PhD Seminars 2002/2003: Prof. R. Brown Semester 3

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1 Seminar 1: 14th May 2003

1.1 A general seminar on higher dimensional material

Simplicial groups whose Moore complex is of length 1 correspond to crossed modules (Loday, 1982). Simplicial groups whose Moore complex is of length 2 correspond to 2-crossed modules (Daniel Condoché).

Consider a sequence $L \xrightarrow{\lambda} M \xrightarrow{\mu} P$ of groups. Let P act on M and on L on the right. $\mu : M \to P$ is a pre-crossed module. M acts on L via P. $L \xrightarrow{\lambda} M$ is a crossed module. $\mu\lambda = 0$. We also have a Peiffer lifting $\{\ ,\ \} : M \times M \to L$. In M, you can write $\langle m, m' \rangle = m'^{-1} m m' (m^{\mu m'})^{-1}$. The point is that $\lambda\{m, m'\} = \langle m, m' \rangle$, and there are also some other axioms. Papers in this area: Conduché (Modules croidés de longeur 2) and R. B. + N. D. Gilbert (Automorphisms of crossed modules and algebraic models of 3-types).

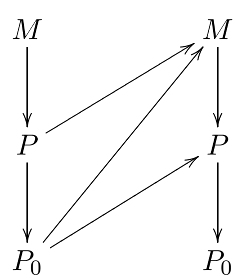
The motivation for the latter paper was 'higher order symmetry':

$$\mathsf{Sets} \xrightarrow{\text{Aut}} \mathsf{Groups} \xrightarrow{\text{Aut}} \mathsf{Crossed\ Modules} \xrightarrow{\text{Aut}} ? \xrightarrow{?} ?$$

The category of groupoids (Gpd) is cartesian closed. $\mathsf{Gpd}(A \times B, C) = \mathsf{Gpd}(A, \mathsf{GPD}(B, C))$. $\mathsf{GPD}(B, C)$ is a groupoid with $\text{Ob}(\mathsf{GPD}(B, C)) = \mathsf{Gpd}(B, C)$. $\mathsf{END}(C) = \mathsf{GPD}(C, C)$ is a monoid in Gpd containing a maximal subgroup object $\mathsf{AUT}(C)$, the wide subgroupoid of $\mathsf{END}(C)$ on $\mathsf{AUT}(C)$ ($\mathsf{AUT}(C)$ is a group in groupoids, also thought of as the crossed module $M(C) \to \text{Aut}(C)$).

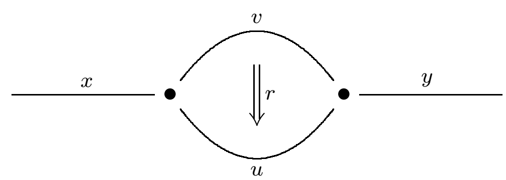
What is $M(C)$? Admissible sections $s : \text{Ob}(C) \to C$ of the domain are equal to 'homotopies' $1_C \simeq f$, where f is an automorphism $\text{cod} \circ s =$ bijection on $\text{Ob}(C)$. Ehresmann strongly uses local admissible sections. He is a pioneer of 'local to global' methods and coined the terms fibre bundle, foliations, germs, jets, holonomy and Lie groupoid.

Consider a crossed module $M \xrightarrow{\mu} P$. XMod is monoidal closed. We have $\mathsf{XMod}(A \otimes B, C) \cong \mathsf{XMod}(A, \mathsf{XMOD}(B, C))$, where $\mathsf{XMOD}(B, C)_0 = \mathsf{XMod}(B, C)$, $\mathsf{XMOD}(B, C)_1 =$ 1-homotopies, and $\mathsf{XMOD}(B, C)_2 =$ 2-homotopies:

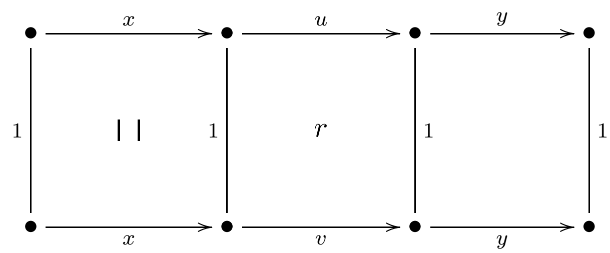


A crossed module of groupoids $P \rightrightarrows P_0$ is a groupoid. Crossed modules of groupoids are equivalent to 2-groupoids and to double groupoids with thin structure/composition. Question: What is the place of cubical methods in higher category theory? Answer: At present, very little! The advantages of using cubes include having multiple composition which we express using double categories.

In presentations of monoids, all relators $r : u \to v$ should lead us to have $xr : xu \to xv$ and $ry : uy \to vy$ for all x and y. This has the following diagram:

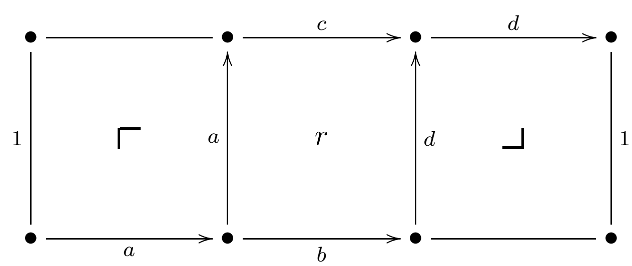


The double category version has the following diagram, where r is $ab \Rightarrow cd$:



We use connections to relate to 2-categories:

$\Phi(r) =$



In the above we replace 'pasting and whiskering' by the calculus of thin elements and connections. A claimed theorem: the boundary of a multiple composition of cubes is a multiple composition of boundaries.