PhD Seminars 2002/2003: Dr. C. D. Wensley Semester 3

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Contents

1 Seminar 1: 4th June 2003

1.1 Joyal's Theory of Species

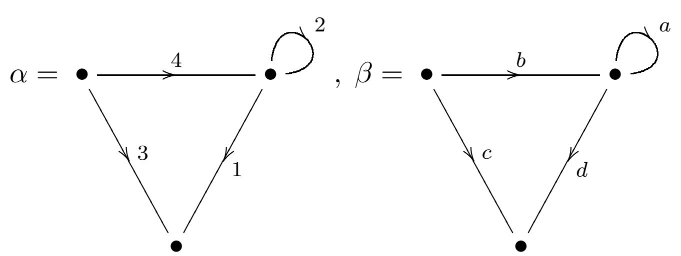
Definition 1.1 A species of structures is a functor $\mathbb{A} : \mathsf{Set}b \to \mathsf{Set}f$ between sets and bijections (groupoids) and sets and functors, $U \mapsto \mathbb{A}[u]$ (a structure). The exponential generating function $\mathbb{A}(x)$ of $\mathbb{A}$ is given by $\mathbb{A}(x) = \sum_{n=0} a_n \frac{x^n}{n!}$, where $a_n = |\mathbb{A}[\mathbf{n}]|$, $\mathbf{n} = [1, 2, \ldots, n]$, $\mathbf{0} = \phi$.

Example 1.2 (1) Exponential species $\mathbb{E}$: $\mathbb{E}[U] = \{U\}$, $\mathbb{E}(x) = e^x$. (2) Permutation species $\mathbb{P}$: $\mathbb{P}[\phi] = \{\phi\}$, $\mathbb{P}[U]$ = set of permutations of U, $|\mathbb{P}[\mathbf{n}]| = n!$ so $\mathbb{P}(x) = 1 + x + \ldots + x^n = (\frac{1}{1-x})$. (3) Circular permutation species $\mathbb{C}$: $\mathbb{C}[\phi] = \phi$, $\mathbb{C}[U]$ = permutations of U comprising a single cycle of length $|U|$, $|\mathbb{C}[\mathbf{n}]| = (n-1)!$ so $\mathbb{C}(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots = \sum_{n\geqslant 1} \frac{x^n}{n} = -\log(1-x)$. (4) Subset species $\mathbb{S}$: $\mathbb{S}[U]$ = set of subsets of U, $|\mathbb{S}[\mathbf{n}]| = 2^n$ so $\mathbb{S}(x) = e^{2x}$. (5) Species $\mathbb{L}$ of linear (total) orders: $\mathbb{L}[U]$ = set of $|U|!$ total orders on U, $\mathbb{L}(x) = (\frac{1}{1-x}) = \mathbb{P}(x)$.

Example 1.3 Let $\mathbb{A}[U] = \phi$ if $|U| \neq 4$ and $\mathbb{A}[U]$ = digraphs with 4 edges labelled by elements of U if $|U| = 4$. We have maps such as the following:

$$\begin{array}{ccc} U = \{1,2,3,4\} & & \mathbb{A}[U] = \{\ldots, \alpha, \ldots\} \\ f = \binom{1234}{dacb} \Big\downarrow \text{24 bijections} & & \Big\downarrow \mathbb{A}[f] \\ V = \{a,b,c,d\} & & \mathbb{A}[U] = \{\ldots, \beta, \ldots\}, \end{array}$$

where α and β are as follows:



The exponential generating function of this species is $\mathbb{A}(x) = \frac{a_4}{24}x^4$.

Definition 1.4 Species $\mathbb{A}$ and $\mathbb{B}$ are isomorphic ($\mathbb{A} \cong \mathbb{B}$) if there exists a natural isomorphism T between the functors $\mathbb{A}$ and $\mathbb{B}$.

In practice, we will give a definition for the morphisms $T_U \in \mathrm{Mor}_{\mathsf{Set}f}(\mathbb{A}[U], \mathbb{B}[U])$, but we shall not bother to verify the identity $\mathbb{B}[f] \circ T_U = T_V \circ \mathbb{A}[f]$ for $U \xrightarrow{f} V$, a morphism in $\mathsf{Set}b$.

If T exists, then for each U and an isomorphism $\theta : U \to U$, there is a bijection $T_U : \mathbb{A}[U] \to \mathbb{B}[U]$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{A}[U] & \xrightarrow{\mathbb{A}[\theta]} & \mathbb{P}[U] \\ {\scriptstyle T_U}\downarrow & & \downarrow{\scriptstyle T_U} \\ \mathbb{B}[U] & \xrightarrow{\mathbb{B}[\theta]} & \mathbb{B}[U] \end{array}$$

Example 1.5 Let I be the identity species, $\mathbb{I}[U] = U$, and define $\mathbb{A}$ to be the species with structures $\mathbb{A}[U] =$ set of subsets of U with $|U| - 1$ elements. Then $\mathbb{I}(x) = \mathbb{A}(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$. Let us define $T : \mathbb{I} \to \mathbb{A}$ by $T_U : U \to \mathbb{A}[U], u \mapsto U \setminus \{u\}$. This is a natural isomorphism so $\mathbb{I} \cong \mathbb{A}$.

So to show that two species $\mathbb{A}$ and $\mathbb{B}$ are not isomorphic, either (i) show that $\mathbb{A}(x) \neq \mathbb{B}(x)$ or (ii) choose a permutation $\theta : U \to U$ for some U so that the permutations $\mathbb{A}[\theta]$ and $\mathbb{B}[\theta]$ have different cycle structures.

Example 1.6 Let us show that $\mathbb{P}$ is not isomorphic to $\mathbb{L}$. Now $\mathbb{P}[\mathbf{3}] = \{I, (12), (13), (23), (123), (132)\}$ $= \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and $\mathbb{L}[\mathbf{3}] = \{$1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1$\} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$. Let us choose $\theta = (12)$ on $\mathbf{3}$. Then $\mathbb{P}[\theta] = (p_3 p_4)(p_5 p_6)$ and $\mathbb{L}[\theta] = (\ell_1 \ell_3)(\ell_2 \ell_4)(\ell_5 \ell_6)$: $\mathbb{P}[\theta]$ is of type $(2^2 1^2)$ and $\mathbb{L}[\theta]$ is of type (2^3) so $\mathbb{P}$ is not isomorphic to $\mathbb{L}$.

Alternatively, let $\mathbb{P}[\mathbf{2}] = \{I, (12)\}$ and let $\mathbb{L}[\mathbf{2}] = \{$2-1, 1-2$\}$. Choose $\theta = (12)$ on $\mathbf{2}$. Then $\mathbb{P}[\theta] =$ the identity permutation on $\mathbb{P}[2]$ and $\mathbb{L}[\theta] =$ a transposition on $\mathbb{L}[2]$. There are 2 possible T_U's: $T_{\mathbf{2}} = () \mapsto$ 2-1 and $(12) \mapsto$ 1-2 or $T'_{\mathbf{2}} = () \mapsto$ 1-2 and $(12) \mapsto$ 2-1. Using $T_{\mathbf{2}}$, we see that the following diagram does not commute:

$$\begin{array}{ccc} () & \longrightarrow & () \\ \downarrow & & \downarrow \\ 2-1 & \longrightarrow & 1-2 \neq 2-1 \end{array}$$

1.1.1 Operation on Species

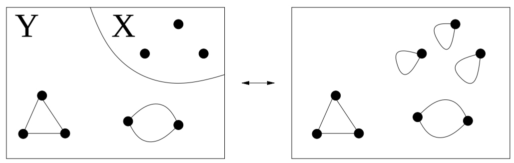
Definition 1.7 The sum $\mathbb{A}+\mathbb{B}$ is the species with structure $(\mathbb{A}+\mathbb{B})[U] = \mathbb{A}[U] \sqcup \mathbb{B}[U]$ (disjoint union). Clearly $(\mathbb{A}+\mathbb{B})(x) = \mathbb{A}(x) + \mathbb{B}(x)$.

Definition 1.8 The product $\mathbb{A}.\mathbb{B}$ is the species with structures $(\mathbb{A}.\mathbb{B})[U] = \{(/U_1/U_2/,\ X,\ Y)$: $/U_1/U2/$ is a composition of U (= an ordered partition), $X \in \mathbb{A}[U_1], Y \in \mathbb{B}[U_2]\}$.

Example 1.9 Let $\mathbb{D}$ be the species of derangements so that $\mathbb{D}[\mathbf{n}] = \{$permutations π of $\{1, \ldots, n\}$ such that $\pi(i) \neq i$ for all $i \in \mathbf{n}\}$, i.e. a derangement is a permutation which has no fixed point. For the following table, carrying on do we get $\lim_{n\to\infty} \frac{n!}{\mathbb{D}[\mathbf{n}]} = e$?

n	0	1	2	3	4	5	...
$\lvert\mathbb{D}[n]\rvert$	1	0	1	2	9	44	...

Let us now prove that $\mathbb{E}.\mathbb{D} \cong \mathbb{P}$. The idea is that any permutation π of U fixes some subset $X \subseteq U$ and deranges the rest, i.e. $U \setminus X$. So given a permutation $\pi : U \to U$, set X = a subset of U of fixed points and set $Y = U \setminus X$. Then the natural transformation $T : \mathbb{E}.\mathbb{D} \leftrightarrow \mathbb{P}$ is described by the following diagram:



i.e. $(/X/Y/, \{X\}, \pi|_Y) \leftrightarrow \pi$.

Remark 1.10 Note that a corollary of the above states that

$$1 + x + x^2 + x^3 + x^4 + \ldots = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) \times \left(1 + 0x + \frac{x^2}{2} + \frac{2x^3}{6} + \frac{9x^4}{24} + \ldots\right),$$

a result which comes from the following table:

1	x	$\frac{x^2}{2}$	$\frac{x^3}{6}$	$\frac{x^4}{24}$	$\frac{x^5}{120}$	...
1	1	1	1	1	1	
		1	3	6	10	
			2	8	20	
				9	45	
					44	

Proposition 1.11 $(\mathbb{A}.\mathbb{B})(x) = \mathbb{A}(x)\mathbb{B}(x)$.

Proof: $\mathbb{A}(x)\mathbb{B}(x) = \sum a_s \frac{x^s}{s!} \sum b_t \frac{x^t}{t!} = \sum_{n=0}^{\infty} (\sum_{s+t=n} \frac{n!}{s!t!} a_s b_t) \frac{x^n}{n!}$, where $\frac{n!}{s!t!}$ = the number of subsets U_1 of U of size s, $a_s = |\mathbb{A}[U_1]|$, and $b_t = |\mathbb{B}[U \setminus U_1]|$. □

Proposition 1.12 *The number of derangements d_n of $\mathbf{n}$ is $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.*

Proof: $\mathbb{P} = \mathbb{E}.\mathbb{D} \Rightarrow \sum_n d_n \frac{x^n}{n!} = (\sum_k (-1)^k \frac{x^k}{k!})(\sum_i x^i)$. (This identity is usually obtained using inclusion/exclusion arguments; it can be shown that $d_n = (n-1)(d_{n-1} + d_{n-2}), n \geqslant 2$). □

2 Seminar 2: 6th June 2003

2.1 Joyal's Theory of Species, Part 2

In the previous seminar, we used a picture to prove that $\mathbb{P} \cong \mathbb{E}.\mathbb{C}$. In general, we have $\mathbb{A} \cong \mathbb{E} \circ \mathbb{C}$, where $\mathbb{C}$-structures are connected $\mathbb{A}$-structures.

Definition 2.1 The substitution $\mathbb{A} \circ \mathbb{B}$ of $\mathbb{B}$ in $\mathbb{A}$ when $\mathbb{B}[\phi] = \phi$ is the species with structures $\mathbb{A} \circ \mathbb{B}[U] = \{(/U_1/\ldots/U_\ell/, X, Y_1, \ldots, Y_\ell) : /U_1/\ldots/U_\ell/$ is a partition of U into ℓ parts, $X \in \mathbb{A}[\{U_1, \ldots, U_\ell\}]$, and $Y_i \in \mathbb{B}[U_i]$ for $1 \leqslant i \leqslant \ell\}$. As an example, $\mathbb{E} \circ \mathbb{C}[U] = \{(U_1/\ldots/U_{ell}/, \{\{U_1, \ldots, U_\ell\}\}, Y_1, \ldots, Y_\ell$ are circular permutations$\}$.

Example 2.2 Let Σ be the species of partitions, $\Sigma[U] = \{/U_1/\ldots/U_\ell/ : U_i \subseteq I,\ U_1 \cup \ldots \cup U_\ell = U,\ U_i \cap U_j = \phi$ for $i \neq j\}$.

U	$\Sigma[U]$
ϕ	$\{\phi\}$
$\{1\}$	$\{/1/\}$
$\{2\}$	$\{/12/,\ /1/2/\}$
$\{3\}$	$\{/123/,\ /12/3/,\ /13/2/,\ /23/1/,\ /1/2/3/\}$

Proposition 2.3 $\Sigma \cong \mathbb{E} \circ \mathbb{E}^*$, *where* $\mathbb{E}^*[\phi] = \phi$ *and* $\mathbb{E}^*[U] = \{U\}$ *for* $U \neq \phi$.

Proof: $(\mathbb{E} \circ \mathbb{E}^*)[U] = \{(/U_1/\ldots/U_\ell/, \{\{U_1, \ldots, U_\ell\}\}, \{U_1\}, \ldots, \{U_\ell\})\}$. This $(\ell+2)$-tuple is uniquely determined by the partition $/U_1/\ldots/U_\ell/$. So we define $T : \Sigma \leftrightarrow \mathbb{E} \circ \mathbb{E}^*$ by $T_U : /U_1/\ldots/U_\ell/ \leftrightarrow (/U_1/\ldots/U_\ell, \{U_1, \ldots, U_\ell\}, \{U_1\}, \ldots, \{U_\ell\})$. This is 'clearly' a natural transformation. $\square$

Theorem 2.4 $\mathbb{A} \circ \mathbb{B}(x) = \mathbb{A}(y)$, *where* $y = \mathbb{B}(x)$. *Example:* $\Sigma(x) = \exp^{(\exp^x - 1)}$.

Theorem 2.5 *(Distributive law for product and substitution of species)* $(\mathbb{A}.\mathbb{B}) \circ \mathbb{C} \cong (\mathbb{A} \circ \mathbb{C}).(\mathbb{B} \circ \mathbb{C})$.

Proof: A structure in $(\mathbb{A}.\mathbb{B}) \circ \mathbb{C}$ has the form $(/U_1/\ldots/U_\ell/, (V_1, V_2, X_1, X_2), Y_1, \ldots, Y_\ell)$, where $/U_1/\ldots/U_\ell/$ is a partition of U into ℓ parts, $Y_i \in \mathbb{C}[U_i]$, (V_1, V_2) is a composition of $\{U_1, \ldots, U_\ell\}$, $X_1 \in \mathbb{A}[V_1]$, and $X_2 \in \mathbb{B}[V_2]$. Let us relabel $\{U_1, \ldots, U_\ell\}$ as $\{V_{11}, \ldots, V_{\ell_1 1}, V_{12}, \ldots, V_{\ell_2 2}\}$, where $\ell_1 + \ell_2 = \ell$, $V_1 = \{V_{11}, \ldots, V_{\ell_1 1}\}$, and $V_2 = \{V_{12}, \ldots, V_{\ell_2 2}\}$. When $V_{i1} = U_j$ set $Z_{i1} = Y_j$, and when $V_{i2} = U_j$ set $Z_{i2} = Y_j$. Then the required natural transformation is given by the rule $(/U_1/\ldots/U_\ell/$, (V_1, V_2, X_1, X_2), $Y_1, \ldots, Y_\ell) \leftrightarrow (V_{11} \cup \ldots \cup V_{\ell_1 1},\ V_{12} \cup \ldots \cup V_{\ell_2 2},\ (/V_{11}/\ldots/V_{\ell_1 1}/,\ X_1,\ Z_{11}, \ldots Z_{\ell_1 1})$, $(/V_{12}/\ldots/V_{\ell_2 2}/, X_2, Z_{12}, \ldots, Z_{\ell_2 2}))$. $\square$

Exercise 2.6 As before, let Σ be the species of partitions. Let $\Sigma_{(\ell)}$ be the species of partitions into ℓ parts. Then

$$\Sigma_{(\ell)}(x) = \sum_{n \geqslant \ell} \left\{ \begin{array}{c} n \\ \ell \end{array} \right\} \frac{x^n}{n!},$$

where $\left\{ \begin{array}{c} n \\ \ell \end{array} \right\}$ is the Stirling number of the second kind. Show that $\mathbb{E}.\mathbb{E}^* \cong \Sigma'_{(2)}$.

2.2 Constituent Parts

The constituent parts are essentially the S_n-orbits.

Definition 2.7 $\mathbb{A} \cong \sum_{n=0}^{\infty} \mathbb{A}_n$, where $\mathbb{A}_n[U] = \mathbb{A}[U]$ if $|U| = n$, and $\mathbb{A}_n[U] = \phi$ otherwise.

Definition 2.8 $\mathbb{A}$ is molecular if $\mathbb{A} \cong \mathbb{B} + \mathbb{C}$. This implies that $\mathbb{B} \cong \phi$ or that $\mathbb{C} \cong \phi$, where ϕ here denotes the empty species, $\phi[U] = \phi$ for all U. Of course $\phi(x) = 0$.

Let $H \leqslant S_n$. Let us define $\mathbb{M}_H$ as follows: $\mathbb{M}_H[\mathbf{n}] = S_n/H$ (cosets); $\mathbb{M}_H[U] = \phi$ if $|U| \neq n$. Facts:

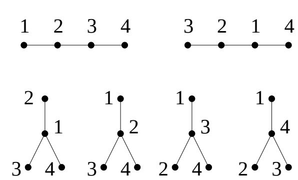
- A transitive S_n-set X is isomorphic to S_n/H, where $H = \operatorname{Stab} x$ for $x \in X$ (with action $(Hg)^{g'} := H(gg'))$;
- H conjugate to $K \Rightarrow S_n/H \cong S_n/K$ as S_n-sets, i.e. the number of isomorphism classes of transitive S_n-sets is equal to the number of conjugacy classes of subgroups.

Example 2.9 Let $\mathbb{A}_n \cong \sum_{[H]\leqslant[S_n]} c_H \mathbb{M}_H$, where $c_H \in \mathbb{N}$ and $[\alpha]$ denotes the conjugacy class of α. It follows that e.g. $\mathbb{P}[\mathbf{3}] \cong \mathbb{M}_{S_3} + \mathbb{M}_{C_2} + \mathbb{M}_{C_3}$ "= {() + (1, 2), (1, 3), (2, 3) + (1, 2, 3), (1, 3, 2)}"; $\mathbb{L}[\mathbf{3}] \cong \mathbb{M}_{I_3} = \{$3-2-1, …, 1-2-3$\}$; $\mathbb{P}[\mathbf{2}] \cong 2\mathbb{M}_{C_2}$; and $\mathbb{L}[\mathbf{2}] \cong \mathbb{M}_{I_2}$.

Definition 2.10 $\mathbb{A}$ is atomic if $\mathbb{A} \cong \mathbb{B}.\mathbb{C}$. This implies that $\mathbb{B} \cong \mathbb{E}_0$ or that $\mathbb{C} \cong \mathbb{E}_0$.

Yeh has shown that every molecular species can be expressed uniquely (up to order) as a product of atomic species.

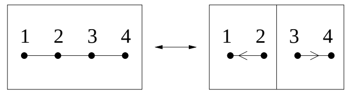
Example 2.11 Let $\mathbb{T}$ denote the species of vertex-labelled trees. $\mathbb{T}[\mathbf{4}]$ is as shown in the diagram below (top row: $\operatorname{Stab} C_2' = \{(), (14)(23)\}$; bottom row: $\operatorname{Stab} S_3 I$):



Conclusion: $\mathbb{T}_4 \cong \mathbb{M}_{C_2'} + \mathbb{M}_{S_3 I}$ (molecular decomposition).

Example 2.12 $\mathbb{M}_{S_3 I} \cong \mathbb{E}_3.\mathbb{X} \cong \mathbb{M}_{S_3}.\mathbb{X}$, where $\mathbb{X}$ ($\cong \mathbb{E}_1$) is defined as follows: $\mathbb{X}(x) = x$, $\mathbb{X}[\mathbf{1}] = \{\phi\}$, $\mathbb{X}[U] = \phi$ if $|U| \neq 1$.

Example 2.13 $\mathbb{M}_{C_2'} \cong \mathbb{E}_2 \circ \mathbb{L}_2$. Exponential Generating Functions: $\mathbb{E}_2(x) = \frac{x^2}{2!}$, $\mathbb{L}_2(x) = x^2 = y$, $\mathbb{E}_2 \circ \mathbb{L}_2(x) = \mathbb{E}_2(y) = \frac{1}{2}y^2 = \frac{1}{2}x^4 = 12(\frac{x^4}{24})$. This can be seen by the diagram shown below and we conclude that even atomic species can be expressed as substitutions in some cases.



2.3 Wreath Products

The wreath product $G \wr H$, where G is a permutation group of degree n, is defined as follows: $G \wr H = G \ltimes (H \times \ldots \times H)$ (H appears n times), where the action is to permute the factors.

Given a permutation representation of H of degree m, write down an $n \times m$ matrix as follows:

$$\begin{pmatrix} 1 & 2 & \ldots & m \\ m+1 & m+1 & \ldots & 2m \\ \vdots & \vdots & & \vdots \\ \ldots & \ldots & & mn \end{pmatrix}$$

Permute each row independently by H and then the set of rows by G. Result: $|G \wr H| = |G| \times |H|^n$.

Example 2.14 Let $G = C_2$ of degree 2 and let $H = I_2$ of degree 2: $\binom{1\,2}{3\,4} \mapsto ()$; $\binom{3\,4}{1\,2} \mapsto (13)(24)$, i.e. $C_2' \cong C_2 \wr I_2$ so that $\mathbb{M}_{C_2'} \cong \mathbb{E}_2 \circ \mathbb{L}_2 \cong \mathbb{M}_{C_2} \circ \mathbb{M}_{I_2}$. Another example: $C_2 \wr C_2 \cong D_8$: $\binom{1\,2}{3\,4} \mapsto ()$, $\binom{2\,1}{3\,4} \mapsto (12)$, $\binom{1\,2}{4\,3} \mapsto (34)$, $\binom{2\,1}{4\,3} \mapsto (12)(34)$, $\binom{3\,4}{1\,2} \mapsto (13)(24)$, $\binom{3\,4}{2\,1} \mapsto (1324)$, $\binom{4\,3}{1\,2} \mapsto (1423)$, and $\binom{4\,3}{2\,1} \mapsto (14)(23)$. Tip: think of substitution as being equivalent to the wreath product.

2.4 Pointed Species

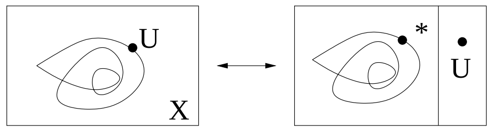
Definition 2.15 $\dot{\mathbb{A}}$ is the species with structures $\dot{\mathbb{A}}[U] = \{(X, u) : X \in \mathbb{A}[U], u \in U\}$. Thus $\dot{\mathbb{A}}(x) = \sum_1^\infty na_n(x^n/n!) = x.\mathbb{A}'(x)$.

Proposition 2.16 *$\dot{\mathbb{A}} \cong \mathbb{O}.\mathbb{A}'$, where $\mathbb{O}$ is the singleton species, $\mathbb{O}[U] = U$ when $|U| = 1$ and $\mathbb{O}[U] = \phi$ otherwise.*

Proof: $(\mathbb{O}.\mathbb{A}')[U] = \{/U_1/U_2/, X, Y : X \in \mathbb{O}[U_1], Y \in \mathbb{A}'[U_2]\} = \{/u/U \setminus \{u\}/, \{u\}, Y\}$, where $Y \in \mathbb{A}[(U \setminus \{u\}) \cup \{*\}]$. Let $\theta : (U \setminus \{u\}) \cup \{*\} \to U$ be the bijection $* \mapsto u$ and $u' \mapsto u'$ for all $u' \neq u$ (this induces a bijection $\theta : \mathbb{A}[(U \setminus \{u\}), \{*\}] \to \mathbb{A}[U]$). Then the morphisms $T_U : (X, u) \mapsto (/u/U \setminus \{u\}/, \{u\}, \theta^{-1}(X))$ define the required natural transformation. □

2.5 Derivative $\mathbb{A}'$ of $\mathbb{A}$

If $\mathbb{A}(x) = \sum_0^\infty a_n \frac{x^n}{n!}$, we require $\frac{d}{dx}(\mathbb{A}(x)) = \mathbb{A}'(x) = a_1 + 2a_2\frac{x}{2!} + 3a_3\frac{x^2}{3!} + \ldots = \sum_{n=0}^\infty a_{n+1}\frac{x^n}{n!}$. Therefore we require $|\mathbb{A}'[\mathbf{n}]| = |\mathbb{A}[\mathbf{n+1}]|$. Another way of thinking about it: $\mathbb{A}(x) = \ldots + a_{n+1}\frac{x^{n+1}}{(n+1)!} + \ldots$, $\mathbb{A}'(x) = \ldots + a_{n+1}\frac{x^n}{n!} + \ldots$; in the diagram below we have $(X, u) \in \dot{\mathbb{A}}[U] \leftrightarrow \mathbb{A}'[U \setminus \{u\}]\mathbb{X}[\{u\}]$:



Definition 2.17 The *successor* U^+ of U is given by $U \cup \{*\}$. The *derivative* $\mathbb{A}'$ of $\mathbb{A}$ is the species with structures $\mathbb{A}'[U] = \mathbb{A}[U^+]$ considered as an S_n-set with $*$ fixed.

Definition 2.18 For any species $\mathbb{A}$, define $\mathbb{A}_n$ by $\mathbb{A}_n[U] = \mathbb{A}[U]$ when $|U| = n$ and $\mathbb{A}_n[U] = \phi$ otherwise. It follows that $\mathbb{A} = \sum_{n=0}^\infty \mathbb{A}_n$.

Cayley's formula: there are n^{n-2} labelled trees with n vertices (the proof uses $\ddot{\mathbb{T}} \cong \mathbb{L}^* \circ \dot{\mathbb{T}}$).