PhD Seminars 2003/2004: Other Seminars in Semester 2

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March 9, 2004

Contents

1 Seminar 1: Richard Lewis (27th February 2004)

1.1 Limits and Colimits

1.1.1 Limits

Consider a functor $F : \mathbb{I} \to \mathcal{C}$. A cone for F is an object V of $\mathcal{C}$ and maps $q_I : V \to FI$ for each $I \in \mathbb{I}$ such that for all $u : I \to S$ we have the following commuting triangle.

$$\begin{tikzcd}
 & V \arrow[dl, "q_I"'] \arrow[dr, "q_S"] & \\
FI \arrow[rr, "Fu"] & & FS
\end{tikzcd}$$

A limit is a universal cone $p_I : \lim F \to FI$ with triangle

$$\begin{tikzcd}
 & \lim F \arrow[dl, "p_I"'] \arrow[dr, "p_S"] & \\
FI \arrow[rr] & & FS
\end{tikzcd}$$

such that whenever $q_I : V \to FI$ is a cone, then there exists a unique map $k : V \to \lim F$ such that the following diagram commutes.

$$\begin{tikzcd}
 & V \arrow[d, "k"] \arrow[ddl, "q_I"'] \arrow[ddr, "q_J"] & \\
 & \lim F \arrow[dl, "p_I"] \arrow[dr, "p_J"'] & \\
FI \arrow[rr, "Fu"'] & & FJ
\end{tikzcd}$$

Example 1.1 For $\mathbb{I} = (\bullet, \bullet)$, $F : \mathbb{I} \to \mathcal{C}$ gives two objects $A, B \in \mathcal{C}$. A cone is

$$\begin{tikzcd}
 & V \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

and a limit is a product

$$\begin{tikzcd}
 & A \times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd} .$$

1.1.2 Colimits

Given $F : \mathbb{I} \to \mathcal{C}$, a cocone is an object $V \in \mathcal{C}$ and a family $(q_I : FI \to V)_{I \in \mathbb{I}}$ such that the diagram

$$\begin{tikzcd}
FI \arrow[rr, "Fu"] \arrow[dr, "q_I"'] & & FJ \arrow[dl, "q_J"] \\
& V &
\end{tikzcd}$$

commutes for all $u : I \to J$ in $\mathbb{I}$. A colimit is a universal cocone $(FI \to \operatorname{colim} F)_{I \in \mathbb{I}}$ with the diagram

$$\begin{tikzcd}
FI \arrow[rr] \arrow[dr] \arrow[ddr] & & FJ \arrow[dl] \arrow[ddl] \\
& \mathrm{colim} F \arrow[d, "k"] & \\
& V &
\end{tikzcd}$$

Example 1.2 For $\mathbb{I} = (\bullet, \bullet)$, $F : \mathbb{I} \to \mathcal{C}$ picks out two objects $A, B \in \mathcal{C}$. A cocone looks like

$$\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl] \\
& V &
\end{tikzcd}$$

and a colimit is a coproduct

$$\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl] \\
& A \coprod B &
\end{tikzcd}.$$

Example 1.3 For $\mathbb{I} = (0 \rightrightarrows 1)$, $F : \mathbb{I} \to \mathcal{C}$ picks out $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$. A cocone is a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[dl, "x"] \\
V &
\end{tikzcd}$$

such that $xg = xf$, and a colimit is a coequaliser for f and g. Note that if $\mathcal{C} =$ Set or if $\mathcal{C} =$ Groups then the coequaliser is $B/_{(f(a) \sim g(a) \text{ for all } a \in A)}$.

Definition 1.4 For a general functor $f : \mathbb{I} \to \mathcal{C}$ with coproducts and coequalisers,

$$\operatorname{colim} F = \text{coequaliser}\left(\coprod_{u:\, I \to S \in \mathbb{I}} FI \overset{a}{\underset{b}{\rightrightarrows}} \coprod_{I} FI \right),$$

where a and b are defined by the following diagrams.

$$\begin{tikzcd}
\coprod_u FI \arrow[r, "\exists! a"] & \coprod_I FI \\
FI \arrow[u] \arrow[r, "Fu"'] & FJ \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_u FI \arrow[r, "b"] & \coprod_I FI \\
FI \arrow[u] \arrow[r, "\mathrm{id}"'] & FI \arrow[u]
\end{tikzcd}$$

1.2 Ends and Coends

1.2.1 Ends

Consider a functor $F : \mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathcal{C}$. We have $\lim F \to F(I, J)$ and there is a functor $I \to F(I, I)$ from which we get the diagram

$$\begin{tikzcd}
F(I,I) \arrow[dr, "f(\mathrm{id}_I, u)"'] & & F(J,J) . \arrow[dl, "F(u, \mathrm{id}_J)"] \\
& F(I,J) &
\end{tikzcd}$$

Instead of cones, we look at expressions like $(q_I : V \to F(I,I))_{I \in \mathbb{I}}$ which yield wedges

$$\begin{tikzcd}
& V \arrow[dl, "q_I"'] \arrow[dr, "q_J"] & & . \\
F(I,I) \arrow[dr] & & F(J,J) \arrow[dl] & \\
& F(I,J) & &
\end{tikzcd}$$

An end for F is a universal wedge $\left(p_I : \int_I F(I,I) \to F(I,I)\right)_{I \in \mathbb{I}}$ with universal property

$$\begin{tikzcd}
& V \arrow[d, "k"] \arrow[ddl, "q_I"'] \arrow[ddr, "q_J"] & & . \\
& \int_I F(I,I) \arrow[dl, "p_I"] \arrow[dr, "p_J"'] & & \\
F(I,I) \arrow[dr] & & F(J,J) \arrow[dl] & \\
& F(I,J) & &
\end{tikzcd}$$

Example 1.5 Let $\mathbb{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{B}$ be a functor. Define $T : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathrm{Set}$ by $T(A,B) = \mathbb{B}(FA, GB)$. Then $\int_A T(A,A) = \mathbb{B}^{\mathbb{A}}(F,G)$ (a verification of this formula was given).

1.2.2 Coends

Given $F : \mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathcal{C}$, we need a cowedge $(q_I : F(I,I) \to V)_{I \in \mathbb{I}}$ and the following diagram must commute for all $u : I \to J$ in $\mathbb{I}$.

$$\begin{tikzcd}
& V & \\
F(I,I) \arrow[ur, "q_I"] & & F(J,J) \arrow[ul, "q_J"'] \\
& F(J,I) \arrow[ul, "F(u,\mathrm{id}_I)"] \arrow[ur, "F(\mathrm{id}_J,u)"'] &
\end{tikzcd}$$

A coend $\int^I F(I,I)$ looks like the following diagram:

$$\begin{tikzcd}
& V & \\
& \int^I F(I,I) \arrow[u, "\exists!k"] & \\
F(I,I) \arrow[uur, "q_I"] \arrow[ur, "p_I"'] & & F(J,J) \arrow[uul, "q_J"'] \arrow[ul, "p_J"] \\
& F(J,I) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example 1.6 Let R be a ring regarded as a 1-object Ab-category. A right R-module A is a functor $R^{\mathrm{op}} \to \mathrm{Ab}$, and a left R-module B is a functor $R \to \mathrm{Ab}$. An abelian group $A \otimes B$ is also a functor $R^{\mathrm{op}} \times R \to \mathrm{Ab}$, $* \mapsto A \otimes B$; and $\int^{* \in R} A \otimes B = A \otimes_R B$ (note that $A \otimes_R B = {}^{A \otimes B}/_{ar \otimes b \sim a \otimes rb}$).

1.3 Homotopy limits and homotopy colimits

Let $F : \mathbb{I} \to \mathcal{S}$ be a functor, where $\mathcal{S}$ = Simplicial Sets. The homotopy limit and colimit of F are given as follows:

$$\operatorname{holim} F = \int_I \underline{\mathcal{S}}(\mathrm{Ner}({}^{\mathbb{I}}/_{I}), FI);$$

$$\operatorname{hocolim} F = \int^I \mathrm{Ner}({}^{I}/_{\mathbb{I}}) \times FI,$$

where $\underline{\mathcal{S}}(A,B)_\Lambda = \{A \times \Delta[n] \to B\}$ (simplicial maps).

Now hocolim F = coequaliser(Ξ), where $\Xi =$

$$\begin{tikzcd}
\coprod_{u\colon I\to J} \mathrm{Ner}({}^J/_{\mathbb{I}}) \times FI \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\
\coprod_I \mathrm{Ner}({}^I/_{\mathbb{I}}) \times FI
\end{tikzcd}$$

and α and β are defined by the following diagrams.

$$\begin{tikzcd}
\coprod_u \mathrm{Ner}({}^J/_{\mathbb{I}}) \times FI \arrow[r, "\alpha"] & \coprod_I \mathrm{Ner}({}^I/_{\mathbb{I}}) \times FI \\
\mathrm{Ner}({}^J/_{\mathbb{I}}) \times FI \arrow[u] \arrow[r, "\mathrm{id}\times Fu"'] & \mathrm{Ner}({}^J/_{\mathbb{I}}) \times JI \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_u \mathrm{Ner}({}^J/_{\mathbb{I}}) \times FJ \arrow[r, "\beta"] & \coprod_I \mathrm{Ner}({}^I/_{\mathbb{I}}) \times FI \\
\mathrm{Ner}({}^J/_{\mathbb{I}}) \times FI \arrow[u] \arrow[r, "\mathrm{Ner}({}^u/_{\mathbb{I}})\times\mathrm{id}"'] & \mathrm{Ner}({}^I/_{\mathbb{I}}) \times FI \arrow[u]
\end{tikzcd}$$

It follows that

$$\text{hocolim}\, F = \left(\coprod_I \mathrm{Ner}({}^I/_{\mathbb{I}}) \times FI \right) /_{\alpha\sim\beta}.$$

Remark 1.7 The meaning of α and β was subsequently discussed, along with possible examples.

2 Seminar 2: Kathryn Hess (2nd March 2004)

2.1 A Homotopy Theory for 2-Categories

2.1.1 2-Categories

A first example: CAT. Objects: CAT_0 = small categories. Morphisms: CAT_1 = functors. 2-Cells: CAT_2 = natural transformations. A trivial example: $\underline{2}$, where $\underline{2}_0 = \{0, 1\}$, $\underline{2}_1 = \{t : 0 \to 1\} +$ identities, and $\underline{2}_2$ = identity 2-cells.

The important example: Bènabou, 1967. Let $\mathcal{A}$ be a 2-category. The 2-category $\mathrm{Ben}(\mathcal{A})$ of cylinders in $\mathcal{A}$ is defined as follows: $\mathrm{Ben}(\mathcal{A})_0 = \mathcal{A}_1$. $\mathrm{Ben}(\mathcal{A})_1$: Let $f, g \in \mathcal{A}_1 = \mathrm{Ben}(\mathcal{A})_0$ ($f : a_0 \to a_1$, $f : b_0 \to b_1$). A morphism from f to g is a triple $(h_0, h_1, \alpha) \in \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2$ such that we have the following diagram ($\alpha : h_1 f \Rightarrow g h_0$).

$$\begin{tikzcd}
a_0 \arrow[r, "h_0"] \arrow[d, "f"'] & b_0 \arrow[d, "g"] \\
a_1 \arrow[r, "h_1"'] & b_1
\end{tikzcd}$$

$\mathrm{Ben}(\mathcal{A})_2$: Suppose that (h_0, h_1, α), $(h'_0, h'_1, \alpha') : f \to g$. A 2-cell from (h_0, h_1, α) to (h'_0, h'_1, α') is a pair $(\beta_0, \beta_1) \in \mathcal{A}_2 \times \mathcal{A}_2$ such that we have the following diagram (with $\beta_0 : h_0 \Rightarrow h'_0$ and $\beta_1 : h_1 \Rightarrow h'_1$ such that $g\beta_0 \circ \alpha = \alpha' \circ \beta_1 f$).

$$\begin{tikzcd}
a_0 \arrow[dr, bend left, "h_0"] \arrow[dr, bend right, "h'_0"'] \arrow[dd, "f"'] & \\
& b_0 \arrow[dd, "g"] \\
a_1 \arrow[dr, bend left, "h_1"] \arrow[dr, bend right, "h'_1"'] & \\
& b_1
\end{tikzcd}$$

Remark 2.1 $\mathrm{Ben}(\mathcal{A}) \cong [[\underline{2}, \mathcal{A}]]$, where $[[\underline{2}, \mathcal{A}]]_0$ = 2-functors $\underline{2} \to \mathcal{A}$, $[[\underline{2}, \mathcal{A}]]_1$ = lax natural transformations, and $[[\underline{2}, \mathcal{A}]]_2$ = modifications.

2.1.2 Motivation from Concurrency Theory

Definition 2.2 Let S and Σ be sets. A labelled transition system where S is the set of states and Σ is the set of labels consists of an indexed set of relations $\{R_\alpha \subseteq S \times S \mid \alpha \in \Sigma\}$. Equivalently, in terms of 2-categories, its a 2-functor $\mathbb{S} : \Sigma^* \to \mathrm{Rel}$ with $\mathbb{S}(*) = S$ and $\mathbb{S}(\alpha) = R_\alpha$ ($\Sigma^*_0 = \{*\}$, Σ^*_1 = free monoid on Σ, Σ_2^* = identities, Rel_0 = sets, Rel_1 = relations ($S, T \in \mathrm{Rel}_0 : \mathrm{Rel}(S,T) = \{R \mid R \subseteq S \times T\}$), and Rel_2 = inclusions of relations).

Definition 2.3 (Hermida) Let $\mathbb{S}, \mathbb{T} : \Sigma^* \to \mathrm{Rel}$ be labelled transition systems with the same labels Σ. We say that $\mathbb{T}$ simulates $\mathbb{S}$ if we have the following diagram:

$$\begin{tikzcd}
 & \mathrm{Ben}(\mathrm{Rel}) \arrow[d, "{(\mathrm{dom}, \mathrm{cod})}"] \\
\Sigma^* \arrow[ur, dashed, "\exists H"] \arrow[r, "{(\mathbb{T},\mathbb{S})}"'] & \mathrm{Rel} \times \mathrm{Rel}
\end{tikzcd} .$$

It follows that a simulation problem becomes a lifting problem which implies that we want a homotopy theory of 2-categories to have a meaningful obstruction theory.

2.1.3 What was already known

- Thomason's model cat structure on CAT.
- The 'folklore' structure on CAT: weak equivalence = equivalence of categories, which is interesting to category theorists.
- Maerdok & Svensson: A model cat structure on 2-Gpd.
- Lack (2002): A model cat structure on 2-Cat analogous to the folklore structure on CAT. **Remark:** This is not right for our purposes (because Ben(Rel) is not a path object on Rel).

Definition 2.4 WE = 2-functors $F : \mathcal{A} \to \mathcal{B}$ such that

- For all $b \in \mathcal{B}_0$, there exists an $a \in \mathcal{A}_0$ such that $F(a) \simeq b$.
- For all $a, a' \in \mathcal{A}_0$, the induced functor $\mathcal{A}(a, a') \to \mathcal{B}(F(a), F(a'))$ is an equivalence of categories.

2.1.4 Thomason model cat structure on 2-cat

Let $L : \mathcal{M} \rightleftarrows \mathbb{C} : R$ be adjoint functors, and let $\mathcal{M}$ be a model category. Then $L \dashv R$ *creates* a model category structure on $\mathcal{C}$ if the following choice defines a model category structure on $\mathcal{C}$: $WE_{\mathcal{C}} = R^{-1}(WE_{\mathcal{M}})$, $\mathrm{Fib}_{\mathcal{C}} = R^{-1}(\mathrm{Fib}_{\mathcal{M}})$, and $\mathrm{Cof}_{\mathcal{C}} = LLP(\mathrm{Fib}_{\mathcal{C}} \cap WE_{\mathcal{C}})$ (LLP = left lifting property).

Proposition 2.5 (Creation Proposition (Kan/Beke)) *Let $L : \mathcal{M} \rightleftarrows \mathbb{C} : R$ be an adjoint pair, where $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is complete and cocomplete. Let $\mathcal{I}$ be the generating cofibrations and let $\mathcal{J}$ be the generating acyclic cofibrations in $\mathcal{M}$. Consider the following conditions.*

(1) For all $X : \lambda \to \mathcal{M}$ *such that for all* $\beta < \beta + 1 < \lambda$, $X_\beta \xrightarrow{j_\beta} X_{\beta+1} \in WE_{\mathcal{M}}$, *then* $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta \in WE_{\mathcal{M}}$ *(‘Transfinite compositions of WE’s are WE’s’).*

(2) For all $X : \lambda \to \mathcal{C}$, $R(\mathrm{colim}_{\beta<\lambda} X_\beta) \cong \mathrm{colim}_{\beta<\lambda} R(X_\beta)$.

(3) For all $j \in \mathcal{J} \subset \mathrm{Mor}\,\mathcal{M}$ *and for all diagrams*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "L(j)"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

we have $R(g) \in WE_{\mathcal{M}}$.

(4) $L(\mathcal{I})$ *and* $L(\mathcal{J})$ *permit the Small Object Argument.*

If the above conditions are satisfied, then $L \dashv R$ *creates a model category on* $\mathcal{C}$.

Remark 2.6 $\mathcal{M}$, a model category, is cofibrantly generated if there exists an $\mathcal{I} \subset \mathrm{Cof}_{\mathcal{M}}$ and a $\mathcal{J} \subseteq \mathrm{Cof}_{\mathcal{M}} \cap WE_{\mathcal{M}}$ such that $\mathrm{Fib}_{\mathcal{M}} = RLP(\mathcal{J})$ and $\mathrm{Fib}_{\mathcal{M}} \cap WE_{\mathcal{M}} = RLP(\mathcal{I})$.

Remark 2.7 Thomason’s structure on CAT is created by $c \cdot Sd^2 : \mathrm{sSet} \rightleftarrows \mathrm{CAT} : \mathrm{Ex}^2 \cdot \mathcal{N}$, where $k \in \mathrm{sSet}$ implies $c(k)$ is a category with $\mathrm{Ob}\, c(k) = k_0$, $\mathrm{Mor}\, c(k)$ generated by 1-simplices modulo

$$\begin{tikzcd}
 & 2 & \\
0 \arrow[ur, "c"] \arrow[rr, "a"'] & & 1 \arrow[ul, "b"']
\end{tikzcd} ,$$

$Sd(k)$ = subdivision, and Ex = its right adjoint.

Theorem 2.8 (HPTW) *The adjoint pair* $c_2 \circ Sd^2 : \mathrm{sSet} \rightleftarrows 2-\mathrm{Cat} : \mathrm{Ex}^2 \circ \mathcal{M}$ *creates a model category structure on 2-Cat.*

Remark 2.9 The key to the proof is to study ‘weak immersions’ which are certain 2-functors.

3 Seminar 3: Richard Lewis (9th March 2004)

3.1 Nerves of Categories

$$(\mathrm{Ner}\,\mathcal{C})_n = \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n\} = \{(f_1, \dots, f_n)\},$$

with face maps $d_i(f_1, \dots, f_n) = (f_1, \dots, f_i f_{i+1}, \dots, f_n)$ $(0 < i < n)$, $d_0(f_1, \dots, f_n) = (f_2, \dots, f_n)$, and $d_n(f_1, \dots, f_n) = (f_1, \dots, f_{n-1})$; and degeneracies $s_i(f_1, \dots, f_n) = (f_1, \dots, f_i, 1_n, \dots, f_n)$.

3.2 Slice Categories

Let $\mathbb{I}$ be a category, and let $I \in \mathbb{I}$. A slice category $({}^{I}/_{\mathbb{I}})$ has objects $u : I \to J$ and maps $(u : I \to J) \longrightarrow (v : I \to K)$ with $\alpha : J \to K \in \mathbb{I}$ such that the following diagram commutes.

$$\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "v"] & \\
J \arrow[rr, "\alpha"'] & & K
\end{tikzcd}$$

Definition 3.1 Consider $w : I \to J \in \mathbb{I}$. $({}^{w}/_{\mathbb{I}}) : ({}^{J}/_{\mathbb{I}}) \to ({}^{I}/_{\mathbb{I}})$ maps objects as $(J \xrightarrow{a} X) \mapsto (I \xrightarrow{w} J \xrightarrow{a} X)$ and morphisms as $(J \xrightarrow{a} X) \xrightarrow{\alpha} (J \xrightarrow{b} Y) \mapsto (I \xrightarrow{wa} X) \xrightarrow{\alpha} (J \xrightarrow{wb} Y)$.

Definition 3.2 Let $T : \mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathcal{S}$, $(I, J) \mapsto \mathrm{Ner}({}^{I}/_{\mathbb{I}}) \times FJ$. Then $\int^* T(I, I) = \int^* \mathrm{Ner}({}^{I}/_{\mathbb{I}}) \times FI$ is the homotopy colimit of F.

Example 3.3 Let $\mathbb{I} =$

$$\begin{tikzcd}
0 \arrow[r, "01"] \arrow[d, "02"'] & 1 \\
2 &
\end{tikzcd}$$

. Then $F : \mathbb{I} \to \mathcal{S}$ looks like the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

.

The homotopy colimit of F is the double mapping cylinder depicted below, whereas the colimit of F is given by $\operatorname{colim} F = \frac{B \coprod C}{f(a) \sim g(a)}$.