

PhD Seminars 2003/2004: Other Seminars in Semester 1

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December 19, 2003

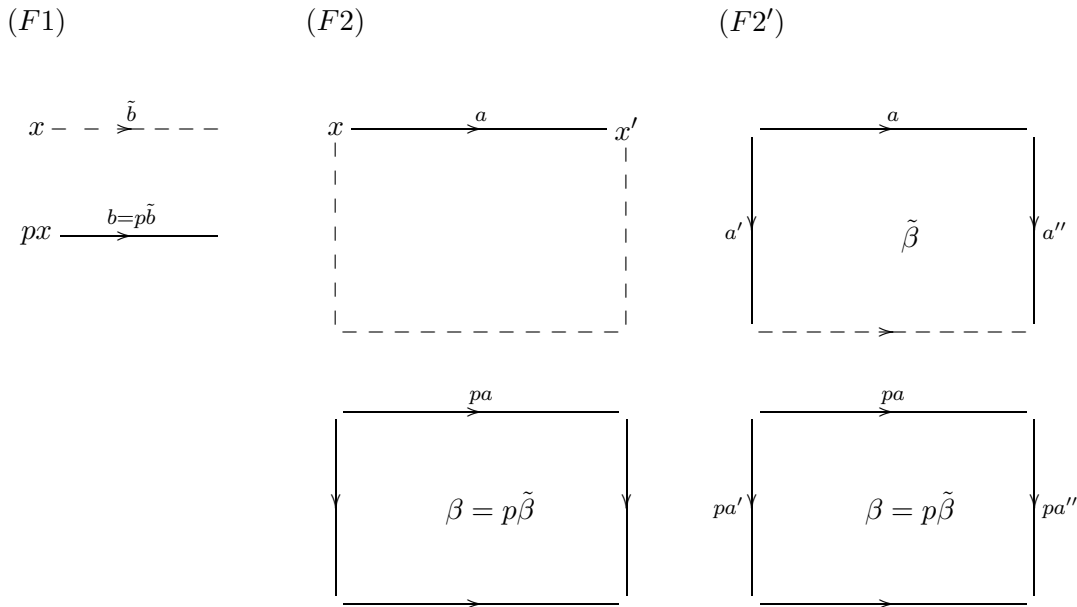
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1 Seminar 1: Heiner Kamps, Hagen (27th October 2003)

1.1 Fibrations of Double Groupoids

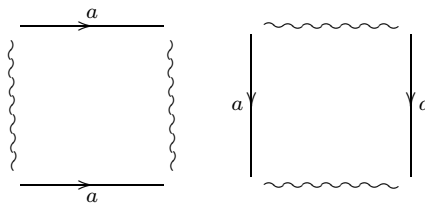
With fibrations of spaces, we have homotopy lifting properties. For example, in dimension 1, we have the path lifting property. In dimension 2, we have the following axioms:



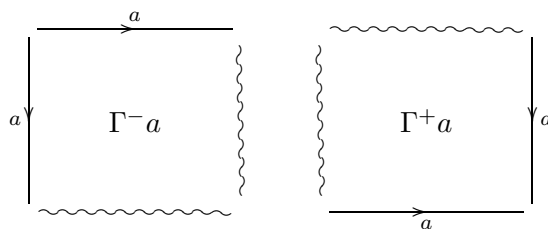
In Top, (F2) \Leftrightarrow (F2') holds while in Simplicial Sets, (F2) \Leftrightarrow (F2') does not hold.

Example 1.1 *Fibrations of groupoids* $p : X \rightarrow Y$.

Example 1.2 *Fibrations of double groupoids.* In dimension 2, a double groupoid $\mathbb{G} = (G_0, G_1, G_2)$ has objects or 0-cells G_0 , arrows or 1-cells G_1 , and squares G_2 . Taking the vertical direction as direction 1 and the horizontal direction as direction 2 by convention, and writing an identity as a squiggly line, we see that we can have horizontal and vertical composition of squares with vertical and horizontal identities as shown below:

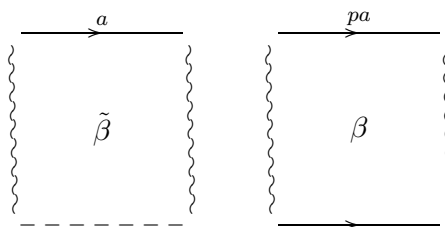


We can also introduce what are known as *connections* (some of which are shown in the diagram below) and transport laws for $\Gamma^-(a + a')$ and $\Gamma^+(a + a')$.



Finally, we can show diagrammatically that (a) (F1) and (F2') \Rightarrow (F2) and (b) (F2) \Rightarrow (F2').

Example 1.3 *Fibrations of 2-groupoids* $\mathbb{C} = (C_0, C_1, C_2)$, where $C_0 = 0$ -cells, $C_1 = 1$ -cells and $C_2 = 2$ -cells. Note that a 2-cell is usually written as an oval but may also (using the notation introduced in this seminar) be written as a vertical identity. We also have an extra axiom (F2'') for $p : \mathbb{C} \rightarrow \mathbb{D}$:



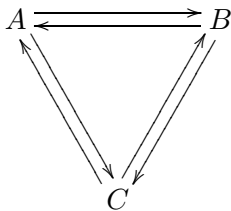
It is possible to show that a fibration of a double groupoid $p : \mathbb{G} \rightarrow \mathbb{H}$ implies a fibration of a 2-groupoid $\gamma(p)$, and also that a fibration of a 2-groupoid $p : \mathbb{C} \rightarrow \mathbb{D}$ implies a fibration of a double groupoid $\lambda(p)$. We describe λ in terms of quintuples

$$\left(\alpha : c \begin{array}{c} a \\ d \end{array} b \right)$$

and describe γ in terms of vertical identities.

Remark 1.4 Brown has found a 6-term exact sequence for fibrations of groupoids, while Hardie & Kieboem have found a 9-term sequence for 2-groupoids.

Remark 1.5 The reason for wanting all of this material is that we are working in the following diagram, where $A =$ crossed complexes $\lambda : \text{Crs}$, $B =$ Cubical ω -groupoids γ with connection, and $C =$ globular ω -groupoids.



2 Seminar 2: Anne Heyworth (6th November 2003)

2.1 Background

String rewriting for monoids has been extended to categories and to Kan extensions over sets. We can do logged rewriting which leads to identities among relations and ‘endorewrites’, and we can model string rewriting by using 2-categories. What is the corresponding situation for Gröbner Bases? We know that Gröbner Bases can be extended to k -categories and to Kan extensions over k -Mods, but what can we model Gröbner Bases by?

2.2 Modelling String Rewriting by Sesquicategories or 2-categories

Let X be a set and let $R \subseteq X^* \times X^*$ be a set of rules. In our model, we have $C_0 = \text{objects } *$, $C_1 = \text{paths } X^*$, and $C_2 = \text{rewrites } \alpha : \ell \rightarrow r \text{ between paths}$. To obtain a sesquicategory, we need the following axioms:

- Whiskering: If $\alpha : \ell \rightarrow r$ then $u\alpha r : ulv \rightarrow urv$;
- Inverse: $\alpha^{-1} : r \rightarrow \ell$;
- Vertical Composition: $\alpha : \ell \rightarrow r$ and $\beta : r \rightarrow q$ leads to $\alpha \cdot \beta : \ell \rightarrow q$;
- Identity: $\alpha \cdot \alpha^{-1} = 1_{(\text{src } \alpha)}$.

To obtain a 2-category, we also need to define Horizontal Composition as follows: $\alpha \circ \beta = \alpha(\text{src } \beta) \cdot (\text{tgt } \alpha)\beta = (\text{src } \alpha)\beta \cdot \alpha(\text{tgt } \beta)$.

Remark 2.1 There is a slight problem with how to represent the log in logged rewriting. Consider the rules $\alpha_1 = e_2^2 - e_2$, $\alpha_2 = e_3^2 - e_3$ and $\alpha_3 = e_3e_1 - e_1e_3$, and consider the following reduction of $p = e_3^2e_1 + e_2^2$: $e_3^2e_1 + e_2^2 \rightarrow e_3e_1 + e_2^2 \rightarrow e_1e_3 + e_2^2 \rightarrow e_1e_3 + e_2$. How do we represent the log? Should it be $p \xrightarrow{*} e_1e_3 + e_2$ by $(\alpha_2e_1) \cdot (\alpha_3) \cdot (\alpha_1)$, or $p \xrightarrow{*} e_1e_3 + e_2$ by $(\alpha_2e_1 + e_2^2) \cdot (\alpha_3 + e_2^2) \cdot (e_1e_3 + \alpha_1)$?

2.3 Modelling Polynomial Rewriting with ???

In this situation, $C_0 = \text{objects } *$ (k -algebra case), $C_1 = \text{paths } k[X^*]$ (all polynomials in the free k -algebra), and $C_2 = \text{rewrites between paths}$. *Idea*: Use the k -Mod structure on rewrites so that $\alpha : a \rightarrow b$ and $\alpha' : a' \rightarrow b'$ leads to $k\alpha : ka \rightarrow kb$ and $\alpha + \alpha' : a + a' \rightarrow b + b'$. Again we have vertical composition, inverses, whiskering, identities and horizontal composition: $\alpha \circ \beta \stackrel{\text{def}}{=} \alpha(\text{src } \beta) \cdot (\text{tgt } \alpha)\beta = (\text{src } \alpha)\beta \cdot \alpha(\text{tgt } \beta)$, but what does this lead to? (*R. Brown then made some comments, suggesting a Brown-Mosa paper as being relevant*).

3 Seminar 3: Jonathan Gratus (11th December 2003)

3.1 Deformation Quantisation

Consider a map $\pi : A \rightarrow C$, where A is a noncommutative associative algebra over \mathbb{C} , C is a commutative algebra over \mathbb{C} , and π is a surjective algebra homomorphism. We have an element $\varepsilon \in A$ with the following properties: (1) If $\pi(f) = 0$ then there exists a unique $g \in A$ such that $f = \varepsilon g$; (2) If $f \in A$ with $f \neq 0$, then there exists a $g \in A$ and an $n \in \mathbb{N}_0$ such that $f = \varepsilon^n g$ and $\pi(g) \neq 0$.

Some examples of C : $C = C^\infty(M)$ (M is a manifold), $C = C^\omega(M)$, $C = \mathbb{C}[x, y, z]/x^2 + y^2 + z^2 = 1$, and $C = \mathbb{C}[p, q]$. Note that for the third example we could have $A = \mathbb{C}[\varepsilon]\langle x, y, z \rangle / \{[x, y] = i\varepsilon z, [y, z] = i\varepsilon x, [z, x] = i\varepsilon y, x^2 + y^2 + z^2 = 1\}$.

3.2 Poisson Structure

The Poisson structure is a map $C \times C \rightarrow C$ (denoted $\{\bullet, \bullet\}$) with the following properties: $\{f, \lambda g \pm h\} = \lambda\{f, g\} + \{f, h\}$, $\{f, g\} = -\{g, f\}$, $\{f, gh\} = \{f, g\}h + \{f, h\}g$, and $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

Take $f, g \in A$. Then $\pi([f, g]) = 0$ and so there exists an h such that $\varepsilon h = [f, g]$, i.e. $h = \frac{1}{\varepsilon}[f, g]$. It follows that we can define $\{\pi(f), \pi(g)\} = \pi(\frac{1}{\varepsilon}[f, g])$.

Remark 3.1 A trick is to use $A = \{\sum_{n,m \geq 0} \alpha a_+^n z^m + \sum \beta a_-^n z^m\}$, where $a_\pm = x \pm iy$, $x = \frac{a_+ + a_-}{2}$, $y = \frac{a_+ - a_-}{2}$, $[z, a_\pm] = \pm \varepsilon a_\pm$, and $a_+ a_- + 2z^2 = 1$.

Let $C = \mathbb{C}[x_1, x_2, \dots, x_n]$ be a quantisation of \mathbb{R}^n , let $\mathcal{F} = C[[\varepsilon]]\langle \hat{x}_1, \hat{x}_2, \dots, \hat{x}_n \rangle$ (where $C[[\varepsilon]] = \{\sum_{r=0}^{\infty} \alpha_r \varepsilon^r\}$), and let $A = \mathcal{F} / \{[x_i, x_j] = \varepsilon C_{ij}\}$ for all i and j .

Theorem 3.2 *The following are equivalent:*

- A is a deformation quantisation of $\mathbb{C}[\mathbf{x}]$ (i.e. $\pi : A \rightarrow \mathbb{C}[\mathbf{x}]$ is surjective);
- The product μ on F^Ω is associative;
- $\phi_\infty([x_1, C_{jk}] + [x_j, C_{ki}] + [x_k, C_{ij}]) = 0$.

Extra definitions needed for the above:

- $F^\Omega = \bigoplus_{s=0}^{\infty} \varepsilon^s F_0^\Omega$, where $F_0^\Omega = \text{Span}\{x_1^{r_1} \dots x_n^{r_n} \mid r_1, \dots, r_n \in \mathbb{N}_0\}$;
- Define ϕ_∞ as $\phi_\infty : F \rightarrow F^\Omega$ (as in ‘put in normal order and in order to commute x_i and x_j use $[x_i, x_j] = \varepsilon C_{ij}$, e.g. $x_2 x_1 = x_1 x_2 + \varepsilon C_{ij}$ ’);
- $\mu : F^\Omega \times F^\Omega \rightarrow F^\Omega$ is defined by $\mu(f, g) = \phi_\infty(fg)$, e.g. $\mu(x_1, x_2) = x_1 x_2$ and $\mu(x_2, x_1) = x_1 x_2 + \varepsilon \phi_\infty(C_{21})$.

4 Seminar 4: Jonathan Gratus (18th December 2003)

4.1 Quantising Surfaces with Arbitrary Genus

In this seminar, Jonathan summarised his paper “Quantising Surfaces with Arbitrary Genus”. He started by recalling some facts from the previous seminar before considering what happens when we replace $C = C^\infty(T^*Q)$ by $C = C^\infty(M)$ — can we ‘quantise’ if M is a surface with some genus and not the sphere or torus?

The example $M = \{(x, y, z) \in \mathbb{R}^3 \mid (p(x) + y^2)^2 + z^2 = 1\}$ was introduced, where $p : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial. By choosing a Poisson structure on M , we can write down most of the structure: A is given by $\mathbb{C}[[\varepsilon]]\langle x, y, z \rangle$, with $[x, y] = i\varepsilon z$, $[x, z] = -i\varepsilon(yw + wy)$, $[y, z] = i\varepsilon\{w \mid \mathbf{p}\} - i\varepsilon^3 \mathbf{q}$, and $w^2 + z^2 = 1$, where $\mathbf{p} = p(x)$, $w = \mathbf{p} + y^2$, $\{w \mid \mathbf{p}\}$ is an ‘ordered derivative’, and \mathbf{q} is a special polynomial $\mathbf{q} = \sum_{r=0}^{\infty} \varepsilon^r q_r(x)$.