PhD Seminars 2003/2004: Dr. C. D. Wensley Semester 1

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1 Seminar 1: 9th October 2003

1.1 Rings with Several Objects

In this series of seminars we will be going through Barry Mitchell's paper "Rings with Several Objects" from Advances in Mathematics, 1972, Volume 8, Pages 1–161.

Definition 1.1 A *Ringoid* (Hilton/Lederman, 1940's – 1960's) is a small additive category.

A category $\mathbb{C}$ is additive if every hom set is an abelian group such that composition is bilinear. So $\mathbb{C}(x,x)$ is a ring with identity 1_x. In the following diagram, $\mathbb{C}(x,y)\otimes\mathbb{C}(y,z)\to\mathbb{C}(x,z)$ is a morphism in $\mathbb{A}\mathrm{b}$:

$$\begin{tikzcd}
& y \arrow[dr, "{\mathbb{C}(y,z)}", "g"'] & \\
x \arrow[ur, "{\mathbb{C}(x,y)}", "f"'] \arrow[rr, "fg", "{\mathbb{C}(x,z)}"'] & & z
\end{tikzcd}$$

Examples:

- A one-object ringoid is a ring with identity.
- Let R be a ring and let I_n be an n-object tree groupoid. Then $R\times I_n$ is a ringoid with $(r,i,j)+(s,i,j)=(r+s,i,j)$.
- With Objects $0,1$; Rings R,S; and $M=$ left-R, right-S module, we have the following diagram (with $rm\in M$, $ms\in M$):

  $$\begin{tikzcd}
  0 \arrow[loop above, "R"] \arrow[r, "M"] & 1 \arrow[loop above, "S"]
  \end{tikzcd}$$

- Let π be a small category (a poset). $\mathbb{Z}_\pi$ has Objects $=\mathrm{Ob}(\pi)$ and $\mathbb{Z}_\pi(p,q)=$ free abelian group on $\pi(p,q)$. The composition is defined to be bilinear and the injection $\pi\to\mathbb{Z}_\pi$ is a functor.

  Example 1.2 Let π and $\mathbb{Z}_\pi$ be as shown in the diagrams below, noting that π also has 4 morphisms $0\to 2$:

  $$\pi = \begin{tikzcd}
  0 \arrow[loop above, "1_0"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & 1 \arrow[loop above, "1_1"] \arrow[r, shift left, "j"] \arrow[r, shift right, "k"'] & 2 \arrow[loop above, "1_2"]
  \end{tikzcd}$$

  $$\mathbb{Z}_\pi = \begin{tikzcd}
  0 \arrow[loop above, "\alpha 1_0"] \arrow[r, "af+bg"] & 1 \arrow[loop above, "\beta 1_1"] \arrow[r, "cj+dk"] & 2 \arrow[loop above, "\gamma 1_2"]
  \end{tikzcd}$$

We have $(af+bg)(cj+dk) = ac(fj) + ad(fk) + bc(gj) + bd(gk)$. **Note:** If π is a monoid, then $\mathbb{Z}_\pi$ is the usual monoid ring (in this example $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \equiv \mathbb{Z}^4$).

Definition 1.3 If $\mathbb{C}$ is a ringoid then $\mathsf{Mat}\ \mathbb{C}$ has objects which are finite sequences of objects and morphisms which are $m \times n$ matrices $[\gamma_{ij}]$ with $\gamma_{ij} \in \mathbb{C}(p_i, q_j)$ (the morphism goes from $\mathbf{p} = (p_1, \ldots, p_m)$ to $\mathbf{q} = (q_1, \ldots, q_n)$). Taking the example $\mathsf{Mat}\ \mathbb{Z}_\pi$, the sequence $(1,0,1) \to (2,1) \to 2$ may give rise to the following matrix multiplication, say:

$$\begin{bmatrix} 3j+4k & 7(1_1) \\ fj-gk & -f+9g \\ -k & 01_1 \end{bmatrix} \times \begin{bmatrix} -3(1_2) \\ 2j-k \end{bmatrix} = \begin{bmatrix} 5j-19k \\ -5fj-12gk+fk+18gj \\ -3k \end{bmatrix}$$

Note that there is no morphism $\mathbf{p} \to \mathbf{q}$ if $\mathbb{C}(p_i, q_j) = \phi$ for some i, j.

2 Seminar 2: 16th October 2003

2.1 Adding Natural Transformations

Suppose $\mathbb{C}$ is an additive category and consider two natural transformations β, $\bar{\beta}$ between two functors $G, G' : \mathbb{B} \to \mathbb{C}$. These can now be added: $c \xrightarrow{y} d$ in $\mathbb{B}$ leads to the following diagram:

$$\begin{tikzcd}
Gc \arrow[r, "Gy"] \arrow[d, shift right, "\beta_c"'] \arrow[d, shift left, "\bar{\beta}_c"] & Gd \arrow[d, shift right, "\beta_d"'] \arrow[d, shift left, "\bar{\beta}_d"] \\
G'c \arrow[r, "G'y"'] & G'd
\end{tikzcd}$$

As β_c and $\bar{\beta}_c$ are elements of $\mathbb{C}(Gc, G'c)$ then $\beta_c + \bar{\beta}_c$ is defined and in this way $\beta + \bar{\beta}$ is defined. What does this say about $\mathbb{C}^{\mathbb{B}}$?

Aside by RB: If $\mathbb{C}$ is a category then $\square\mathbb{C}$ is the double category of commuting squares in $\mathbb{C}$ summarised by the following diagrams:

$$\begin{tikzcd}
S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\
V \arrow[r, shift left] \arrow[r, shift right] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{C} \arrow[d, shift left] \arrow[d, shift right] \\
\mathbb{C} \arrow[r, shift left] \arrow[r, shift right] & Ob(\mathbb{C})
\end{tikzcd}$$

If $\mathbb{C}$ is additive what happens to $\square\mathbb{C}$?

2.2 Additive Functors

Let $\mathbb{B}$ and $\mathbb{C}$ be additive categories and let us find an additive $G : \mathbb{B} + \mathbb{C}$ such that $G(x+y) = Gx + Gy$ and $\mathbb{C}^{\mathbb{B}}$ is the additive functor category.

$$\begin{tikzcd}[column sep=large]
\mathbb{A} \arrow[r, bend left=70, "F", ""{name=F1, below}] \arrow[r, "F'"{name=F2}] \arrow[r, bend right=70, "F''"', ""{name=F3, above}] & \mathbb{B} \arrow[r, bend left=70, "G", ""{name=G1, below}] \arrow[r, "G'"{name=G2}] \arrow[r, bend right=70, "G''"', ""{name=G3, above}] & \mathbb{C}
\arrow[from=F1, to=F2, "\alpha"]
\arrow[from=F2, to=F3, "\alpha'"]
\arrow[from=G1, to=G2, "\beta"]
\arrow[from=G2, to=G3, "\beta'"]
\end{tikzcd}$$

The idea is to define the composition $\alpha \bullet \beta : FG \to F'G'$ compatible with the following diagrams ($a \xrightarrow{x} b$ is in $\mathbb{A}$):

$$\begin{tikzcd}
Fa \arrow[r, "Fx"] \arrow[d, "\alpha_a"'] & Fb \arrow[d, "\alpha_b"] \\
F'a \arrow[r, "F'x"'] & F'b
\end{tikzcd}$$

$$\begin{tikzcd}
& GFa \arrow[rr, "GFx"] \arrow[dl, "\beta_{Fa}"'] \arrow[dd, "G\alpha_a"] \arrow[dddl, dashed, "(\alpha\bullet\beta)_a"] & & GFb \arrow[dl, "\beta_{Fb}"'] \arrow[dd, "G\alpha_b"] \\
G'Fa \arrow[rr, "G'Fx"] \arrow[dd, "G'\alpha_a"'] & & G'Fb \arrow[dd, "G'\alpha_b"] & \\
& GF'a \arrow[rr, "G'F'x"] \arrow[dl, "\beta_{F'a}"] & & GF'b \arrow[dl, "\beta_{F'b}"] \\
G'F'a \arrow[rr, "G'F'x"'] & & G'F'b &
\end{tikzcd}$$

We define $(\alpha \bullet \beta)_a := \beta_{Fa} G'\alpha_a = G\alpha_a \beta_{F'a}$.

2.2.1 Properties of $\bullet$

- $1_{\mathbb{A}} \bullet \alpha = \alpha = \alpha \bullet 1_{\mathbb{B}}$:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend left=60, "1_{\mathbb{A}}", ""{name=U, below}] \arrow[r, bend right=60, "1_{\mathbb{A}}"', ""{name=D}] & \mathbb{A} \arrow[r, bend left=60, "F", ""{name=U2, below}] \arrow[r, bend right=60, "F''"', ""{name=D2}] & \mathbb{B}
\arrow[from=U, to=D, "1_{\mathbb{A}}"]
\arrow[from=U2, to=D2, "\alpha"]
\end{tikzcd}$$

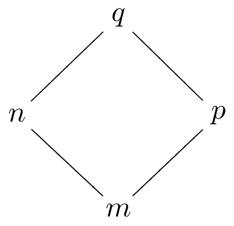
- $(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$.
- $1_F \bullet 1_G = 1_{FG}$.
- $(\alpha\alpha') \bullet (\beta\beta') = (\alpha \bullet \beta)(\alpha' \bullet \beta')$.
- $\alpha \bullet (\beta + \bar{\beta}) = \alpha \bullet \beta + \alpha \bullet \bar{\beta}$ ($\mathbb{C}$ additive).
- $(\alpha + \bar{\alpha}) \bullet \beta = \alpha \bullet \beta + \bar{\alpha} \bullet \beta$ ($\mathbb{B}$, $\mathbb{C}$, G, G' additive).

Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring. For example, with $k[x_1, \ldots, x_n]$, $R_n =$ homogeneous polynomials of degree n for $n \geqslant 0$, and $R_n = 0$ for $n < 0$. The associated additive $\mathbb{C}$ has objects $\mathbb{Z}$ and $\mathbb{C}(m, n) = R_{n-m}$. Further, if R is a ring then $R_{\pi(p,q)}$ is the free R-module on $\pi(p, q)$.

2.3 Ring of an additive category

Let $\mathbb{C}$ be small and additive. Then $[\mathbb{C}] = \mid \mathbb{C} \mid \times \mid \mathbb{C} \mid$ matrices $[\alpha_{pq}]$, where $\alpha_{pq} \in \mathbb{C}(p, q)$ and we have a finite number of non-zero entries in each row and column. It follows that $[\mathbb{C}]$ is a ring with identity and the usual matrix operations. Further, $[\alpha_{pq}][\beta_{qr}] = [\sum_q \alpha_{pq}\beta_{qr}]$.

Take as an example π a finite poset and R a ring. Then $[R_\pi] = \mid \pi \mid \times \mid \pi \mid$ matrices $[r_{pq}]$ with $r_{pq} = 0$ if $!(p \leqslant q)$:



$$\begin{bmatrix} r_{mm} & r_{mn} & r_{mp} & r_{mq} \\ 0 & r_{nn} & 0 & r_{nq} \\ 0 & 0 & r_{pp} & r_{pq} \\ 0 & 0 & 0 & r_{qq} \end{bmatrix}$$

2.4 Tensor Products

For ordinary categories, we have $(\mathbb{A}^{\mathbb{C}_1})^{\mathbb{C}_2} = \mathbb{A}^{(\mathbb{C}_1 \times \mathbb{C}_2)}$. If $\mathbb{C}_1$ and $\mathbb{C}_2$ are additive, then so is $\mathbb{C}_1 \times \mathbb{C}_2$. But the formula $(\mathbb{A}^{\mathbb{C}_1})^{\mathbb{C}_2} = \mathbb{A}^{(\mathbb{C}_1 \times \mathbb{C}_2)}$ is not valid when $\mathbb{A}$ is additive. Instead, $\mathbb{C}_1 \otimes \mathbb{C}_2$ has objects $\mid \mathbb{C}_1 \mid \times \mid \mathbb{C}_2 \mid$ and $\mathsf{Mor}((p,q),(p',q')) = \mathbb{C}_1(p,p') \otimes \mathbb{C}_2(q,q')$ with bilinear composition $(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_1x_2) \otimes (y_1y_2)$. It follows that $(\mathbb{A}^{\mathbb{C}_1})^{\mathbb{C}_2} = \mathbb{A}^{\mathbb{C}_1 \otimes \mathbb{C}_2}$.