories. We have $\mathbb{Z}U : \mathbb{Z}\pi \to \mathbb{Z}\bar{\pi}$ and the induced functor $F \in \mathbf{a}^{\pi}$, $q \in \mathrm{Ob}(\bar{\pi})$. Consider the following diagram for which we want the 'best' $F'$ (the triangle will probably never commute).



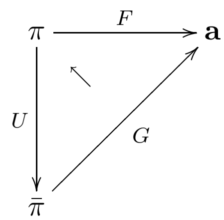

There are two ways of interpreting 'best'.

(1) Suppose $G : \bar{\pi} \to \mathbf{a}$. We could ask for $\mathbf{a}^{\pi}(F, GU) \cong \mathbf{a}^{\bar{\pi}}(F', G)$, where $F' = \mathrm{Lan}_U F$ is the left

Kan extension of $F$ along $U$.




(2) Here $\mathbf{a}^{\pi}(GU, F) \cong \mathbf{a}^{\bar{\pi}}(G, F')$, where $F'$ is the right Kan extension.




It follows that

$$F'(q) = \mathrm{Lan}_U F(q) = \int_p F(p) \overset{\otimes}{_k} \mathbb{Z}\bar{\pi}(Up, q).$$

# 6 Seminar 6: 12th March 2004

## 6.1 2-Categories

Let $\mathcal{A}, \mathcal{B}, \ldots$ be small categories, let $A, A', \ldots$ be objects in $\mathcal{A}$, let $F, F' : \mathcal{A} \to \mathcal{B}$ and $G, G' : \mathcal{B} \to \mathcal{C}$ be functors, and let $\alpha : F \to F'$ and $\beta : G \to G'$ be natural transformations.

$$\mathcal{A} \underset{F'}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{G'}{\overset{G}{\rightrightarrows}} \mathcal{C} \qquad (\Downarrow \alpha,\ \Downarrow \beta)$$

Consider $A \in \mathcal{A}$. Then $\alpha A : FA \to F'A$ is in $\mathcal{B}$. Applying $G$ and $G'$, we get $G\alpha A : GFA \to GF'A$ and $G'\alpha A : G'FA \to G'F'A$ (which are in $\mathcal{C}$). If $B \in \mathcal{B}$, we get $\beta B : GB \to G'B$, and taking $B$ to be $FA$ and $F'A$ in turn, we obtain $\beta FA : GFA \to G'FA$ and $\beta F'A : GF'A \to G'F'A$. Putting all of this together, we obtain the following diagram.

$$\begin{array}{ccc}
GFA & \xrightarrow{G\alpha A} & GF'A \\
{\scriptstyle \beta FA}\downarrow & & \downarrow{\scriptstyle \beta F'A} \\
G'FA & \xrightarrow[G'\alpha A]{} & G'F'A
\end{array}$$

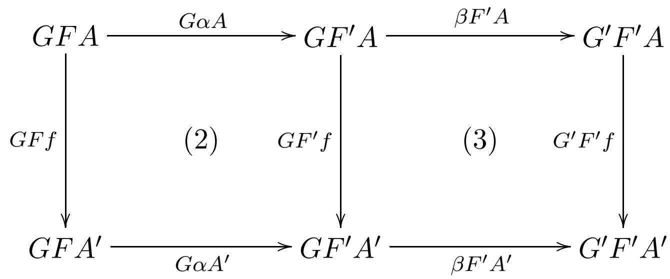
**Remark 6.1** $\beta$ is a natural transformation so that if $F : B \to B'$, we get the following diagram (take $B = FA \xrightarrow{F=\alpha A} F'A = B'$).

$$\begin{array}{ccc}
GB & \xrightarrow{Gf} & GB' \\
{\scriptstyle \beta B}\downarrow & \text{(comm.)} & \downarrow{\scriptstyle \beta B'} \\
G'B & \xrightarrow[G'f]{} & G'B'
\end{array}$$

Now define $\beta\#_0\alpha : GF \to G'F'$ to be given by $(\beta\#_0\alpha)A = \beta F'A \cdot G\alpha A$ $(= G'\alpha A \cdot \beta FA)$. **Claim:** $\beta\#_0\alpha$ is a natural transformation. **Proof of Claim:** Given $f : A \to A'$, we need to check the commutativity of the following diagram:

$$\begin{array}{ccc}
GFA & \xrightarrow{(\beta\#_0\alpha)A} & G'F'A \\
{\scriptstyle GFf}\downarrow & (1) & \downarrow{\scriptstyle G'F'f} \\
GFA' & \xrightarrow[(\beta\#_0\alpha)A']{} & G'F'A'
\end{array}$$

Expanding, we get the diagram shown below, where (2) commutes because it is the image under $G$ of $\alpha$ (which is a natural transformation), and (3) commutes because $\beta$ is a natural transformation.

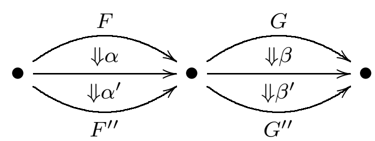



**Remark 6.2** $\#_0$ and $\#_1$ are known as horizontal and vertical compositions respectively.

Naturally, we would expect $\#_0 : \text{Cat}(\mathcal{A},\mathcal{B}) \times \text{Cat}(\mathcal{B},\mathcal{C}) \to \text{Cat}(\mathcal{A},\mathcal{C})$ to be a functor ($\text{Cat}(\mathcal{A},\mathcal{B})$ *is* a category). It turns out that $\#_0$ is a functor iff

$$(\beta'\#_1\beta)\#_0(\alpha'\#_1\alpha) = (\beta'\#_0\alpha')\#_1(\beta\#_0\alpha),$$

the Godement interchange law.



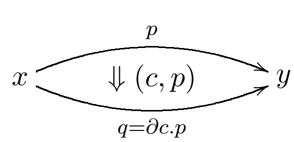

**Definition 6.3** A 2-category $\mathcal{C}$ has the following properties:

- $\mathcal{C}$ is a category;
- For each $A, B \in \mathcal{C}$, $\mathcal{C}(A,B)$ has a category structure;
- $1_A \in \mathcal{C}(A,A)$ is an object in this category;
- $\mathcal{C}(A,B) \times \mathcal{C}(B,C) \xrightarrow{\#_0} \mathcal{C}(A,C)$ is a functor.

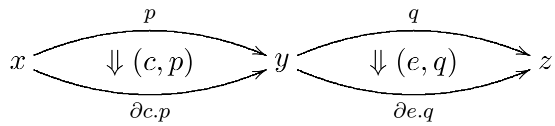
**Example 6.4** Let $\mathcal{M}$ be a crossed module of groups $C \xrightarrow{\partial} P$, where $C$ and $P$ are objects over some set $O$ of objects, $\partial$ is the identity on objects, and we have an action of $P$ on $C$ satisfying (CM1) and (CM2) as shown below.

- (CM1) $\partial({}^p c) = p\partial c p^{-1}$;
- (CM2) ${}^{\partial c}c' = cc'c^{-1}$, the Peiffer rule.

Write $x, y, \ldots$ for objects in $O$. It follows that $\mathcal{X}(\mathcal{M})(x,y)$ is a category with objects $p : x \to y$ in $P(x,y)$, and if $p, q \in P(x,y)$ then $(\mathcal{X}(\mathcal{M})(x,y))(p,q) = \{c \in C \mid \partial c.p = q\} \times \{p\} = \{(c,p) \mid \partial c.p = q\}$.



Define $(d, \partial c.p)\#_1(c,p) := (dc, p)$ and $(e,q)\#_0(c,p) := (e^q c, qp)$.




We need to check that $\partial(e^q c, qp) = \partial e.q.\partial c.p$ (CM1) and we can then check that $\mathcal{X}(\mathcal{M})(x,y)$ *is* a category and that $\mathcal{X}(\mathcal{M})$ *is* a 2-category (in fact a 2-Gpd). The interchange law is then equivalent to CM2.