PhD Seminars 2003/2004: Prof. T. Porter Semester 1

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Contents

1 Seminar 1: 20th October 2003

1.1 Biproduct Diagrams

Definition 1.1 The diagram

$$\begin{tikzcd} A \arrow[r, shift left, "i_1"] & C \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

is a biproduct diagram for objects A and B in an additive category $\mathcal{A}$ if $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_C$.

Proposition 1.2 *A and B have a biproduct in $\mathcal{A}$ iff they have a product in $\mathcal{A}$.*

Corollary 1.3 *A and B have a product C in $\mathcal{A}$ iff they also have C as a coproduct.*

Proof: (Of the Corollary) Let the product of A and B in $\mathcal{A}$ be denoted by C. It follows that C is also a biproduct in $\mathcal{A}$ of A and B. If $\begin{tikzcd} A \arrow[r, shift left, "i_1"] & C \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$ is a biproduct diagram in $\mathcal{A}$, it is also a biproduct diagram in $\mathcal{A}^{op}$ with the i's and p's interchanging roles. Therefore C is a biproduct of A and B in $\mathcal{A}^{op}$, i.e. it is a coproduct of A and B in $\mathcal{A}$. □

Proof: (Of the Proposition) Assume that $\begin{tikzcd} A \arrow[r, shift left, "i_1"] & C \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$ is a biproduct diagram. Then $p_2 i_1 = p_2(i_1 p_1 + i_2 p_2) i_1 = p_2 i_1 p_1 i_1 + p_2 i_2 p_2 i_1 = p_2 i_1 + p_2 i_1 = 0$. Similarly, $p_1 i_2 = 0$. Now suppose that $A \xleftarrow{f_1} D \xrightarrow{f_2} B$ is given. **Claim:** $A \xleftarrow{p_1} C \xrightarrow{p_2} B$ is a product diagram.

Let $h : D \to C$ be given by $h = i_1 f_1 + i_2 f_2$. The following diagram commutes: $p_1 h = p_1 i_1 f_1 + p_1 i_2 f_2 = f_1$, etc.

$$\begin{tikzcd} A & D \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "h"] & B \\ & C \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \end{tikzcd}$$

Suppose that $h' : D \to C$ is such that $p_1 h' = f_1$, $p_2 h' = f_2$. Then $h' = (i_1 p_1 + i_2 p_2) h' = i_1 p_1 h' + i_2 p_2 h' = i_1 f_1 + i_2 f_2 = h$. This shows that C 'is' $A \times B$.

Conversely, suppose that $A \xleftarrow{p_1} C \xrightarrow{p_2} B$ *is* a product diagram as follows:

$$\begin{tikzcd} A & A \arrow[l, "="'] \arrow[r, "0"] \arrow[d, "\exists!\ i_1"] & B \\ & C \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \end{tikzcd}$$

We therefore get $i_1 : A \to C$, $i_2 : B \to C$ (similarly), $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$, $p_2 i_1 = 0$, and $p_2 i_2 = 0$.

To finish, let us look at the diagram

$$\begin{tikzcd}
A & C \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "i_1p_1+i_2p_2"] & B \\
& C \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Does this commute? Yes, because $i_1p_1 + i_2p_2 = 1_C$, etc. □

Remark 1.4 The biproduct in Ab, R-Mod, etc. is the "direct sum $\oplus$".

1.2 Abelian Categories

An Abelian category has the following:

- A Null Object;
- Binary Biproducts;
- Every arrow has a kernel and a cokernel;
- Every monic is a kernel and every epic is a cokernel.

Remark 1.5 The first two items imply that $\mathcal{A}$ is an additive category. Further, given $f_1, f_2 : A \to B$, $f_1 + f_2$ is $A \overset{\text{diag}}{\longrightarrow} A \oplus A \overset{f_1 \oplus f_2}{\longrightarrow} B \oplus B \overset{\text{codiag}}{\longrightarrow} B$ (see MacLane).

1.3 Kernels & Cokernels

A kernel (and dually a cokernel) is defined by the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
K \arrow[u, "\iota"] \arrow[urr, "0"] & K' \arrow[ul, dashed, "\iota'"] \arrow[ur, dashed, "0"'] \arrow[l, dashed, dash, "\exists!"] &
\end{tikzcd}$$

Kernels, null and biproducts imply that all finite limits exist. Alternatively, noting that null and biproducts implies products, we see that products and equalisers imply that all finite limits exist ($\text{eq}(f, g) = \ker(f - g) \longrightarrow A \overset{\longrightarrow}{\longrightarrow} B$, where the two arrows from A to B are f and g). Similarly, cokernels, null and biproducts imply that all finite colimits exist.

Remark 1.6 Any $f:\ A \to B$ factors as

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\
& \bullet \arrow[ur, "m"'] &
\end{tikzcd},$$

where e is an epic and m is monic.

1.4 Projective Objects

$a : B \to C$ is epic (written $B \twoheadrightarrow C$) if given $b, c:\ C \to D$ such that $ba = ca$, we have $b = c$.

Definition 1.7 P is projective if $a\bar{f} = f$ in the following diagram:

$$\begin{tikzcd}
& P \arrow[d, "f"] \arrow[dl, dashed, "\exists \bar{f}"'] \\
B \arrow[r, two heads, "a"] & C
\end{tikzcd}$$

Example 1.8 Free abelian groups are projective in $\mathbb{A}$b.

2 Seminar 2: 4th November 2003

2.1 Derived Functors of lim

Let $\mathbb{I}$ be a small category and consider $M : \mathbb{I} \to \mathbb{A}\mathrm{b}$, an $\mathbb{I}$-module / $\mathbb{I}$-diagram / $\mathbb{Z}(\mathbb{C})$-module ($\mathbb{A}\mathrm{b}$ could be replaced by R-mod). There is a naturally occurring functor $\mathbb{A}\mathrm{b} \xrightarrow{c} \mathbb{A}\mathrm{b}^{\mathbb{I}}$, $A \mapsto c(A)$, where $c(A)(i) = A$ and $c(A)(i \to j) = \mathrm{id}_A$. Categorically, $\lim M \cong \mathrm{NatTrans}(c(\mathbb{Z}), M)$, illustrated by the following diagram:

$$\begin{tikzcd}
 & & {} \arrow[dr] & & {} \\
 & & & M(i) \arrow[ur] \arrow[dr] & \\
A \arrow[r, dashed, "\exists!"] \arrow[urrr, "q_i"] \arrow[drrr, "q_j"'] & \lim M \arrow[urr, "p_i"'] \arrow[rrr, "p_k"] \arrow[drr, "p_j"] & & & M(k) \\
 & & & M(j) \arrow[ur] \arrow[dr] & \\
 & & {} \arrow[ur] & & {}
\end{tikzcd}$$

Result: $\mathbb{A}\mathrm{b}^{\mathbb{I}}(c(A), M) \overset{\varphi_A}{\cong} \mathbb{A}\mathrm{b}(A, \lim M)$. In particular, taking $A = \mathbb{Z}$, we get $\lim M \cong \mathbb{A}\mathrm{b}(\mathbb{Z}, \lim M)$ $\overset{\varphi_{\lim M}}{\cong}$ $\mathbb{A}\mathrm{b}^{\mathbb{I}}(c(\mathbb{Z}), M)$, $\mathrm{id}_{\lim M} \to p_i c(\lim M)$. This gives us a description of $\lim M$: if $x \in \lim M$, then $\bar{x} : c(\mathbb{Z}) \to M$.

$$\begin{tikzcd}
i \arrow[dd, "\alpha"] & \mathbb{Z} \arrow[r, equal] \arrow[dd, "="'] & c(\mathbb{Z})(i) \arrow[r, "\bar{x}(i)"] \arrow[dd, "c(\mathbb{Z})(\alpha)"] & M(i) \arrow[dd] \\
 & & & \\
j & \mathbb{Z} \arrow[r, equal] & c(\mathbb{Z})(j) \arrow[r, "\bar{x}(j)"'] & M(j)
\end{tikzcd}$$

Knowing that $\bar{x}(i)(1) \in M(i)$, write $x(i) = \bar{x}(i)(1)$ so that $p_i(x) = x(i)$; we have $x \in \lim N \leftrightarrow x = (x(i))_{i \in \mathrm{Ob}(\mathbb{I})}$, where if $\alpha : i \to j$ then $M(\alpha)(x(i)) = x(j)$. Further, $\lim M \cong \{(x(i)) \mid x(i) \in M(i)$ and $\forall \alpha : i \to j,\ M(\alpha)x(i) = x(j)\}$.

Example 2.1 Suppose that $\mathbb{I} = \mathbb{N}$ so we are considering the sequence $1 \leftarrow 2 \leftarrow \ldots \leftarrow n \leftarrow n{+}1 \leftarrow \ldots$. For $M : \mathbb{N} \to \mathbb{A}$, we have the sequence $M(1) \leftarrow M(2) \leftarrow \ldots \leftarrow M(n) \leftarrow M(n+1) \leftarrow \ldots$. Assume that each $M(\alpha)$ is an inclusion so that M is a descending sequence of subgroups of $M(1)$. Then $\lim M = \cap_n M(n)$.

Example 2.2 Let $\mathbb{I} = \begin{tikzcd} 0 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 1 \end{tikzcd}$ so that $M = \begin{tikzcd} M(0) \arrow[r, shift left, "M(\alpha)"] \arrow[r, shift right, "M(\beta)"'] & M(1) \end{tikzcd}$. If $x \in \lim M$ (= equalisers of $M(\alpha)$ and $M(\beta)$), then $x = (x(0), x(0))$ and $M(\alpha)(x(0)) = x(1)$, $M(\beta)(x(0)) = x(1)$. Therefore,

$M(\alpha)(x(0)) = M(\beta)(x(0))$ and $\mathbf{x} = (x(0), M(\alpha)(x(0)))$ so that $\lim M \cong \{x(0) \in M(0) \mid M(\alpha)(x(0)) = M(\beta)(x(0))\}$.

Example 2.3 Let G be a group and let $\mathbb{G}$ be the corresponding small category. $M \in \mathbb{A}\mathrm{b}^{\mathbb{G}}$ is a left G-module $(M(g) : \ M(*) \to M(*))$ and $\lim M = \{m(*) \mid \forall g \in G,\ M(g)(m(*)) = m(*)\}$ is a fixed subgroup of $M(*)$.

2.2 Exact Sequences

We note that lim turns exact sequences into 'left exact' sequences. To see what this means, let us first go through some definitions.

Definition 2.4 $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact at B if $\ker(\beta) = \mathrm{im}\ (\alpha)$.

Definition 2.5 $0 \to A \xrightarrow{\alpha} B$ is exact at A iff α is a monomorphism ($\ker(\alpha) = 0$).

Definition 2.6 $B \xrightarrow{\beta} C \to 0$ is exact at C if β is an epimorphism ($\mathrm{im}\ (\beta) = C$).

Definition 2.7 $0 \to A \to B \to C \to 0$ is exact if it is exact at A, B and C.

Definition 2.8 $a : \ M' \to M$ is a monomorphism iff for each i, $a(i) : \ M'(i) \to M(i)$ is a monomorphism (there is a similar definition for an epimorphism $b : M \to M'$).

Notice that $\ker(a)(i) = \ker(a(i)) \subseteq M'(i)$ and that $\mathrm{im}\ (b)(i) = \mathrm{im}\ (b(i)) \subseteq M''(i)$. Therefore, $0 \to M' \xrightarrow{a} M \xrightarrow{b} M'' \to 0$ is exact iff for all i, $0 \to M'(i) \xrightarrow{a(i)} M(i) \xrightarrow{b(i)} M''(i) \to 0$ is exact.

Remark 2.9 $\lim a : \ \lim M' \to \lim M$ will be monic, but $\lim b : \ \lim M \to \lim M''$ need not be epic.

Example 2.10 Let $\mathbb{I} = \mathbb{N}$ again, and let $M(n) = \mathbb{Z}$, $M(n+1) \xrightarrow{\times 2} M(n)$. We see that $\lim M = \{(x(i)) \mid x(i) \in \mathbb{Z},\ \forall n\ 2x(n+1) = x(n)\}$ so that $x(1)$ must be divisible by 2^k for all k, i.e. $x(1) = 0$ and so $\lim M = \{0\}$. Consider the sequence $0 \to M \xrightarrow{\times 3} M \to$ '$c(\mathbb{Z}_3)$' $\to 0$. Expanded, this is as follows:

$$\begin{tikzcd}
n & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 3"] & \mathbb{Z} \arrow[r] & \mathbb{Z}_3 \arrow[r] & 0 \\
n+1 \arrow[u] & 0 \arrow[r] & \mathbb{Z} \arrow[u, "\times 2"] \arrow[r, "\times 3"'] & \mathbb{Z} \arrow[u, "\times 2"] \arrow[r] & \mathbb{Z}_3 \arrow[u, "\cong"] \arrow[r] & 0
\end{tikzcd}$$

Taking the limit, we get the sequence $0 \to 0 \to 0 \to \mathbb{Z}_3 \to 0$. As this is not exact on the right, we conclude that lim destroys exactness.

For any additive functor that destroys exactness, one can try to measure that 'destruction' by using derived functors. Starting with $0 \to A \to B \to C \to 0$, we go via an F to get the sequence $0 \to F(A) \to F(B) \to F(C) \to ?$ and want to continue exactness to the right to get a sequence $0 \to F(A) \to F(B) \to F(C) \to F^1(A) \to F^1(B) \to F^1(C) \to F^2(A) \to \dots$ in a universal way.

Example 2.11 Let M be a non-projective module over R. Then $\mathrm{Hom}(M, \_) : R\text{-Mod} \to \mathbb{A}\mathrm{b}$ is left exact but not exact, and the derived functors are called $\mathrm{Ext}^n(M, \_)$. For groups, homology and cohomology are examples of derived functors.

Some questions to finish:

- Can we define $\lim^{(i)} M$, the derived functors of lim? (Yes).
- How can we construct them in a useful way?
- What does $\lim^{(i)} M$ tell you about M and $\mathbb{I}$?
- When is $\lim^{i} M = 0$?

3 Seminar 3: 13th November 2003

3.1 Derived Functors of lim, Part 2

Suppose that $T : R\text{–Mod} \overset{\text{additive}}{\longrightarrow} \mathbb{A}\mathrm{b}$ is left exact, i.e. if $0 \to L \to M \to N \to 0$ is exact then $0 \to TL \to TM \to TN$ is exact. Given $M \in R\text{–Mod}$, let $M \to I_0$ be a monic with I_0 injective.

(I is injective :

$$\begin{tikzcd}
 & & I & & \\
0 \arrow[r] & A \arrow[r] \arrow[ur, "\text{given}"] & B \arrow[r] \arrow[u, dashed, "\exists"'] & C \arrow[r] & 0
\end{tikzcd}$$

is exact).

Example 3.1 Let $R = \mathbb{Z}$.

$$\begin{tikzcd}
 & & I & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times n"] \arrow[ur, "\varphi"] & \mathbb{Z} \arrow[r] \arrow[u, dashed, "\exists\bar{\varphi}"'] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Let $\varphi(1) = k \in I$. Then there exists a $\bar{\varphi} : \mathbb{Z} \to I$, where $\bar{\varphi}(n) = k$ as well. But $\bar{\varphi}(n) = n\bar{\varphi}(1)$, so there exists an element j $(= \bar{\varphi}(1))$ with $nj = k$.

Let $C_0 = \mathrm{coker}(M \to I_0) \overset{\text{pick}}{\longrightarrow} I_1$, with I_1 injective. Let us continue to find an injective resolution $I_\bullet$: $M \to (I_0 \to I_1 \to I_2 \to \ldots)$. Apply T to $I_\bullet$ to get a (C_0) chain complex $T(I_\bullet)$, and define $(R^nT)(M) = H^n(T(I_\bullet)) = \ker(T\partial)/\mathrm{im}\,(T\partial)$. One now checks that $(R^nT)(M)$ is independent of the choice of $I_\bullet$.

If $0 \to L \to M \to N \to 0$ is exact, then we get a long exact sequence $0 \to TL \to TM \to TN \to R^1T(L) \to R^1T(M) \to \ldots$. This depended on there being 'enough injectives'. In general, we have $\lim : \mathbb{A}\mathrm{b}^{\mathbb{I}} \to \mathbb{A}\mathrm{b}$. Does this have enough injectives?

Let $\mathbb{A} = R$-Mod and consider $U : \mathbb{A}^{\mathbb{I}} \to \mathbb{A}^{|\mathbb{I}|}$, the category of $|\,\mathbb{I}\,|$-indexed families of objects from $\mathbb{A}$, where $|\,\mathbb{I}\,|$ is the discrete category on the same objects as $\mathbb{I}$. U has a right adjoint $C : \mathbb{A}^{|\mathbb{I}|} \to \mathbb{A}^{\mathbb{I}}$, $N = \{N(i) : i \in |\,\mathbb{I}\,|\}$, where $C(N)(i) = \prod_{\alpha\colon i \to k} N(k)$. We can obtain $\mathbb{A}\mathrm{b}^{|\mathbb{I}|}(U(M), N) \cong \mathbb{A}\mathrm{b}^{\mathbb{I}}(M, C(N))$, and we also get the following properties.

(i) $\eta(M) := M \to C(U(M))(i)$ is the unit of the adjunction (if $x \in M(n)$, then $(\eta(M)(n)(x))_\alpha = M(\alpha)(x)$).

(ii) U and C preserve exactness.

(iii) $\overset{\lim}{\longleftarrow} C(N) = \prod_{i\in|\mathbb{I}|} N(i)$ so that $\overset{\lim}{\longleftarrow} C$ is exact.

(iv) U preserves monomorphisms so that C preserves injectives.

$$\begin{tikzcd}
\psi = \mathbb{A}^{|\mathbb{I}|}(U(B), N) \arrow[r, "\cong"] \arrow[d, "U(\lambda)^*"'] & \bar{\psi} = \mathbb{A}^{\mathbb{I}}(B, C(N)) \arrow[d, dash, "\lambda^*"] \\
\underline{\varphi}\ (= U(\lambda)^*(\psi) = \psi U(\lambda)) = \mathbb{A}^{|\mathbb{I}|}(U(A), N) \arrow[r, "\cong"'] & \varphi\ (= \lambda^*\bar{\psi} = \bar{\psi}\lambda) = \mathbb{A}^{\mathbb{I}}(A, C(N))
\end{tikzcd}$$

Definition 3.2 $\lim^{(k)}(M) = (R^k \lim)(M)$ for $k \geqslant 0$.

What about calculation? The answer is that we use the Roos complex, writing $F = CU$, iterating $F^{\bullet}M$, and then applying lim to get a complex $\prod^q M = \overset{\lim}{\longleftarrow} F^{q+1}(M) = \prod\{M(U) \mid U = (i_0 \overset{\alpha_1}{\longleftarrow} i_1 \leftarrow \ldots \overset{\alpha_q}{\longleftarrow} i_q),\ M(u) := M(i_0)\}$.

Theorem 3.3 (Roos, André, Bousfield-Kan) $H^q(\prod^* M) \cong \lim^{(q)} M$.

Example 3.4 If $\mathbb{I} = \mathbb{G}$ ($\mathbb{G}$ = a group G as a category), then $\lim^q M \cong H^q(G, M)$.

4 Seminar 4: 20th November 2003

4.1 Relative Injectives

If M is a direct summand of some $C(N)$, then it has the following property: Given an exact sequence ε in $\mathbb{A}^{\mathbb{I}}$ ($\varepsilon : 0 \to A \to B \to C \to 0$) such that $U(\varepsilon)$ is split, then any map $A \to M$ extends over B.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl, dashed, "\exists"] & C \arrow[r] & 0 \\
 & M & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A}^{\mathbb{I}}(C, C(N)) \arrow[r] & \mathbb{A}^{\mathbb{I}}(B, C(N)) \arrow[r, "\text{onto}"] \arrow[d, dash, "\cong"'] & \mathbb{A}^{\mathbb{I}}(A, C(N)) \arrow[d, dash, "\cong"] \\
 & \mathbb{A}^{|\mathbb{I}|}(U(B), N) \arrow[r, shift left, "\text{onto}"] & \mathbb{A}^{|\mathbb{I}|}(U(A), N) \arrow[l, shift left, dashed, "\text{split epimorphism}"]
\end{tikzcd}$$

We say M is relatively injective (relative to the class of U-split exact sequences).

If M is relatively injective, then $\lim^{(i)} M = 0$ for all $i > 0$. Since $M \oplus K \cong C(N)$, then $\lim^{(i)} M \oplus \lim^{(i)} K \cong \lim^{(i)} C(N)$ so it is sufficient to prove that $\lim^{(i)} C(N) = 0$ for $i > 0$.

If M is a direct summand of some $C(N)$, then $\lim^q M = 0$ for $q > 0$. It is sufficient to show that $\lim^q C(N) = 0$ for $q > 0$. One method is to construct a contracting homotopy on $\prod^\bullet C(N)$. In our case, $\prod^\bullet C(N) = \lim F^\bullet C(N)$ is cocontractible as a coaugmented cochain complex: $\lim C(N) \rightleftarrows \lim F^\bullet C(N)$, $H^0(\prod^\bullet C(N)) \cong \lim C(N)$, and $\lim^q C(N) = H^q(\prod^\bullet C(N)) \cong 0$ for $q > 0$.

Example 4.1 If G is a *finite* group and if M is a G-module, M is relatively injective iff for any $0 \to A \to B \to C \to 0$ of G-modules which splits at the abelian group level, then

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\
{} &
\end{tikzcd}$$

iff there is an endomorphism $h : M \to N$ such that $\sum_{\alpha \in G} M(\alpha^{-1}) h M(\alpha) = 1$. Thus if $|G| = n$ and if M has trivial G-action, then M is relatively injective iff there exists an $h : M \to M$ such that $nh(m) = m$. ($\Rightarrow$: If $p \mid n$, M p-divisible and p torsion free, then $H^q(G, M) = 0$ for all $q > 0$. p-divisible: iff for all m there exists a solution to $px = m$).

4.2 The Snake Lemma

Suppose we have

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, "i_0"] \arrow[d, "\alpha"] & B_0 \arrow[r, "p_0"] \arrow[d, "\beta"] & C_0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "i_1"'] & B_1 \arrow[r, "p_1"'] & C_1 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Then there is a $\delta : \ker\gamma \to \operatorname{coker}\alpha$ such that $0 \to \ker\alpha \xrightarrow{i_0} \ker\beta \xrightarrow{p_0} \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \xrightarrow{i_1} \operatorname{coker}\beta \xrightarrow{p_1} \operatorname{coker}\gamma \to 0$ is exact.

Sketch Proof. Let $c \in \ker\gamma$ so that $c \in C_0$ and $\gamma c = 0$. $c \in C_0 \Rightarrow \exists b \in B_0$ such that $p_0 b = c$. But $p_1\beta b = \gamma p_0 b = \gamma c = 0$, so $\beta b \in \ker p_1 = \text{im } i_1$ and there is some $a_1 \in A_1$ such that $i_1 a_1 = \beta b$. Suppose $b' \in B_0$ and $p_0 b' = c$ (as well). Then there exists an $a_1' \in A_1$ so that $i_1 a_1' = \beta b'$. But now $b - b' \in \ker p_0 = \text{im } i_0$ so there exists an $a_0 \in A_0$ so that $i_0 a_0 = b - b'$. Further, $\alpha a_0 \in A_1$ so that $i_1\alpha a_0 = \beta i_0 a_0 = \beta i_1 a_0 = \beta b - \beta b' = i_1(a_1 - a_1')$. But i_1 is a monomorphism so that $\alpha a_0 = a_1 - a_1'$. Therefore, $a_1 + \alpha A_0 = a_1' + \alpha A_0$, and if we assign $\delta(c) = a_1 + \alpha A_0$, we see that it is well defined and also a homomorphism.

Exercise 4.2 (a) Check that if $\delta c = 0$ then $c \in p_0(\ker\beta)$. (b) Check exactness.

5 Seminar 5: 25th November 2003

5.1 Derived Functors of lim, Part 3

Let $M : \mathbb{N} \to \mathbb{A}$ be an inverse sequence ($\mathbb{A}$ is abelian with exact products; $i \geqslant j$ is written $i \to j$). It follows that we have something like $M(1) \xleftarrow{p_1^2} M(2) \leftarrow \dots \leftarrow M(i) \xleftarrow{p_i^{i+1}} M(i+1) \leftarrow \dots$. Form $\prod_{i\in\mathbb{N}} M(i)$ and the homomorphism $\partial_M : \prod_{i\in\mathbb{N}} M(i) \to \prod_{i\in\mathbb{N}} M(i)$ as $\partial_M(x_1, x_2, \dots, x_m, \dots) = (x_1 - p_1^2 x_2, x_2 - p_2^3 x_3, \dots, x_m - p_m^{m+1} x_{m+1}, \dots) =$ an 'identity-shift'.

Lemma 5.1 $\ker \partial_M \cong \mathrm{Lim} M$.

Proof: **(Proved for $\mathbb{A} = \mathbf{Ab}$)** $\underline{x} \in \ker \partial_M \Leftrightarrow x_1 = p_1^2 x_2, \dots, x_m = p_m^{m+1} x_{m+1}, \dots$. Consider the following diagrams:

$$\begin{tikzcd}
 & & {} \\
 & & M(i) \arrow[u] \\
N \arrow[r, dashed, "\exists ! f"] \arrow[urr, "f_i"] \arrow[drr, "f_{i+1}"'] & \mathrm{Lim} M \arrow[ur, "p_i"] \arrow[dr, "p_{i+1}"] & \\
 & & M(i+1) \arrow[uu, "p_i^{i+1}"'] \\
 & & {} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & {} \\
 & & M(i) \arrow[u] \\
N \arrow[r, dashed] \arrow[urr, "f_i"] \arrow[drr, "f_{i+1}"'] & \ker \partial_M \arrow[ur, "p_i"] \arrow[dr, "p_{i+1}"] & \\
 & & M(i+1) \arrow[uu, "p_i^{i+1}"'] \\
 & & {} \arrow[u]
\end{tikzcd}$$

We must show that $p_i^{i+1} f_{i+1} = f_i$. But if $y \in N$, then we must have $f(y) = (f_i(y))$ since we need $p_i f = f_i$. But $f : N \to \prod M(i)$ sends y to an element of $\ker \partial_M$ since $f_i(y) = p_i^{i+1} f_{i+1}(y)$, i.e. $f_i(y) - p_i^{i+1} f_{i+1}(y) = 0$ so that $\partial_M f(y) = 0$. $\square$

Definition 5.2 $\underline{\mathrm{Lim}}^{(1)} M = \operatorname{coker} \partial_M$.

Corollary 5.3 (Of the Snake Lemma) *If* $0 \to L \to M \to N \to 0$ *is a short exact sequence in* $\mathbb{A}^{\mathbb{N}}$, *then* $0 \to \mathrm{Lim} L \to \mathrm{Lim} M \to \mathrm{Lim} N \to \underline{\mathrm{Lim}}^{(1)} L \to \underline{\mathrm{Lim}}^{(1)} M \to \underline{\mathrm{Lim}}^{(1)} N \to 0$ *is exact.*

Example 5.4 For the dyadic solenoids (DS), $M(i) = \mathbb{Z}$ for each i and $p_i^{i+1}(m) = 2m$. We have $0 \to \mathrm{DS} \xrightarrow{\times p} \mathrm{DS} \to c(\mathbb{Z}/p\mathbb{Z}) \to 0$ with p coprime to 2, and an exact sequence $0 \to 0 \to 0 \to \mathbb{Z}/p\mathbb{Z} \to \underline{\mathrm{Lim}}^{(1)}\mathrm{DS} \xrightarrow{\times p} \underline{\mathrm{Lim}}^{(1)}\mathrm{DS} \to 0 \to 0$.

Let us now prove that $\underline{\mathrm{Lim}}^{(1)}M \cong \mathrm{Lim}^{(1)}M$ ($\cong H^1(\prod^\bullet M)$). Now $\prod^p M = \prod\{M(i_0) \mid i_0 \leqslant i_1 \leqslant \ldots \leqslant i_p\}$, $\prod^1 M = \prod\{M(n_0) \mid n_0 \leqslant n_1\}$, and $\prod^0 M = \prod M(n)$.

$$\begin{tikzcd}
\prod^0 M(n) \arrow[r, "\partial^1"] \arrow[dd, dash] & \prod^1 M \arrow[r, "\partial^2"] & \prod^2 M \\
 & \ker\partial^2 \arrow[u] \arrow[d, dashed] \arrow[dr] & \\
\prod M(n) \arrow[r, "\partial_M"'] & \prod M \arrow[r] & \mathrm{coker}\,\partial_M
\end{tikzcd}$$

Notice that $\mathrm{Lim}^{(1)}M = \ker\partial^2/\mathrm{im}\,\partial^1$ and that $(\partial^2\underline{m})_{n_0 \leqslant n_1 \leqslant n_2} = p_{n_0}^{n_1}(m_{n_1,\,n_2}) - m_{n_0,\,n_2} + m_{n_0,\,n_1}$.

$$\begin{tikzcd}
 & n_1 \arrow[dl, "m_{n_0n_1}"'] & \\
n_0 & & n_2 \arrow[ul, "m_{n_1n_2}"'] \arrow[ll, "m_{n_0n_2}"]
\end{tikzcd}$$

If $m \in \ker\partial^2$, define $\psi(\underline{m}) = (m_{n,\,n+1})_n \in \prod M(n)$. But $\partial^2\underline{m} = 0$ so that $p_n^{n+1}(m_{n+1,\,n+2}) - m_{n,\,n+1} + m_{n,\,n+1} = 0$. Now if $\underline{m} = \partial^1\underline{u}$, then $\partial^2\underline{m} = 0$ is guaranteed. Further, $(\psi(\underline{m}))_n = p_n^{n+1}u_{n+1} - u_n \in \mathrm{im}\,\partial_M$ and this means that ψ induces a homomorphism $\ker\partial^2/\mathrm{im}\,\partial^2 \to \prod M/\mathrm{im}\,\partial M$ which is a natural isomorphism.

Theorem 5.5 *If we have* $M : \mathbb{N} \to \mathbb{A}$, *then* $\lim^{(p)} M = 0$ *for all* $p \geqslant 2$.

5.2 A silly easy situation!

If k is a field, $\mathbb{I}$ directed and $M : \mathbb{I} \to \mathrm{Vect}_k$, where Vect_k denotes finite dimensional vector spaces, then $\mathrm{Lim}^{(i)}M = 0$ for all $i > 0$. Let us look at the case where $\mathbb{I} = \mathbb{N}$. We have $M(1) \leftarrow M(2) \leftarrow \ldots \leftarrow M(n) \leftarrow$, $0 \to L \xrightarrow{i} M \xrightarrow{f} N \to 0$, and in $0 \to \mathrm{Lim}L \to \mathrm{Lim}M \to \mathrm{Lim}N \to 0$, we are required to prove that the map $\mathrm{Lim}M \to \mathrm{Lim}N$ is onto. Consider $(y_n) \in \mathrm{Lim}N$. Pick $x_n \in M(n)$ with $f_n(x_n) = y_n$.

$$\begin{tikzcd}
L(n) \arrow[r] & M(n) \arrow[r, "f_n"] & N(n) \\
L(n+1) \arrow[r] \arrow[u] & M(n+1) \arrow[r, "f_{n+1}"'] \arrow[u, "p_n^{n+1}"] & N(n+1) \arrow[u, "p_n^{n+1}"]
\end{tikzcd}$$

(x_n) may not satisfy $p_n^{n+1}(x_{n+1}) = x_n$. But $f_n p_n^{n+1}(x_{n+1}) = p_n^{n+1} f_{n+1}(x_{n+1}) = p_n^{n+1} y_{n+1} = y_n = f_n(x_n)$, and $p_n^{n+1}(x_{n+1}) - x_n \in L(n) = \ker f_n$. In $L(n)$, we have $L(n) \supseteq p_n^{n+1} L(n+1) \supseteq \ldots \supseteq p_n^{n+\ell} L(n+\ell) \supseteq \ldots$. But dim $L(n)$ is finite, so there exists an ℓ such that for all $\ell' > \ell$ we have $p_n^{n+\ell'}(L(n+\ell')) = p_n^{n+\ell}(L(n+\ell))$.

Remark 5.6 Any inverse system of finite dimensional vector spaces satisfies the Mittag-Leffler condition, i.e. for any $i \in \mathbb{I}$, there exists an $i' \xrightarrow{u} i$ such that if $i'' \xrightarrow{u'} i$, then $\mathrm{Im}(M(u)) = \mathrm{Im}(M(uu'))$.

6 Seminar 6: 27th November 2003

6.1 Induced Diagrams

In order to prove the result from the previous seminar, we must first introduce some new theory.

- Given $M: \mathbb{J} \to \mathbb{A}$ and $L: \mathbb{J}' \to \mathbb{J}$, composition gives $ML: \mathbb{J}' \to \mathbb{A}$.
- can: $\text{Lim}\, M \to \text{Lim}\, ML$ is a canonical map.
- If $\underline{m} \in \text{Lim}\, M \subseteq \prod M(j)$ such that for each $\alpha: i \to j$ in $\mathbb{J}$, we have

$$\begin{tikzcd}
 & M(i) \arrow[dd, "M(\alpha)"] \\
\text{Lim}\, M \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & \\
 & M(j)
\end{tikzcd}$$

  i.e. $M(\alpha)m_i = m_j$.

- If $\underline{m}' \in \text{Lim}\, ML$, then for $\beta : j' \to k$ we have $ML(\beta)n_{j'} = n_k$.
- $\text{can}(\underline{m})_{j'} = m_{L(j')} \in ML(j')$ and we have $\mathbb{A}^{\mathbb{J}} \xrightarrow{\mathbb{A}^L} \mathbb{A}^{\mathbb{J}'}$ which has left and right adjoints if $\mathbb{A}$ has enough colimits/limits (these are the Kan extension functors as shown below).

$$\begin{tikzcd}
\mathbb{J}' \arrow[r] \arrow[d, "L"'] & \mathbb{A} \\
\mathbb{J} \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

Example 6.1 For $\mathbb{J} \to \{*\}$, $\mathbb{A}^{\{*\}} \xrightarrow{k} \mathbb{A}^{\mathbb{J}}$ has left and right adjoints corresponding to colimits and limits.

Question: When is can an isomorphism?

Definition 6.2 A category J is connected if given any two objects j, $k \in \mathbb{J}$, there is a finite zigzag $j = j_0 \to j_1 \to j_2 \to \ldots \to j_{2n-1} \to j_{2n} = k$ joining them.

Definition 6.3 A functor $L: \mathbb{J}' \to \mathbb{J}$ is initial (or cofinal) if for each $k \in \mathbb{J}$, $L \downarrow k$ is non-empty and connected.

Remark 6.4 $L \downarrow k$ is the comma category whose objects are $u:\ L(j) \to k$ and whose morphisms are $(u:\ L(j') \to k) \longrightarrow (u':\ L(j'') \to k)$ (the middle arrow refers to a morphism $f:\ j' \to j''$ in $\mathbb{J}'$) such that the following diagram commutes:

$$\begin{tikzcd}
L(j') \arrow[dr, "u"] \arrow[dd, "L(f)"'] & \\
 & k \\
L(j'') \arrow[ur, "u'"'] &
\end{tikzcd}$$

Lemma 6.5 (The 'Initial' Lemma) *If $L:\ \mathbb{J}' \to \mathbb{J}$ is initial, then* $\mathrm{can}:\ \mathrm{Lim}\, M \to \mathrm{Lim}\, ML$ *is an isomorphism (in fact if* $\mathrm{Lim}\, ML$ *exists then so does* $\mathrm{Lim}\, M$*).*

Proof: Set $C = \mathrm{Lim}\, ML$. L is initial means that given a $k \in \mathbb{J}$, there is some $u:\ L(j') \to k$. Let us pick one of these and define $\tau_k:\ C \to M(k)$ to be given by $C \xrightarrow{p'_j} M(j') \xrightarrow{M(u)} M(k)$. **Claim:** If $u':\ L(j'') \to k$ is another choice then the following diagram commutes:

$$\begin{tikzcd}
 & ML(j') \arrow[dr, "M(u)"] & \\
C \arrow[ur, "p_{j'}"] \arrow[dr, "p_{j''}"'] & & M(k) \\
 & ML(j'') \arrow[ur, "M(u')"'] &
\end{tikzcd}$$

Proof of Claim. There is a zigzag joining u and u'. Assume to start with that it is of length 1.

$$\begin{tikzcd}
L(j') \arrow[dr, "u"] \arrow[dd, "L(f)"'] & \\
 & k \\
L(j'') \arrow[ur, "u'"'] &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
 & ML(j') \arrow[dr, "M(u)"] \arrow[dd, "ML(f)"] & \\
C \arrow[ur, "p_{j'}"] \arrow[dr, "p_{j''}"'] & & M(k) \\
 & ML(j'') \arrow[ur, "M(u')"'] &
\end{tikzcd}$$

commutes and the result now follows by induction on the length of the zig-zag. **End of Proof of Claim.**

This also shows that $\tau_k : C \to M(k)$ gives a cone on M. Consider

$$\begin{tikzcd}
k \arrow[dd, "g"] \\
\\
k'
\end{tikzcd}
\qquad
\begin{tikzcd}
& ML(j') \arrow[r, "M(u)"] \arrow[rdd, "M(gu)"] & M(k) \arrow[dd] \\
C \arrow[ru, "p_{j'}"] \arrow[rd, "p_{j''}"'] & & \\
& ML(j'') \arrow[r, "M(u')"'] & M(k')
\end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd}
& M(k) \arrow[dd, "M(g)"] \\
C \arrow[ru, "\tau_k"] \arrow[rd, "\tau_{k'}"'] & \\
& M(k')
\end{tikzcd}$$

We note that $gu : L(j') \to k'$ is joined by a zig-zag to $u' : L(j'') \to k'$ (and so on); hence we get $\tau : C \to \mathrm{Lim}\, M$.

If $\lambda_k : D \to M(k)$ is a cone on M, then

$$\begin{tikzcd}
& ML(j') \arrow[dd] \\
D \arrow[ru, "\lambda_{L(j')}"] \arrow[rd, "\lambda_{L(j'')}"'] & \\
& ML(j'')
\end{tikzcd}$$

is a cone on ML and hence we know that there exists an unique $\lambda : D \to C$ such that $p_j \lambda = \lambda_{L(j)}$, i.e. $C \cong \mathrm{Lim}\, M$ as required. □

Definition 6.6 A (non-empty) category $\mathbb{I}$ is said to be *filtering* if (a) given i, $i' \in \mathbb{I}$ there exists a j and maps $i \to j$, $i' \to j$; (b) given u, $v : i \to j$, there exists a k and a $w : j \to k$ such that $wu = wv$ ($\bullet \overset{v}{\underset{u}{\rightrightarrows}} \bullet \overset{w}{\dashrightarrow}$). $\mathbb{I}$ is cofiltering if $\mathbb{I}^{\mathrm{op}}$ is filtering.

Proposition 6.7 *Let $L : J' \to J$ be a functor with J cofiltering. Then L is initial iff*

- *(I1) Given any $k \in J$, there exists a $j' \in J'$ such that $L(j') \to k$;*
- *(I2) Given u, $v : L(j') \to k$ in J', there exists a $h : j'' \to j'$ in J' such that $uL(h) = vL(h)$ (we have $L(j'') \xrightarrow{L(u)} L(j') \overset{v}{\underset{u}{\rightrightarrows}} k$).*

Moreover, if L is initial, then J' is cofiltering. (An idea of the proof was then given).

Theorem 6.8 *If $M : \mathbb{I} \to \mathbb{A}$ satisfies the Mittag-Leffler condition (for any $i_0 \in \mathbb{I}$ there is a $j_0 \in \mathbb{I}$ and an $\alpha : j_0 \to i$ such that if $\beta : j \to j_0$ then the natural map $p : \operatorname{im} M(\beta\alpha) \to \operatorname{im} M(\alpha)$ is an isomorphism) then there is a diagram M' with* $\mathrm{Lim}\, M = \mathrm{Lim}\, M'$ *and M' has all $M(\alpha)$'s being epimorphisms.*

7 Seminar 7: 2nd December 2003

(A recap of the previous seminar was given first).

Take I^2 to mean all morphisms in I. For example, $\mathbb{N}^2 = \{(i,j) \mid j \geqslant i\}$ and $(i,j) \geqslant (i',j')$ if $i \geqslant i'$ and $j \geqslant j'$. Consider $\Delta : I \to I^2$, $\Delta(i) = \mathrm{id}(i)$ (initial/cofinal). In the example, we have $\mathbb{N} \to \mathbb{N}^2$, i.e. $i \mapsto (i,i)$. Let $\overline{M} : I^2 \to \mathbb{A}$ be given by $\overline{M}(u) = \mathrm{Im}(M(u))$ (if $I = \mathbb{N}$ and $m \geqslant n$, then $(n,m) \in \mathbb{N}^2$ and $\overline{M}(n,m) = \mathrm{Im}(M(m) \to M(n))$).

Let $J \subset \mathbb{I}^2$. If $i \in \mathbb{I}$, then $(u : i' \to i \in J) \Leftrightarrow \forall\, u' : i'' \to i' \in \mathbb{I}$ we have $\mathrm{Im}(M(u)) = \mathrm{Im}(M(uu'))$. Recall that J is initial in $\mathbb{I}^2$ iff M is Mittag-Leffler, and let $N : J^2 \to \mathbb{A}$ be given by $N(u) = \mathrm{Im}(M(u))$. If $(i' \xrightarrow{u} i) \xrightarrow{f} (j' \xrightarrow{v} j) \in J$ then $N(f)$ is onto.

$$\begin{tikzcd}
i' \arrow[r, "u"] \arrow[d, "f_0"'] & i \arrow[d, "f_1"] \\
j' \arrow[r, "v"'] & j
\end{tikzcd}$$

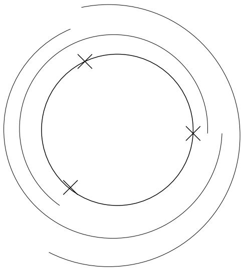
It follows that $\lim_J N \cong \lim_I \overline{M} \cong \lim_I M$.

Over $\mathbb{N}$, if $0 \to L \to M \to N \to 0$ is exact, then if L is ML it follows that $0 \to \lim L \to \lim M \to \lim N \to 0$ is also exact. Note that we can replace this by equivalent projective systems with L 'epi' since the original is ML.

Lemma 7.1 *If L is 'epi' then $\prod_{i\in\mathbb{N}} L(i) \xrightarrow[\Phi]{\mathrm{Id-Shift}} \prod_{j\in\mathbb{N}} L(j)$ is onto $((\Phi(\underline{\ell}))_n = \ell_n - L(n+1,n)(\ell_{n+1}))$.*

Example 7.2 (Čech Constructions) Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X, $\mathcal{U} = \{U_i : i \in \Lambda\}$. A simplicial complex K is made up of a set of vertices $V(K)$ and a subset Z of $\mathcal{P}(V(K))$ consisting of non-empty sets such that if $\sigma \in \Sigma_K$ and if $\tau \subset \sigma$ $(\tau \neq \phi)$, then $\tau \in \Sigma_K$. Now form a simplicial complex $N(X,\mathcal{U})$, the nerve of $\mathcal{U}$, with vertices $\langle U \rangle$, $U \in \mathcal{U}$. We see that $\{U_0, \ldots, U_n\} \in \Sigma_{N(X,\mathcal{U})} \Leftrightarrow \cap_{i=0}^n U_i$ is non-empty (write $\langle U_0, \ldots, U_n \rangle$ if it is a simplex).

Example 7.3 Take $X = S^1$.



We have $O_1 = (0, \frac{4\pi}{3})$, $O_2 = (\frac{2\pi}{3}, 2\pi)$, and $O_3 = (\frac{4\pi}{3}, \frac{2\pi}{3})$.

$$\begin{tikzcd}
& \langle O_1 \rangle \arrow[dl, dash, "\langle O_1{,}O_2 \rangle"'] \arrow[dr, dash, "\langle O_1{,}O_3 \rangle"] & \\
\langle O_2 \rangle \arrow[rr, dash, "\langle O_2{,}O_3 \rangle"'] & & \langle O_3 \rangle
\end{tikzcd}$$

Example 7.4 The Warsaw Circle was considered (consider the graph of $\sin \frac{1}{x}$ in a specified interval, say $(-a, a)$, and then join the endpoints at $-a$ and a with a curve).

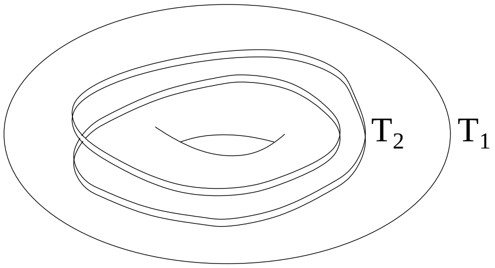
Result: The diagram

$$\begin{tikzcd}
& N(X,\mathcal{U}') \arrow[dr] & \\
N(X,\mathcal{U}'') \arrow[ur] \arrow[rr] & & N(X,\mathcal{U})
\end{tikzcd}$$

is commutative up to homotopy, and $H_*(N(X,\mathcal{U}')) \to H_*(N(X,\mathcal{U}))$ is well defined.

In Čech homology, for $\xleftarrow[\mathcal{U}]{\lim} H_*(NX,\mathcal{U}))$, there is no exact sequence for pairs (X, A). However, if we use finite dimensional vector spaces V as coefficients, then $\overset{\lim}{\longleftarrow} H_*(X,\mathcal{U} \setminus \otimes V)$ does give you an exact sequence.

Example 7.5 (Dyadic Solenoid)



$\lim \mathrm{DS} = \cap T_i$ for the sequence $\to \ldots T_3 \to T_2 \to T_1$ partially shown in the picture above (which is totally arcwise disconnected).

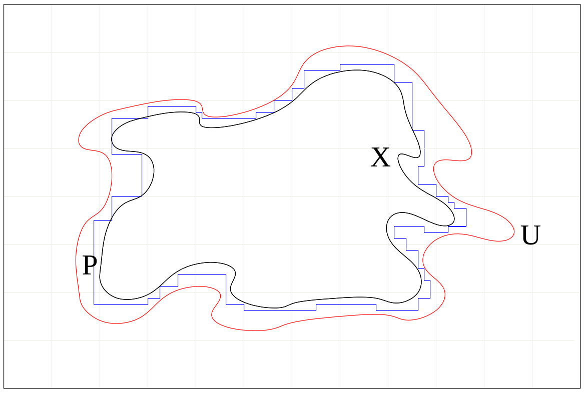
Let $\mathcal{U}_{V_n}$ be an open cover by open balls of diameter $\frac{1}{n}$, say. Then we have $H_1(N(\mathrm{DS}, \mathcal{U})) \cong \mathbb{Z}$ and the diagram below, where $m > n$ and the map $S^1 \to S^1$ of covering spaces is not an inclusion.

$$\begin{tikzcd}
N(\mathrm{DS}, \mathcal{U}_{\frac{1}{n}}) \arrow[r, "\cong"] & S^1 \\
N(\mathrm{DS}, \mathcal{U}_{\frac{1}{m}}) \arrow[r, "\cong"] \arrow[u] & S^1 \arrow[u]
\end{tikzcd}$$

Let X be a compact metric space, $X \subset I^\omega$. Information in the limit of refinements on X corresponds to information 'towards ∞' in $I^\omega \setminus X$.

8 Seminar 8: 4th December 2003

8.1 Hilbert Cubes



Let $X \subset I^\omega = \prod[0, \frac{1}{n}]$. Subdivide the cube as shown above, and take a neighbourhood U as shown in red. Then take a polyhedral approximation P within U as shown in blue. There is an $\epsilon < \frac{1}{n}$ and $I^\omega \cong \prod_{m=1}^n [0, \frac{1}{m}] \times I^\omega$.

Theorem 8.1 (Borsuk (1968) / Christie (1944)) *Take a base point in* X*, and let* $U \subset P \subset X$*, where* P *is a polyhedral approximation. Let us look at a map of fundamental groups* $\varphi_P^Q : \pi_1(Q, x_0) \to \pi_1(P, x_0)$*, where* Q *is a better approximation. Then* $\overset{\lim}{\longleftarrow} \pi_1(P, x_0) = \{(\alpha_P) \mid \alpha_P \in \pi_1(P, x_0), \varphi_P^Q(\alpha_Q) = \alpha_P\}$*.*

Define $\check{\Lambda}(X, x_0) = \overset{\lim}{\longleftarrow} \pi_1(P, x_0)$. Now $\check{H}_q(X) \cong \overset{\lim}{\longleftarrow} H_q(P)$, where $\check{H}_q(X)$ is the Čech homology.

Remark 8.2 In the Dyadic Solenoid example from the previous seminar, $H_1(P) = C_\infty$ (or $\mathbb{Z}$).

Let us now consider a squaring map $(\ )^2 : C_\infty \to C_\infty$ which we shall think of as $\times 2 : \mathbb{Z} \to \mathbb{Z}$.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times 3"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/3\mathbb{Z} \\
\mathbb{Z} \arrow[u, "\times 2"] \arrow[r, "\times 3"'] & \mathbb{Z} \arrow[u, "\times 2"] \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[u, "\times 2", "\cong"'] \\
{} \arrow[u, dashed] & {} \arrow[u, dashed] & {} \arrow[u, dashed]
\end{tikzcd}$$

In the diagram above, $\overset{\lim}{\longleftarrow} (\mathbb{Z}, \times 2) = \{(\alpha_n),\ \alpha_n \in \mathbb{Z},\ 2\alpha_{n+1} = \alpha_n\ \forall n + 1\}$.

Remark 8.3 Čech homology does *not* have exact sequences in general.

Now consider the diagram

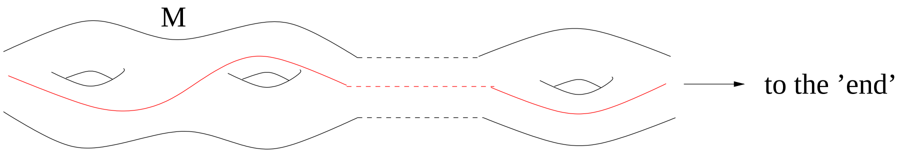
$$\begin{tikzcd}
0 \arrow[r] & \check{H}_1(DS) \arrow[r] \arrow[d, equal] & \check{H}_1(DS) \arrow[r] \arrow[d, equal] & \check{H}_1(\text{something!}) \arrow[r] & H_2(DS) \arrow[r] \arrow[d, equal] & 0\,. \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

In the limit for the diagram, we have

$$\begin{tikzcd}
0 \arrow[r, "\times 3"] & 0 \arrow[r, "\text{not surjective}"] & \mathbb{Z}/3\mathbb{Z} \arrow[r, "\text{(can continue)}"] & \underleftarrow{\lim}^{(1)} (\mathbb{Z}, \times 2) \arrow[r] & \underleftarrow{\lim}^{(1)} (\mathbb{Z}, \times 2) \arrow[r] & \cdots\,.
\end{tikzcd}$$

8.2 Proper Homotopy Theory

Definition 8.4 For spaces X and Y, a continuous map $f: X \to Y$ is *proper* if for compact $K \subset Y$, $f^{-1}(K)$ is compact.



Normally in homotopy we use base points, but here they aren't of much use. Instead we replace 'base point in X' by 'base ray $* : [0, \infty) \to X$' (for example as shown in red in the above diagram).

There are different notions of 'fundamental group'. Ed Brown: $\pi_1^{\mathrm{EB}}(X, *) \cong [(S^1, *), (M, *)]$ (proper homotopy classes). Waldhausen: $\pi_1^{\mathrm{W}}(M, *) = [(S^1 \times [0, \infty)), (M, *)]$ (proper homotopy classes).

Remark 8.5 Chapter 3 of the Handbook of Algebraic Topology was given out, namely "Proper Homotopy Theory" by Prof. T. Porter. Some discussion of the handout then took place (Puppe sequences, etc.)