PhD Seminars 2002/2003: Prof. T. Porter Semester 2

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1 Seminar 1: 10th January 2003

1.1 Tensor Products

Let R be a commutative ring, let M, N and P be R-modules, and let $f : M \times N \to P$ be bilinear: $f(m,-)$, $N \to P$ is linear and $f(-,n)$, $M \to P$ is linear for each m, n. Why bother do this? Well, suppose we have $\varphi : M \to \mathrm{Hom}_R(N,P)$, where $\varphi(m) : N \to P$ and so $\varphi(m)(n) \in P$. Consider $\bar{\varphi} : M \times N \to P$, $\bar{\varphi}(m,n) = \varphi(m)(n)$. Then $\varphi \longleftrightarrow \bar{\varphi}$ and

$$\mathrm{Hom}(M, \mathrm{Hom}(N,P)) \cong \mathrm{Bilin}(M \times N, P) \tag{1}$$

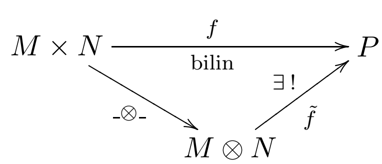
(Hom is the vector space of $n \times p$ matrices).

Remark 1.1 $\bar{\varphi}$ is not a homomorphism:

$$\begin{aligned} \bar{\varphi}((m,n) + (m',n')) &= \bar{\varphi}(m+m', n+n') \\ &= \bar{\varphi}(m, n+n') + \bar{\varphi}(m', n+n') \\ &= \bar{\varphi}(m,n) + \bar{\varphi}(m,n') + \bar{\varphi}(m',n) + \bar{\varphi}(m',n') \\ &\quad \text{(OOPS!)} \end{aligned}$$

Example 1.2 Let $R = \mathbb{R}$, $M = \mathbb{R}^m$, etc. In this case, the dimension of the LHS of (1) is mnp and the dimension of $\mathrm{Hom}(M \times N, P)$ is $(m+n)p$.

We now want to replace $M \times N$ by some new R-module $M \otimes N$ such that $\mathrm{Bilin}(M \times N, P) \cong \mathrm{Hom}(M \otimes N, P)$. Construction: Form a set X of symbols $x \otimes y$ for $x \in X$, $y \in N$; form a free R-module on X, $R^{(X)}$; and divide out by the three relations $(x+x') \otimes y = x \otimes y + x' \otimes y$, $x \otimes (y+y') = x \otimes y + x \otimes y'$ and $r(x \otimes y) = rx \otimes y = x \otimes ry$ to get $M \otimes N$. There is a natural universal bilinear map $M \times N \xrightarrow{-\otimes-} M \otimes N$, $(m,n) \mapsto m \otimes n$, so that we have the following ($\tilde{f}(m \otimes n) = f(m,n)$ is an R-mod morphism):



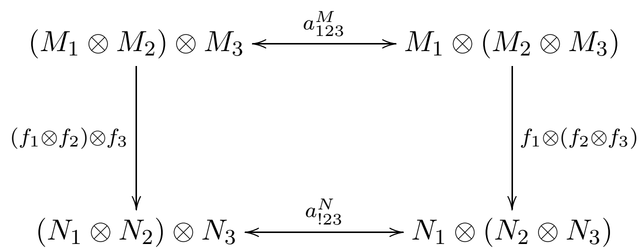
1.2 Examples

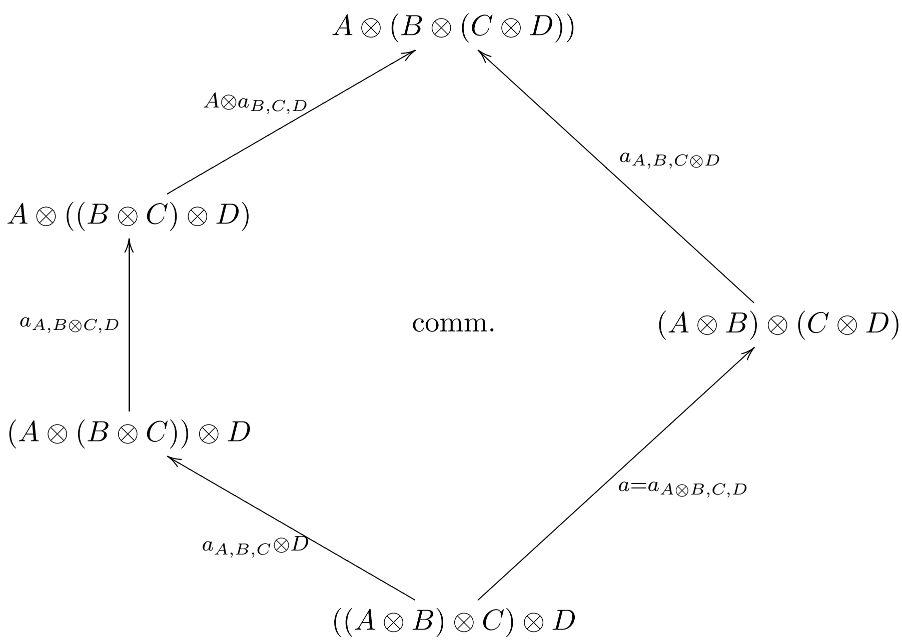
Example 1.3 If $R = \mathbb{R}$, then $\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}$ (check basis).

Example 1.4 Let $R = \mathbb{Z}$, and consider $A = C_{37} \otimes C_{59}$, where we have $C_{37} = < a \mid 37a = 0 >$ and $C_{59} = < b \mid 59b = 0 >$. A is generated by $a \otimes b$, but $a \otimes b = (8 \times 37 - 5 \times 59)(a \otimes b) = 8(37a \otimes b) - 5(a \otimes 59b) = 0 \otimes 0$, so that $A = \{0\}$.

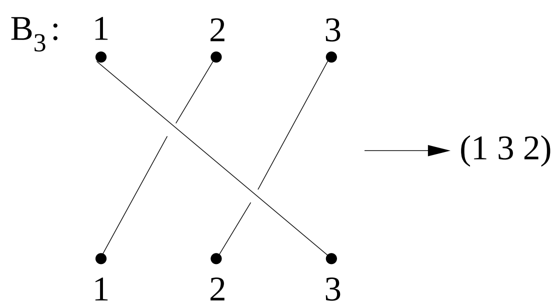
Example 1.5 $C_{10} \otimes C_{15} \cong C_5$ because the generator $a \otimes b$ has order dividing 5. In general, $C_r \otimes C_s \cong C_{\gcd(r,s)}$.

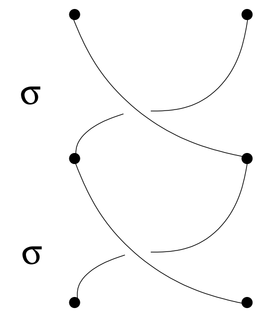
We have shown that $\mathbb{A} = R$-mod is a monoidal closed category: there is a functor $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ ("$M \times N \to M \otimes N$") so if we have $f : M \to M'$ and $g : N \to N'$ then $f \otimes g : M \otimes N \to M' \otimes N'$. What about $(M_1 \otimes M_2) \otimes M_3 \cong M_1 \otimes (M_2 \otimes M_3)$, where $\cong$ '=' $\overrightarrow{a_{123}}$, a natural isomorphism? Given $f_i : M_i \to N_i$, we have the following diagrams (including the important 'pentagon' diagram):





Remark 1.6 R-Mod is symmetric monoidal. Consider the natural isomorphism $s : M \otimes N \to N \otimes M$ ($s(m \otimes n) = n \otimes m$), where $s^2 = \mathrm{id}$. There is an action of S_n on $A_1 \otimes \ldots \otimes A_n$ given by permuting the factors. Also, we have braiding, i.e. the action of the braid group, which has a link with knot theory (below, we have $\sigma : M \otimes N \to N \otimes M$ with $\sigma^2 \neq \mathrm{id}$ in general).





Remark 1.7 Take $\mathcal{X}$ to be a crossed module $M \to P$, and take $C = \mathcal{C}(\mathcal{X})$ to be the Cat^1-group $C = P \ltimes M$. C is a category (groupoid). Write $(p, m) \otimes (p', m') = (p, m).(p', m') = (pp', m^{p'} m')$ (using group multiplication in C). C is a monoidal category!

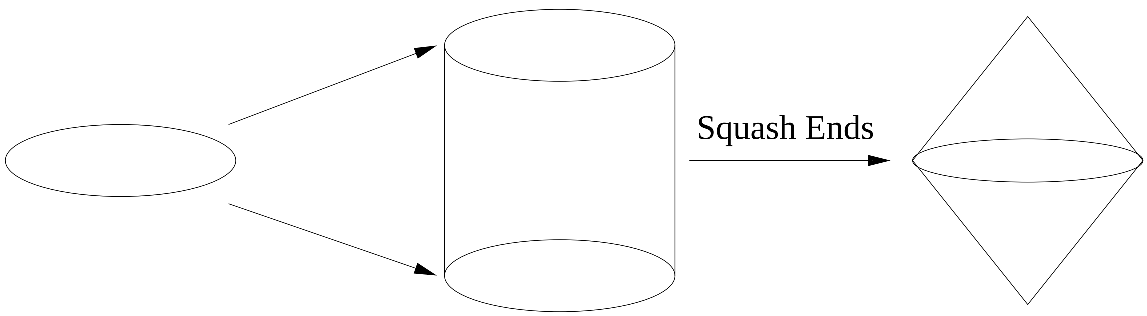
2 Seminar 2: 17th January 2003

2.1 Tying Up Loose Ends

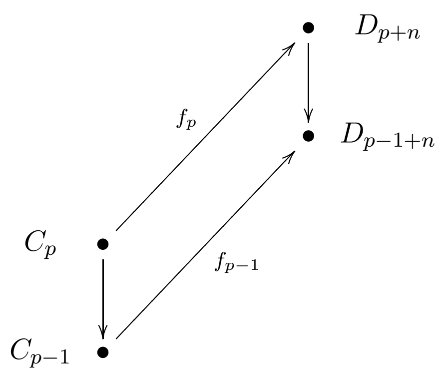
Recall the formulae $(I \otimes C)_n = C_n \oplus C_n \oplus C_{n-1}$ and $(D \otimes C)_n = \bigoplus_{p+q=n} D_p \otimes C_q$. Now $\partial(d \otimes c) = \partial^D d \otimes c + (-1)^{\deg(d)} d \otimes \partial^C c$ and

$$I:\ 0 \longleftarrow \underset{\substack{\text{dim. } 0 \\ \text{gen. } e_0^0,\ e_1^0}}{\mathbb{Z} \oplus \mathbb{Z}} \xleftarrow{\ \partial\ } \underset{\substack{\text{dim. } 1 \\ \text{gen. } e^1}}{\mathbb{Z}} \longleftarrow 0\ .$$

For a suspension, topologically we have the diagram shown below while algebraically we have $(\Sigma C)_n = C_{n-1}$ with $(-1)^D \partial_p$ as differential.



In the last seminar, we discussed $\mathrm{Hom}(K \otimes L, M) \cong \mathrm{Hom}(K, \mathrm{Hom}(L, M))$ in modules. Let us now consider $\mathcal{H}\mathrm{om}(C, D)_n =$ the module of degree n graded maps from C to $D = \prod_p \mathrm{Hom}(C_p, D_{p+n})$, with $(\partial_n^{\mathcal{H}}(f))_p = \partial^D f - (-1)^n f \partial^C$. The following diagram illustrates the case $\partial^D f_p - (-1)^n f_{p-1}\partial$:



For $\partial:\ \mathcal{H}\mathrm{om}(C, D)_n \to \mathcal{H}\mathrm{om}(C, D)_{n-1}$ we have

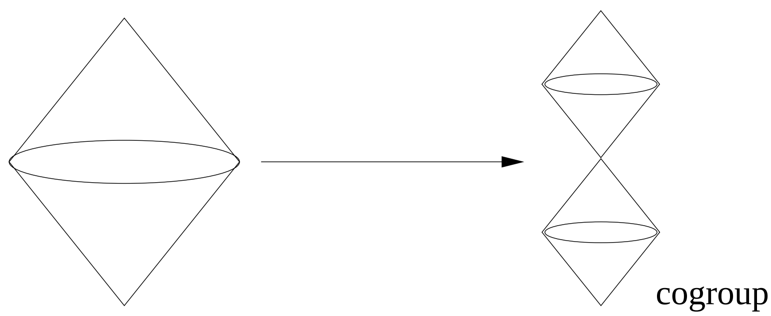
$$\begin{aligned}
\partial^{\mathcal{H}}\partial^{\mathcal{H}}(f) &= \partial^{\mathcal{H}}(\partial f - (-1)^n f \partial) \\
&= \partial^D\partial^D f - (-1)^n \partial^D f \partial^C - (-1)^{n-1}\partial^D f \partial^C - f\partial^C\partial^C \\
&= 0 \ \ (\text{because } \partial^? \partial^? = 0).
\end{aligned}$$

Lemma 2.1 $\mathcal{H}\text{om}(C \otimes D, \mathcal{E}) \cong \mathcal{H}\text{om}(C, \mathcal{H}\text{om}(D, \mathcal{E}))$. *Further,* $\mathcal{H}\text{om}(I, D) = D^I$ *and this is a co-cylinder.*

Now $(D^I)_n \simeq D_n \oplus D_n \oplus D_{n+1}$ and if ΩD consists of the loops in D then we have

$$\Omega D \longrightarrow D^I \underset{\longrightarrow}{\overset{\longrightarrow}{}} D.$$

Further, $\Omega D \times \Omega D \xrightarrow{\bullet} \Omega D$ is the restriction of $D^I \overset{\times}{\underset{D}{}} D^I \to D^I$ ($\overset{\times}{\underset{D}{}}$ is a pullback), and $(\Omega D, \bullet)$ 'is' a group object in Ch, where $\text{Ch}(C, \Omega D) \cong \text{Ch}(\Sigma C, D)$ (ΩD is a group up to homotopy and ΣC is a cogroup up to homotopy).



$$\Sigma C \longrightarrow \Sigma C \vee \Sigma C$$

$\text{Ch}(\Sigma C, D)$ is a group by the above structure, and the interchange law implies that this is just the bog-standard addition in this case — but that does not mean that this is useless!

2.2 Group Cohomology: Introduction

2.2.1 The Group Extension Problem

An extension of a group G by a group N is a short exact sequence

$$\mathcal{E}: \quad 1 \longrightarrow N \underset{\lhd}{\overset{i}{\longrightarrow}} E \overset{\pi}{\longrightarrow} G \longrightarrow 1$$

so that $G \cong E/N$. The group extension problem is to classify the possible E's (with corresponding i's and π's).

Given G and N, 'classify' means 'up to equivalence' in the sense that $\mathcal{E}$ and $\mathcal{E}$' are equivalent in the following diagram if there is a suitable isomorphism $\varphi: E \to E'$ compatible with subgroup and quotient ('(*) commutes'):

$$\begin{array}{ccccccccc}
\mathcal{E}: & 1 & \longrightarrow & N & \overset{i}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & G & \longrightarrow 1 \\
& & & \| & & \downarrow \varphi & & \| & \\
\mathcal{E}': & 1 & \longrightarrow & N & \underset{i'}{\longrightarrow} & E' & \underset{\pi'}{\longrightarrow} & G & \longrightarrow 1
\end{array}$$

Consider the case where N is abelian. We claim that N inherits a G-action. To prove this, let us first recall that E acts on N by conjugation: $i({}^e n) = ei(n)e^{-1}$. As $i(N)$ acts trivially on N, N therefore inherits an action of $E/i(N) = G$. Explicitly, pick a transversal or section for π, i.e. a function $s : G \to E$ such that $\pi_s = \mathrm{Id}_G$ ('$s(g)$ is a coset representative for g'), and then ${}^g n = s(g)ns(g)^{-1}$ and this is independent of the choice of s (if s' is another transversal, then $s'(g) = s(g)n'_g$ for some $n'_g \in N$, and so $s'(g)ns'(g)^{-1} = s(g)n'_g n {n'_g}^{-1} s(g)^{-1} = {}^g n$). We can now see that ${}^{(g_2 g_1)}n = {}^{g_2}({}^{g_1}n)$ is an action because $s(g_2 g_1) = s(g_2)s(g_1)n'$ for some $n' = n'_{g_1 g_2} \in N$.

Further, N is a G-module: ${}^g(n_1 n_2) = {}^g n_1 {}^g n_2$ or $(g.(n_1 + n_2) = gn_1 + gn_2)$. Also, $\mathcal{E}$ is *central* if $N \subseteq Z(E)$ and the action is then trivial. In any case we get a homomorphism $G \to \mathrm{Aut}(N)$.

3 Seminar 3: 31st January 2003

3.1 Split Extensions

Recall the diagram

$$\mathcal{E}: \quad 1 \longrightarrow N \underset{\triangleleft}{\overset{i}{\longrightarrow}} E \overset{\pi}{\longrightarrow} G \longrightarrow 1.$$

In general, it is not possible to pick $s : G \to E$ to be a homomorphism. But if it is, then $\mathcal{E}$ is said to be split, and it follows that $\mathcal{E} \cong N \rtimes G$.

3.2 Semidirect Products

If G and N are groups and we have a left action $\alpha : G \to \mathrm{Aut}(N)$, $\alpha(g)(n) = {}^g n$, the semidirect product $N \rtimes G$ (or more precisely $N \underset{\alpha}{\rtimes} G$) has elements $\{(n, g) \mid n \in N,\ g \in G\}$, i.e. the same as $N \times G$, but if α is non-trivial, $N \underset{\alpha}{\rtimes} G$ need not be 'the same' as $N \times G$. The multiplication is given by $(n_1, g_1)(n_2, g_2) = (n_1 {}^{g_1}n_2, g_1 g_2)$.

Lemma 3.1 *This is a group.*

Proof: Associativity: EER. Identity: $(1, 1)$. Inverses: we want $(a, b)(n, g) = (1, 1)$ and $(a^b n, bg) = (1, 1)$ so that $b = g^{-1}$ and $a^{g^{-1}} n = 1$; $a = {}^{g^{-1}} n^{-1}$, i.e. $(n, g)^{-1} = ({}^{g^{-1}} n^{-1}, g^{-1})$. □

Lemma 3.2

$$1 \longrightarrow N \overset{i}{\longrightarrow} N \rtimes G \underset{s}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 1$$

is a split extension, where $i_N(n) = (n, 1)$, $\pi_G(n, g) = g$, *and* $s_G(g) = (1, g)$.

Proposition 3.3 *Suppose $\mathcal{E}$ is an arbitrary split extension, then $\mathcal{E}$ is equivalent to a split extension of the above type.*

Proof: Consider the following diagram again:

$$1 \longrightarrow N \overset{i}{\longrightarrow} N \rtimes G \underset{s}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 1.$$

For $e \in G$, $e = e.s\pi(e)^{-1}.s\pi(e)$.

Let us define $\varphi : E \to N \rtimes G$ by $\varphi(e) = (\varphi_1(e), \varphi_2(e))$, where $i(\varphi_1(e)) = e(s\pi(e))^{-1} \in \ker \pi$ and $\varphi_2(e) = \pi(e)$. Claim:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \overset{i}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \| = & & \downarrow \varphi & & \| = & & \\
1 & \longrightarrow & N & \underset{i_N}{\longrightarrow} & N \rtimes G & \underset{\pi_G}{\longrightarrow} & G & \longrightarrow & 1
\end{array}$$

is compatible and φ is an isomorphism. Moreover φ is compatible with the splittings.

Now the right hand square is obvious, and for the left hand square, $\varphi i(n) = (\varphi_1(i(n), 1))$ since $\pi i(n) = 1_G$; and the element $\varphi_1(i(n))$ satisfies $i\varphi_1(i(n)) = i(n)(s\pi i(n))^{-1} = i(n)$ — so as i is a monomorphism, then $\varphi_1(i(n)) = n$ as required.

The inverse isomorphism of φ is $(n, g) \to i(n)s(g)$ $(N \rtimes G \to E)$. This is a homomorphism:

$$(n_1\, {}^{g_1}n_2, g_1g_2) \to i(n_1\, {}^{g_1}n_2)s(g_1g_2)$$

and

$$(n_1, g_1)(n_2, g_2) \to i(n_1)s(g_1)i(n_2)s(g_2) = i(n_1)s(g_1)i(n_2)s(g_1)^{-1}s(g_1g_2) = i(n_1)i({}^{g_1}n_2)s(g_1g_2)$$

by the definition of the action. □

3.3 Classifying Split Extensions

Each splitting s yields a φ. Fix one so as to rewrite the extension as

$$1 \longrightarrow N \longrightarrow N \rtimes G \longrightarrow G \longrightarrow 1$$

and consider an arbitrary splitting $s : G \to N \rtimes G$. This is a homomorphism and must take the form $s(g) = (d(g), g)$. Note: d is rarely a homomorphism but s is always a homomorphism: $s(g_1g_2) = s(g_1)s(g_2) = (d(g_1), g)(d(g_2), g_2) = (d(g_1))({}^{g_1}d(g_2)), g_1g_2)$, i.e. $d(g_1g_2) = d(g_1)^{g_1}d(g_2)$.

As N is abelian, written additively $d(g_1g_2) = d(g_1) + g_1.d(g_2)$ and d is a G-derivation. Now consider two (N abelian) splittings s_1 and s_2: they are N-conjugate if there is some $n \in N$ such that $s_1(g) = i(n)s_2(g)i(n)^{-1}$:

$$\begin{array}{ccc} \bullet & \xrightarrow{s_1(g)} & \bullet \\ {\scriptstyle i(n)}\downarrow & & \downarrow{\scriptstyle i(n)} \\ \bullet & \xrightarrow[s_2(g)]{} & \bullet \end{array}.$$

In this case the corresponding derivations $d_1, d_2 : G \to N$ have difference $(d_2 - d_1)(g) = gn - n = (g-1)n$; and the mapping $g \to gn - n$ is a derivation $g_1g_2 \to g_1g_2n - n = g_1g_2n - g_1n + g_1n - n = g_1n - n + g_1(g_2n - n)$ called a principal derivation.

Lemma 3.4 *The derivations $d : G \to N$ form an abelian group* $\mathrm{Der}(G, N)$ *where $(d_1 + d_2)g = d_1g + d_2g$. The principal derivations form a subgroup $P(G, N)$ of* $\mathrm{Der}(G, N)$.

Proposition 3.5 *N-conjugacy classes of splittings of $\mathcal{E}$ are in $1-1$ correspondence with the elements of* $\mathrm{Der}(G, N)/P(G, N)$.

Remark 3.6 This group is $H^1(G, N)$ as we'll see later.

4 Seminar 4: 14th February 2003

4.1 Classification of Extensions with N Abelian

Consider that we are given G, that N is a G-module, and that $\mathcal{E}: \ 1 \to N \to E \xrightarrow{i} G \xrightarrow{\pi} 1$ is an extension. We choose a transversal $s: \ G \to E$ (but not necessarily a splitting) so that if $g_1, g_2 \in G$ then we have $s(g_1 g_2) \neq s(g_1)s(g_2)$ in general (but it can be true in some cases). However, $\pi(s(g_1 g_2)) = \pi(s(g_1))\pi(s(g_2))$ so that $s(g_1 g_2)s(g_2)^{-1}s(g_1)^{-1} \in i(N)$.

Now define $f: \ G \times G \to N$ by $i(f(g_1, g_2)) = s(g_1)s(g_2)s(g_1, g_2)^{-1}$, where f is the factor set of the extension relative to s. If we choose $s(1_G) = 1_N$ then $f(1, g) = 0 = f(g, 1)$ and we say that s is normalised.

How can we build E from G, N, etc.? We know that as a set $E \leftrightarrow N \times G$ and we have a map $N \times G \to E$, $(n, g) \mapsto i(n)s(g)$. Further, $(n_1, g_1)(n_2, g_2) = i(n_1)s(g_1)i(n_2)s(g_2)$. Thinks: ${}^{g_1}n_2$ is defined by $i({}^{g_1}n_2) = s(g_1)i(n_2)s(g_1)^{-1}$. Therefore,

$$\begin{aligned}
(n_1, g_1)(n_2, g_2) &= i(n_1)s(g_1)i(n_2)s(g_2) \\
&= i(n_1)i({}^{g_1}n_2)s(g_1)s(g_2) \\
&= i(n_1{}^{g_1}n_2)if(g_1, g_2)s(g_1, g_2) \\
&\leftrightarrow (n_1 + g_1.n_2 + f(g_1, g_2), \ g_1 g_2),
\end{aligned}$$

i.e. we can retrieve the multiplication in E from G, N and a factor set.

An arbitrary $f: G \times G \to N$ may not give a group multiplication as the above may not be associative:

$$\begin{aligned}
((n_1, g_1)(n_2, g_2))(n_3, g_3) &= (n_1 + g_1.n_2 + f(g_1, g_2), \ g_1 g_2)(n_3, g_3) \\
&= (n_1 + g_1.n_2 + f(g_1, g_2) + g_1 g_2.n_3 + f(g_1 g_2, g_3), \ g_1 g_2 g_3) \\
&= (n_1 + g_1.n_2 + g_1 g_2.n_3 + f(g_1, g_2) + f(g_1 g_2, g_3), \ g_1 g_2 g_3)
\end{aligned}$$

whilst

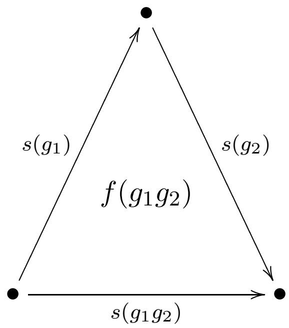
$$\begin{aligned}
(n_1, g_1)((n_2, g_2)(n_3, g_3)) &= (n_1, g_1)(n_2 + g_2 n_3 + f(g_2, g_3), \ g_2 g_3) \\
&= (n_1 + g_2.n_3 + g_1 g_2.n_3 + g_1 f(g_2, g_3) + f(g_1, g_2 g_3), \ g_1 g_2 g_3).
\end{aligned}$$

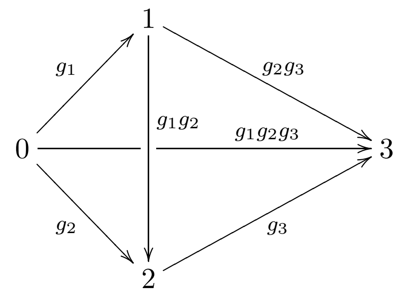
So associativity corresponds to the condition

$$f(g_1, g_2) + f(g_1 g_2, g_3) = g_1 f(g_2, g_3) + f(g_1, g_2 g_3),$$

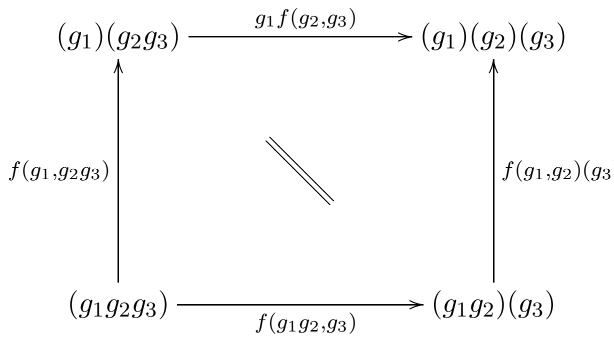
the cocycle condition.

Note that f and the cocycle condition may be visualised as follows:





The paths from 0 to 3 can be represented as follows:



Now suppose that $s' : G \to E$ is another (normalised) transversal, with $s'(g) = (c(g), g)$ giving $f' : G \times G \to N$. We have

$$\begin{aligned} i(f'(g_1, g_2)) &= s'(g_1)s'(g_2)s'(g_1g_2)^{-1}, \\ s'(g_1g_2) &\leftrightarrow i(c(g_1g_2))s(g_1g_2), \\ s'(g_1) &\leftrightarrow i(c(g_1))s(g_1), \text{ etc.} \end{aligned}$$

Note: $if(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$ so that $s(g_1)s(g_2) = if(g_1, g_2)s(g_1, g_2)$ and similarly for the dashed version.

Now

$$\begin{aligned} s'(g_1)s'(g_2) &= (if'(g_1, g_2)ic(g_1g_2))s(g_1g_2) \text{ and} \\ s'(g_1)s'(g_2) &= ic(g_1)s(g_1)i(c(g_2))s(g_2) \\ &= ic(g_1)i({}^{g_1}c(g_2))s(g_1)s(g_2) \\ &= i(c(g_1)^{g_1}c(g_2)f(g_1, g_2))s(g_1g_2). \end{aligned}$$

So $f'(g_1, g_2)c(g_1g_2) = c(g_1)^{g_1}c(g_2)f(g_1, g_2)$ or $f'(g_1, g_2) - f(g_1, g_2) = c(g_1) + g_1.c(g_2) - c(g_1g_2)$ or $f' - f = \delta c$, where $\delta c(x, y) = c(x) - c(xy) + xc(y)$. It follows that if this is satisfied then the extensions are isomorphic.

Definition 4.1 $f : G \times G \to N$ is a cocycle if it satisfies the cocycle condition and the set of cocycles form an abelian group $Z^2(G, N)$.

Lemma 4.2 *Given any* $c: G \to N$*, we have* $\delta c \in Z^2(G, N)$.

Definition 4.3 If $B^2(G, N) = \{\delta c \mid c: G \to N\} < Z^2(G, N)$, then

$$H^2(G, N) = \frac{Z^2(G, N)}{B^2(G, N)},$$

the second cohomology group of G with coefficients in N. Further, we have Equivalence Classes of Extensions $\leftrightarrow H^2(G, N)$.

5 Seminar 5: 21st February 2003

5.1 Standard Resolution

Let G be a group and A a $\mathbb{Z}(G)$-module. We define $C^n(G,A) = \{f : G^n \to A\}$, an abelian group. We also define $dC^n(G,A) \to C^{n+1}(G,A)$ as follows:

$$df(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1}) + \left(\sum_{i=1}^{n}(-1)^i f(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1})\right) + (-1)^{n+1} f(g_1,\ldots,g_n).$$

The above formula is (claimed to be) modelled on the cocycle condition, etc. Now $f \in C^2(G,A)$, $f : G \times G \to A$ is potentially a factor set. With $df(g_1,g_2,g_3) = g_1 f(g_2,g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1,g_2)$, f is a cocycle iff $df = 0$. For $n = 1$, we have $c : G \to A$, $dc(g_1,g_2) = g_1 c(g_2) - c(g_1,g_2) + c(g_1)$. In general, $f \in C^n(G,A)$ is an n-cocycle if $df = 0$ and $Z^n(G,A)$ is the abelian group of such n-cocycles. Further, if $c \in C^{n-1}(G,A)$, then $dc \in C^n(G,A)$ is called a coboundary and $B^n(G,A) = \mathrm{Im}(d : C^{n-1}(G,A) \to C^n(G,A))$.

Lemma 5.1 $dd = 0$ *(Proof left as exercise).*

Corollary 5.2 $B^n(G,A) \subseteq Z^n(G,A)$.

Definition 5.3 $H^n(G,A) = Z^n(G,A)/B^n(G,A)$.

5.1.1 Low Dimensions

Consider the case $n = 0$. We have $B^0(G,A) = 0$ and $C^0(G,A) \cong A$. If we define $f_a : \{1\} \to A$, then $df_a : G \to A$ is defined by $df_a(g) = gf_a - f_a$. Now

$$\begin{aligned} f_a \in Z^0(G,A) &\iff gf_a - f_a = f_0 \;\forall\, g \in G \\ &\iff g.a - a = 0 \;\forall\, g \in G \\ &\iff a \in A^G = \{b \mid gb = b \;\forall\, g \in G\}. \end{aligned}$$

It follows that $H^0(G,A) \cong A^G$.

For the case $n = 1$, what should $H^1(G,A)$ be? Well, $C^1(G,A) = \{f : G \to A\}$ and $B^1(G,A) = \{df_a \mid f_a \in C^0(G,A)\}$, with $(df_a)(g) = (g-1)a$. Now recall that $\partial : G \to A$ is a principal derivation if $\partial f = ga - a$ for some $a \in A$. So if $f : G \to A$, $df : G \times G \to A$, then $(df)(g_1,g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$, and therefore $df = 0 \iff f(g_1 g_2) = g_1 f(g_2) + f(g_1) \iff f$ is a derivation, i.e. $Z^1(G,A) = \mathrm{Der}(G,A)$ and $H^1(G,A)$ classifies split extensions.

$H^2(G,A)$ is as before, i.e. the group of equivalence classes of extensions of G by A. $H^3(G,A)$ was described by MacLane in terms of 'abstract kernels', but later by Huebschmann in terms that generalised to all n. Then about the same time, similar results were given by Conrad, Holt and in retrospect earlier by Lue.

5.2 BG, The Bar Resolution

The idea here is to replace 'functions' by module homomorphisms using the natural isomorphism

$$\mathrm{Sets}(X, U(A)) \cong \mathbb{Z}(G)\mathrm{-Mod}(\mathbb{Z}G^{(X)}, A),$$

where $\mathbb{Z}G^{(X)}$ is the free G-module on X.

Let $X_n = \{(g_0, \dots, g_n) \mid g_i \in G\} \cong G^{n+1}$. Consider that we are given a G-action on X_n by left multiplication $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$, and let $\mathbb{F}_n$ be the free G-module on this G-set (a typical element of $\mathbb{F}_n$ is $\sum_{\mathbf{x}\in X} n_{\mathbf{x}}\mathbf{x}$, with $n_{\mathbf{x}} \in \mathbb{Z}G$ having finite support and $g.\sum n_{\mathbf{x}}\mathbf{x} = \sum n_{\mathbf{x}}(g.\mathbf{x})$). Now define $d_i : \mathbb{F}_n \to \mathbb{F}_{n-1}$ by $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$ and $\partial = \sum(-1)^i d_i$. We have $\mathbb{F}_0 \cong \mathbb{Z}G$ and we set $\mathbb{F}_{-1} = \mathbb{Z}$ with trivial action.

Claim: The elements $(1, g_1, g_1g_2, \dots, g_1 \dots g_n)$ form a basis for $\mathbb{F}_n$. To see this, write $(\bar{g}_0, \bar{g}_1, \dots, \bar{g}_n) = \bar{g}_0(1, \bar{g}_0^{-1}\bar{g}_1, \bar{g}_0\bar{g}_2, \dots, \bar{g}_0\bar{g}_n)$ and record the 'incremental' g's in the notation $[\, g_1 \mid g_2 \mid \dots \mid g_n \,]$ to get

$$d_i[\, g_1 \mid \dots \mid g_n \,] = \begin{cases} g_1[\, g_2 \mid \dots \mid g_n \,] & (i = 0) \\ [\, g_1 \mid \dots \mid g_{i-1} \mid g_i g_{i+1} \mid g_{i+2} \mid \dots \mid g_n \,] & \\ [\, g_1 \mid \dots \mid g_{n-1} \,] & (i = n) \end{cases}.$$

It follows that $C^n(G, A) \cong \mathrm{Hom}(\mathbb{F}_n, A)$.

Remark 5.4 $g_i = (\bar{g}_{i-1})^{-1}\bar{g}_i$.

6 Seminar 6: 26th February 2003

6.1 Bar Resolution, Part 2

Recall that we are considering BG, the unnormalised bar resolution; $\mathbb{F}_n$ is generated by $[\,g_1 \,|\, \ldots \,|\, g_n\,]$ as a free G-module; and we have the following sequence and definitions:

$$\ldots \mathbb{F}_2 \xrightarrow{\partial_2} \mathbb{F}_1 \xrightarrow{\partial_1} \mathbb{F}_0 = \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z};$$
$$\partial_2[\,g_1 \,|\, g_2\,] = g_1[\,g_2\,] - [\,g_1 g_2\,] + [\,g_1\,];$$
$$\partial_1[\,g\,] = g[\ \ ] - [\ \ ] \longleftrightarrow g - 1.$$

Let us introduce the notation $\varepsilon(\sum n_g e_g) = \sum n_g$. $IG \sim$ augmentation ideal $= \ker \varepsilon$. Resolution: $H_n(BG) = 0$ if $n > 0$.

6.2 Contracting Homotopy

In general, C is a chain complex ($\partial\partial = 0$), D is another one, and we have

$$C \underset{f_1}{\overset{f_0}{\rightrightarrows}} D.$$

Further, $h : f_0 \simeq f_1$ ($h : I \otimes C \to D$, $he_0 = f_0$, $he_1 = f_1$) is equivalent to a chain homotopy $\{h'_n : C_n \to D_{n+1}\}$ such that $f_0 - f_1 = \partial h' + h'\partial$. Now

$$C \text{ is contractible} \quad \equiv \quad C \underset{0}{\overset{\mathrm{Id}}{\rightrightarrows}} C \text{ are homotopic } (\Downarrow h')$$
$$\implies \quad H_n(C) = 0 \ \ \forall\, n.$$

Now suppose that $c \in Z_n(C) = \ker \partial_n$, $c = \partial h'(c) + h'\partial c \Rightarrow c \in B_N(c)$, where h' is the contracting homotopy (and $h'\partial c = 0$). Let $h : BG_n \to BG_{n+1}$, $\mathbb{F}_n \to \mathbb{F}_{n+1}$. The following is $\mathbb{Z}$ linear but not $\mathbb{Z}G$ linear:

$$h(g_0, \ldots, g_n) = (1, g_0, \ldots, g_n).$$

Exercise 6.1 What is $h(g[\,g_1 \,|\, \ldots \,|\, g_n\,])$? Answer: $[\,g_1 \,|\, g_2 \,|\, \ldots \,|\, g_n\,] \longleftrightarrow (1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_n)$ so we must have

$$\begin{array}{ccc}
g[\,g_1 \,|\, g_2 \,|\, \ldots \,|\, g_n\,] & \longleftrightarrow & (g, gg_1, \ldots, gg_1 g_2 \ldots g_n) \\
\downarrow h & & \downarrow h \\
[\,g \,|\, g_1 \,|\, g_2 \,|\, \ldots \,|\, g_n\,] & \longleftarrow & (1, g, gg_1, \ldots, gg_1 \ldots g_n)
\end{array}$$

6.3 Simplicial Resolutions

Consider

$$\text{Sets} \underset{U}{\overset{F}{\rightleftarrows}} \text{Groups},$$

where F is the free group functor and U is the underlying set functor.

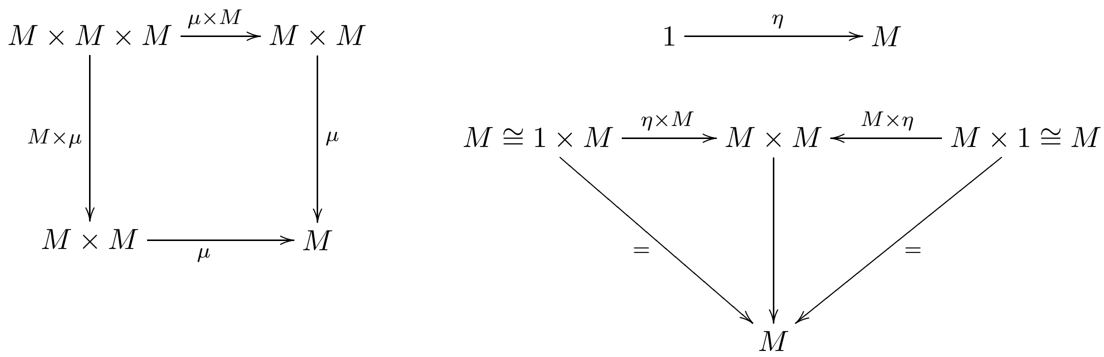
- $\text{Sets}(X, U(G)) \xrightarrow{\cong} \text{Groups}(F(X), G)$
  (F is left adjoint to U).
- $\text{Set}(U(G), U(G)) \cong \text{Groups}(FU(G), G)$.
- $\varepsilon_G : FU(G) \to G$ is a homomorphism that takes off the 'cling film'.
- $\text{Set}(X, UF(X)) \cong \text{Groups}(FX, FX)$.
- $\eta_X : X \to UF(X)$ is an inclusion of generators.
- All of the above leads to the following diagram:

$$\begin{array}{ccc} FUF(X) & \xrightarrow{\varepsilon_{F(X)}} & F(X) \\ {\scriptstyle F\eta_X}\uparrow & \nearrow_{=} & \\ F(X) & & \end{array}$$

Let us now define $\mu(X) : UFUF(X) \overset{U\varepsilon_{F(X)}}{\longrightarrow} UF(X)$. For simplicity, let $T = UF$ so that we have a unit $\eta : I \to T$ and a multiplication $\mu : T^2 \to T$ (T : Sets $\to$ Sets). It follows that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle =} & \\ & & T & & \end{array}$$

Remark 6.2 The above discusses the Monad generated by the (F, U)-adjunction. We can compare this to the situation where we consider monoids, where we have $\mu : M \times M \to M$, $\mu(m, m') = m.m'$. In the diagrams below, the one on the left represents the associative law while the first one on the right picks out the 1_M ($1 =$ 'one element set'):

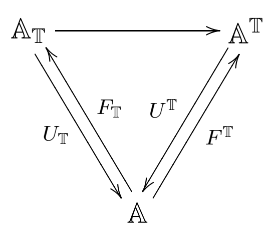


6.4 The Kleisli and Eilenberg-Moore Categories (an aside)

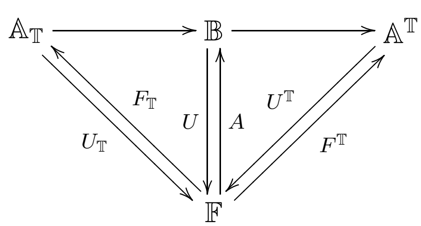
Given any adjoints

$$\mathbb{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{B},$$

we obtain a monad ($T = UF$, $\mu = U\varepsilon_F$, η). Conversely, if we are given a monad $\mathbb{T} = (T, \mu, \eta)$ with $T : \mathbb{A} \to \mathbb{A}$, there are *two* categories to consider, both of which generate $\mathbb{T}$. The first one is the Kleisli category of $\mathbb{T}$, $\mathbb{A}_\mathbb{T}$; and the second one is the Eilenberg-Moore category of $\mathbb{T}$, $\mathbb{A}^\mathbb{T}$. The situation is summarised by the following diagram:



Starting with $\mathbb{A} \rightleftarrows \mathbb{B}$, we get a comparison with the following diagram:



Remark 6.3 In the group case, the Kleisli category consists of all free groups, and the Eilenberg-Moore category consists of all groups.

Question: What is $\mathbb{A}_\mathbb{T}$? *Answer:* $\mathbb{A}_\mathbb{T}$ is the objects of $\mathbb{A}$, where $\mathbb{A}_\mathbb{T}(X, Y) = \mathbb{A}(X, TY)$. It follows that given $X \xrightarrow{f} TY$ and $Y \xrightarrow{g} TZ$, we have

$$X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu Z} TZ.$$

Remark 6.4 For $T = UF : \text{Sets} \to \text{Sets}$, $\text{Sets}_{\mathbb{T}} \cong$ the Category of Free Groups.

Exercise 6.5 Let M be a monoid. Assume that we have

$$M\text{--Sets} \underset{U}{\overset{F}{\rightleftarrows}} \text{Sets},$$

$(a : M \times X \to X) = (X, a)$, $U(X, a) = X$, and $F(X) = (M \times X, \mu \times X)$. Our object in this exercise is to prove the following isomorphism, which we shall call (*):

$$\text{Sets}(X, U(Y, a)) \cong M\text{--Sets}(F(X), (Y, a)).$$

Now we know the definitions of f and $\bar{f}$: $f : X \to U(Y, a) = Y$ and $\bar{f} : (M \times X, \mu \times X) \to (Y, a)$; and we also have the following diagram:

$$\begin{array}{ccc} M \times M \times X & \xrightarrow{M \times \bar{f}} & M \times Y \\ \Big\downarrow{\scriptstyle \mu \times X} & & \Big\downarrow{\scriptstyle a} \\ M \times X & \xrightarrow[\bar{f}]{} & Y \end{array}$$

We want $\bar{f}(\mu \times X) = a(M \times \bar{f})$. The natural thing to do is to try $\bar{f}(m, x) = a(m, f(x)) = m.f(x)$ so that $\bar{f}(m, x) = \bar{f}(m.(1, x))$ and therefore $\bar{f}(1, x) = f(x)$. We proceed now by first proving that (*) is an isomorphism (given ε and η), then describing $T = UF : \text{Sets} \to \text{Sets}$, and finally describing $\text{Sets}_{\mathbb{T}}$.

7 Seminar 7: 28th February 2003

7.1 Simplicial Resolutions, Part 2

Let us define $T^\bullet(X) = T^{n+1}(X)$. We have the following diagram, representing a cosimplicial set (this is for the reader to check), where T, η and μ are as in the previous seminar (before the aside):

$$\begin{array}{ccccccc}
\ldots T^3(X) & \underset{\overset{\longleftarrow}{T\eta T,\ T^2\eta}}{\overset{\mu T,\ T\mu}{\rightrightarrows}}\ \overset{\eta T^2}{\longleftarrow} & T^2(X) & \underset{\overset{\longleftarrow}{T\eta}}{\overset{\mu}{\longrightarrow}}\ \overset{\eta T}{\longleftarrow} & T(X) & & \\
2 & & 1 & & 0 & & \text{Dimension}
\end{array}$$

Let us now dualise by reversing the roles of F and U so that we obtain $T = FU$ (Groups to Groups), a counit $\varepsilon : T \to 1$, and a comultiplication $m : T \to T^2$, $FU \xrightarrow{F\eta U} FUFU$. To see what m does, let us set up some temporary notation.

If $g \in G$, write $(g) = g$ as a generator of $T(G)$ and also as the single letter word in $T(G)$. If $\omega \in T(G)$ then $\omega = (g_1)^{\varepsilon_1} \ldots (g_n)^{\varepsilon_n}$, $\varepsilon_i = \pm 1$. *Question:* What is $m(G)(\omega)$? *Answer:* We have $\eta(X) : X \to UF(X)$, $x \mapsto (x)$, $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \mapsto (x_1)^{\varepsilon_1} \ldots (x_n)^{\varepsilon_n}$, so $m(G)(\omega) = ((g_1))^{\varepsilon_1} \ldots ((g_n))^{\varepsilon_n}$, and therefore m is a comultiplication for the comonad. We also have the associative law, visualised as follows:

$$\begin{array}{ccc}
T & \xrightarrow{m} & T^2 \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mT} \\
T^2 & \xrightarrow[Tm]{} & T^3
\end{array}$$

Now if G is a group and we define $T_n(G) = T^{n+1}(G)$, we have the following augmented simplicial set:

$$\cdots \underset{\longleftarrow}{\longrightarrow} T^3(G) \underset{\overset{\longleftarrow}{mT,\ Tm}}{\overset{\varepsilon T^2,\ T\varepsilon T,\ T^2\varepsilon}{\rightrightarrows}} T^2(G) \underset{\overset{\longleftarrow}{m}}{\overset{\varepsilon T,\ T\varepsilon}{\rightrightarrows}} T(G) \xrightarrow{\varepsilon} G$$

This has a free group in all dimensions and has a trivial homotopy above dimension 0 ($\pi_o(T_\bullet(G)) \cong G$). Further, $UT_\bullet(G)$ is a contractible simplicial set:

$$\begin{array}{ccccc}
UT^2(G) & \leftrightarrows & UT(G) & \underset{s_{-1} = \eta U(G)}{\leftrightarrows} & U(G) \\
\| & & \| & & \\
UFUFU(G) & \underset{\eta UFU(G) = s_{-1}}{\overset{UF\eta U}{\leftleftarrows}} & UFU & &
\end{array}$$

Remark 7.1 The above is closely related to transversals/splittings in the extension theory discussed earlier. Also, $\eta UFU(G) = s_{-1}$ in the above diagram acts like an additional degeneracy and gives a contracting homotopy for $(UT_\bullet(G))$, but notice that s_{-1} is *not* a homomorphism.

7.2 Cochain Complexes

Let A be an abelian group. Then Groups$(T_\bullet(G), A)$ is a cosimplicial abelian group and so yields a cochain complex $C^\bullet(G, A)$ with the following diagram ($d^k = \text{Groups}(d_k^{(T_\bullet(G))}, A)$ and the cohomology is the cohomology of G with coefficients in A):

$$\begin{array}{ccc}
C^n(G,A) & \cong & \text{Group}(T^{n+1}(G),A) \\
\Big\downarrow{\scriptstyle \partial^n} & & \Big\downarrow{\scriptstyle \sum(-1)^k d^k} \\
C^{n+1}(G,A) & \cong & \text{Group}(T^{n+2}(G),A)
\end{array}$$

General references for this material include the monograph by J. Duskin in the AMS Memoirs and 'The Triple Seminar' in the Springer LNS, about 1967. Further, from work by André and Quillen, we can build a simplicial resolution step-by-step, e.g. start with a presentation $< X \mid R >$ for G and obtain the following (for the (*) see papers by A. Mutlu and T. Porter):

$$(*)\ \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} F(R \sqcup s_0(X)) \begin{array}{c}\xrightarrow{d_1}\\ \xrightarrow{d_0}\\ \xleftarrow{s_0}\end{array} F(X) \longrightarrow G.$$

7.3 Crossed Resolutions

Definition 7.2 A *crossed complex* C of groupoids is a sequence of morphisms of groupoids over C_0 (a single vertical arrow implies source = target):

$$\begin{array}{ccccccccc}
\longrightarrow C_n & \xrightarrow{\delta_n} & \cdots \longrightarrow & C_3 & \xrightarrow{\delta_3} & C_2 & \xrightarrow{\delta_2} & C_1 \\
\Big\downarrow{\scriptstyle \beta} & & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta} & & {\scriptstyle \delta^1}\Big\downarrow\Big\downarrow{\scriptstyle \delta^0} \\
C_0 & & \cdots & C_0 & & C_0 & & C_0
\end{array}$$

Here $\{C_n\}_{n \geqslant 2}$ is a sequence of families of groups with indexing map β so that $C_n = \{C_n(p) \mid p \in C_0\}$, where $C_n(p) = \beta^{-1}(p)$ and δ^0 and δ^1 are the source and target maps for the groupoid C_1.

There is an action of C_1 on the C_n for $n \geqslant 2$ (i.e. if $p \xrightarrow{a}$ in C_1 and $x \in C_n(p)$ then $x^a \in C_n(q)$) such that

1. Each δ_n is a morphism of groupoids over C_0;
2. $C_2 \to C_1$ is a crossed module;

3. C_n (for $n \geqslant 3$) is a G-module (so each $C_n(p)$ is abelian);

4. $\delta :\ C_n \to C_{n-1}$ for $n \geqslant 3$ is an operator morphism;

5. $\delta\delta :\ C_n \to C_{n-2}$ is trivial for $n \geqslant 3$; and

6. $\delta_2 C_2$ acts trivially on all C_n for $n \geqslant 3$.

8 Seminar 8: 7th March 2003

8.1 Examples of Crossed Complexes

Example 8.1 Let $G_\bullet$ be a simplicial groupoid, let $NG_\bullet$ be a Moore complex and let D_n be the subgroup of G_n generated by the degenerate elements.

Fact: $NG_\bullet$ is a crossed complex (of groupoids) $\iff$ for each $n \geqslant 1$, $NG_n \cap D_n$ consists just of identities. Further, given any $G_\bullet$, define

$$C(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}$$

for $n \geqslant 0$ and define $C(G)_{n+1} \xrightarrow{\partial} C(G)_n$ induced by d_0 — it follows that $(C(G), \partial)$ is then a crossed complex.

Definition 8.2 Given a group G and a crossed complex of groups C augmented over G,

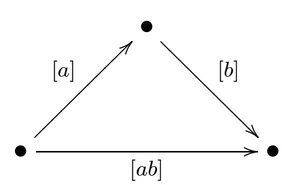
$$C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \overset{\varphi}{\dashrightarrow} G$$

($\varphi\partial$ = trivial), $((C, \partial), \varphi)$ is a crossed resolution of G if the sequence (above) is exact. Further, it is a free crossed resolution if (C, ∂) is a free crossed complex, i.e. C_1 is a free group, $C_2 \to C_1$ is a free crossed module, and all C_n for $n > 2$ are free $\mathbb{Z}G$-modules.

Example 8.3 Take $G = \langle X : R \rangle = \mathcal{P}$, $C_1 = F(X)$, $C_2 = C(\mathcal{P}) \xrightarrow{\partial} F(X)$, and let $\pi = \ker(\partial : C(\mathcal{P}) \to F(X))$ be the module of identities (a G-module). Then the following is a free resolution of π by G-modules:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C_4 & \longrightarrow & C_3 & & \longrightarrow & & C_2 & \xrightarrow{\partial} & C_1 \longrightarrow G. \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & \pi & & & & \end{array}$$

Example 8.4 (The Standard Crossed Resolution, $F^{\mathrm{st}}(G)$). Let $F_1^{\mathrm{st}}(G)$ be the free group on $[a]$ for $a \in G$ and let $F_2^{\mathrm{st}}(G)$ be the free $F_1^{\mathrm{st}}(G)$-module on $\omega : G \times G \to F_1^{\mathrm{st}}(G)$ defined by $\omega(a,b) = [a][b][ab]^{-1}$ (go round the diagram clockwise):



For $n \geqslant 3$, take $F_n^{\mathrm{st}}(G)$ to be the free G-module on G^n with $\partial_3[a,b,c] = [a,bc][ab,c]^{-1}[a,b][b,c]^{[a]^{-1}}$ (right action, with $[\ldots]$ not referring to commutators here). If $n \geqslant 4$, we define $\partial_n[a_1, a_2, \ldots, a_n]$

$$= [a_2, \ldots, a_n]^{a_1^{-1}} + \left(\sum_{i=1}^{n-1} (-1)^i [a_1, a_2, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n] \right) + (-1)^n [a_1, a_2, \ldots, a_{n-1}].$$

We now have the following situation:

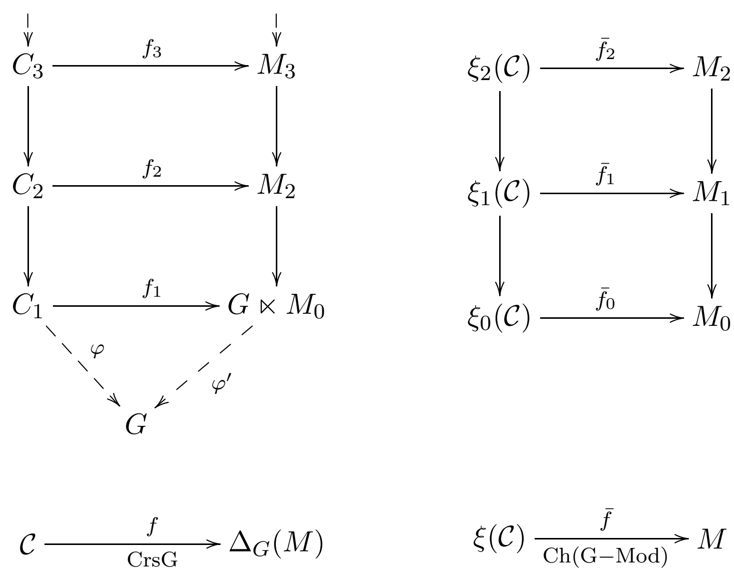
(free)		(free)		(free)
Simplicial Resolutions	$\xrightarrow[?]{C}$	Crossed Resolutions	$\dashrightarrow_{?}$	(Chain) Resolutions
Bar		*Standard*		*Bar*

Let us first look at CrsComp (CrsG) $\to$ Chain Complexes.

Proposition 8.5 *There is a functor Δ : Ch(G-Mod) $\to$ CrsG given by: if $M = \ldots \to M_n \to \ldots \to M_2 \to M_1 \to M_0$ in Ch(G-Mod), then $\Delta_G(M)_1 = G \ltimes M_0$ and $\Delta_G(M)_n = M_{n-1}$ for $n \geqslant 2$. Further, $\varphi : \Delta_G(M)_1 \to G$ is a projection, $\varphi(g,m) = g$; and we have maps $d_n^{\Delta_G(M)} = d_{n-1}$ for $n \geqslant 3$, $d_2^{\Delta_G(M)} : M_1 \to G \ltimes M_0$, and $d_2^{\Delta_G(M)}(m) = (1, d_1(m))$ (the details are left to the reader — specify the actions).*

Proposition 8.6 *Δ_G has a left adjoint.*

Proof: The input data (what we've got) is shown below on the left, while the output data (what we want) is shown below on the right:

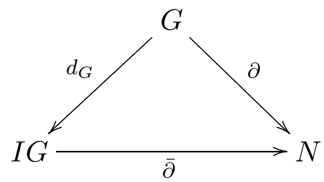


First, let us examine $f_1(c) = (\varphi(c), \partial_\varphi(c))$, where $\partial_\varphi :\ C_1 \to M_0$ is not a homomorphism. *Question:* What is $\partial_\varphi(c, c')$? Well, $f_1(c, c') = (\varphi(cc'), \partial_\varphi(c, c'))$ and $f_1(c)f_1(c') = (\varphi(c), \partial_\varphi(c))(\varphi(c'), \partial_\varphi(c')) = (\varphi(cc'), \partial_\varphi(c).\varphi(c') + \partial_\varphi(c'))$, a φ-derivation (the meaning of which we shall now investigate). $\square$

8.2 φ-Derivations

Consider $G \xrightarrow{\varphi} H$ and let M be a right H-module. A function $\partial : G \to M$ is a φ-derivation if $\partial(g_1, g_2) = \partial(g_1)\varphi(g_2) + \partial(g_2)$ (Example: $G = H$ and $\varphi = \mathrm{Id}_G$ gives an ordinary derivation).

Let us now take $M = I(G)$ to be the augmentation ideal $\ker(\mathbb{Z}(G) \xrightarrow{\text{aug}} \mathbb{Z})$, and let $d_G : G \to I(G)$ be defined by $d_G(g) = g - 1$. Suppose $\partial : G \to N$ is a G-derivation, and let us define $\bar{\partial} : IG \to N$ by $\bar{\partial}(g-1) = \partial(g)$, a G-homomorphism. Now $\{g - 1 \,|\, g \in G\}$ forms a basis for IG as a $\mathbb{Z}$-module because $(g-1)g' = gg' - g' = (gg'-1) - (g'-1)$ and $\bar{\partial}((g-1)g') = \partial(gg') - \partial(g') = \partial(g)g' + \partial(g') - \partial(g') = \partial(g)g'$. Therefore,



commutes and d_G is the universal solution to 'homomorphising' G-derivations.

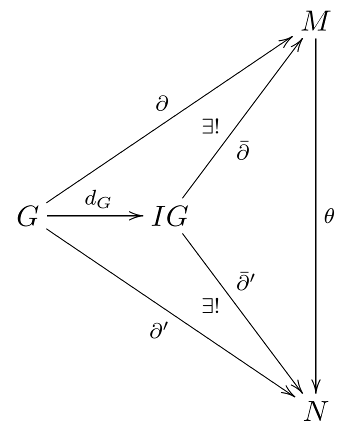
Next time, we shall discuss φ-derivations in more detail.

9 Seminar 9: 14th March 2003

In the previous seminar we found a correspondence between $\partial : G \to M_G$ (derivations) and $IG \to M_G$ (homomorphisms). Now $\mathrm{Der}_G(G, M)$ is the abelian group of G-derivations from G to M, which is isomorphic to $\mathrm{Hom}_G(IG, M)$, the abelian group of homomorphisms $IG \to M$. Note also that $\mathrm{Der}_G(G, M) \cong \mathrm{Hom}_{/G}(G, G \ltimes M)$.

Definition 9.1 $(IG,\ d_g :\ g \to g - 1)$ is the *universal* G-derivation.

Let $\mathrm{Der}_\varphi(G, M_H) = \{\partial : G \to M_H\}$ (a set of φ-derivations). Note that if $\theta : M_H \to N_H$ is a H-module morphism and if $\partial :\ G \to M_H$ is a φ-derivation then so is $\theta\partial :\ G \to N_H$, i.e. assuming that $\theta\partial = \partial'$ we claim that $\theta\bar{\partial} = \bar{\partial}'$ in the following diagram:



Justification: $\partial' = \theta\partial = (\theta\bar{\partial})d_G$ and $\partial' = \bar{\partial}d_G$. But $\bar{\partial}'$ is the unique homomorphism such that $\partial' = \bar{\partial}' d_G$.

Let us now go back to $\mathrm{Der}_\varphi(G, M_H)$. We want to find a universal φ-derivation $d_\varphi :\ G \to D_\varphi$, and D_φ must be a H-module. The following will be a sequence of attempts at constructing this φ-derivation.

(1) Let F be the free H-module on symbols $d_\varphi(g)$, $g \in G$. Let R be the sub H-module of F generated by all elements of the form $d_\varphi(g_1 g_2) - d_\varphi(g_1)\varphi(g_2) - d_\varphi(g_2)$. Set $D_\varphi = F/R$, $d_\varphi :\ G \to D_\varphi$ given by '$d_\varphi(g) = d_\varphi(g)$'. We must now check the Universal Property: we have a φ-derivation $G \xrightarrow{\partial} M$ and we want a $D_\varphi \xrightarrow{\bar{\partial}\,?} M$ such that $\partial = \bar{\partial} d_\varphi$. On generators, $\bar{\partial}(d_\varphi(g)) = \partial(g)$; and $\bar{\partial}(R) = 0$. But $\bar{\partial}$ is unique so that $G \to D_\varphi$ must be the universal φ-derivation.

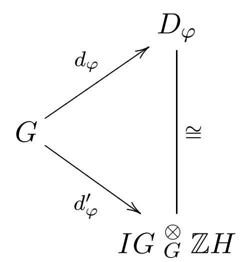
(2) $\varphi : G \to H$ gives $\mathbb{Z}\varphi : \mathbb{Z}G \to \mathbb{Z}H$, which means that there are lots of functors around! In general, if we have $\varphi :\ R \to S$ and N is an S-module, then N can be given an R-module structure by $n.r := n\varphi(r)$ (let us call this result $\varphi^*(N)$). It follows that $N \to \varphi^*(N)$ is the same as Mod-$S \to$ Mod-R. Therefore, there is an isomorphism $\mathrm{Hom}_R(M, \varphi^*(N)) \cong \mathrm{Hom}_S(M \overset{\otimes}{R} S, N)$.

Remark 9.2 $M \overset{\otimes}{R} S$ has generators $m \otimes s$ ($m \in M$, $s \in S$); relators $(m_1 + m_2) \otimes s = m_1 \otimes s + m_2 \otimes s$, $m \otimes (s_1 + s_2) = m \otimes s_1 + m \otimes s_2$, and $mr \otimes s = m \otimes \varphi(r)s$; and the S-module structure is induced by $(m \otimes s_1)s_2 = m \otimes s_1 s_2$. For the intuition of $M \underset{R}{\otimes} S$, we can crudely think of the R as $\mathbb{R}$ and the S as $\mathbb{C}$ so that the elements of $M \underset{R}{\otimes} S$ are (in this restricted environment) expressions of the form $u + iv$ with $u, v \in M$.

It follows that $\mathrm{Hom}_G(IG, \varphi^*(N)) \cong \mathrm{Hom}_H(IG \underset{G}{\otimes} \mathbb{Z}H, N)$.

Lemma 9.3 *If we have $\varphi : G \to H$ and if N is a H-module, then $\mathrm{Der}_\varphi(G, N) \cong \mathrm{Der}_G(G, \varphi^*(N))$ (the proof is obvious because $\partial(g_1 g_2) = \partial(g_1)\varphi(g_2) + \partial(g_2)$).*

Theorem 9.4 *$D_\varphi \cong IG \underset{G}{\otimes} \mathbb{Z}H$ such that the commutativity of the following diagram follows from $d'_\varphi(g) = (g-1) \otimes 1_{\mathbb{Z}H}$:*



Proof:

(i) $\mathrm{Der}_\varphi(G, N) \cong \mathrm{Der}_G(G, \varphi^*(N)) \cong \mathrm{Hom}_G(IG, \varphi^*(N)) \cong \mathrm{Hom}_H(IG \underset{G}{\otimes} \mathbb{Z}G, N)$ so that $D_\varphi \cong IG \underset{G}{\otimes} \mathbb{Z}G$.

(ii) $d'_\varphi(g_1 g_2) = (g_1 g_2 - 1) \otimes 1 = (g_1 - 1)g_2 \otimes 1 + (g_2 - 1) \otimes 1 = (g_1 - 1) \otimes \varphi(g_2) + (g_2 - 1) \otimes 1 = d'_\varphi(g_1)\varphi(g_2) + d'_\varphi(g_2)$.

(iii) $\mathrm{Der}_\varphi(G, IG \underset{G}{\otimes} \mathbb{Z}H) \cong \mathrm{Der}_G(G, \varphi^*(IG \underset{G}{\otimes} \mathbb{Z}H)) \cong \mathrm{Hom}_G(IG, \varphi^*(IG \underset{G}{\otimes} \mathbb{Z}H)) \cong \mathrm{Hom}_H(IG \underset{G}{\otimes} \mathbb{Z}H, IG \underset{G}{\otimes} \mathbb{Z}H)$ (the (universal) φ-derivation d'_φ corresponds to the identity in the final expression).

□

9.1 Fox Derivatives

Take F to be the free group on X and define $\frac{\partial}{\partial x} : F \to \mathbb{Z}F$ (given $x \in X$) as follows: $\frac{\partial}{\partial x} y = 1$ if $x = y$, $\frac{\partial}{\partial x} y = 0$ if $x \neq y$, and extend by the derivation formula. For example, $\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}(x)yx + \frac{\partial}{\partial x}(yx) = yx + \frac{\partial}{\partial x}(y)x + \frac{\partial}{\partial x}(x) = yx + 1$. In the next seminar, we'll discuss the relationship between Fox derivatives and universal derivations.

10 Seminar 10: 2nd April 2003

10.1 Commutative Algebra and Algebraic Geometry

Working in the classical setting, we shall consider an algebraic closed field k.

Definition 10.1 A variety V is defined to be a closed algebraic subset of k^n,

$$V = \{(x_1, \ldots, x_n) \mid f_1(x_1, \ldots, x_n) = 0, \ \ldots, \ f_m(x_1, \ldots, x_n) = 0\},$$

where the f_i are polynomials over k.

If we look at $A = (f_1, \ldots, f_m) \subseteq k[x_1, \ldots, x_n]$ (the ideal generated by the f_i's in the above definition), if $\mathbf{x} \in V$ and $g \in A$ then it follows that $g(\mathbf{x}) = 0$ and $V = V(A) = \{\mathbf{x} \in k^n \mid g(\mathbf{x}) = 0 \ \ \forall\, g \in A\}$. Note that this does not depend on A being finitely generated because $k[x_1, \ldots, x_n]$ is a Noetherian ring.

If Σ is a closed algebraic set, $\Sigma \subseteq k^n$, let us define $I(\Sigma) = \{g \in k[x_1, \ldots, x_n] \mid g(\mathbf{x}) = 0 \ \ \forall\, \mathbf{x} \in \Sigma\}$, an ideal in $k[x_1, \ldots, x_n]$. It follows easily that $\Sigma = V(I(\Sigma))$ but is it true to say that $I(V(A)) = A$? Inventing an example to see that the equality does not hold, let $n = 2$, $m = 1$ and $A = ((x_1 - x_2^2)^2)$ so that $V(A) =$ the parabola (t^2, t), but $I(V(A)) = ((x_1 - x_2^2))$. In fact, instead of equality, the following result tells us what to expect:

Theorem 10.2 *(Hilbert Nullstellensatz): $I(V(A)) = \sqrt{A} = \{f \mid f^r \in A$ for some $r\}$. (For the proof it is clear that $\sqrt{A} \subseteq I(V(A))$; the rest of the proof is omitted).*

By the above theorem, it follows that there is a one-to-one correspondence between the V's (the closed algebraic subsets) and the radical ideals A ($\sqrt{(A)} = A$). This sets up a lattice isomorphism $A \subset B \Rightarrow V(A) \supset V(B)$ and $\Sigma_1 \subset \Sigma_2 \Rightarrow I(\Sigma_1) \supseteq I(\Sigma_2)$. Furthermore, we have the two properties $V(\Sigma A_\alpha) = \bigcap_\alpha V(A_\alpha)$ and $V(A \cap B) = V(A) \cup V(B)$, properties which suggest that the $V(A)$'s form a collection of closed sets for some topology on k^n, a topology which we shall now discuss.

10.2 The Zariski Topology on k^n

The Zariski topology on k^n is defined as follows: $\mathcal{T}_{\text{Zar}}(k^n) = \{$complements of $V(A)$'s in $k^n\}$. As an example, consider k^1. If A is an ideal in $k[x]$, then there exists an f such that $A = (f)$. Since k is algebraically closed, we have $f = f(x) = c(x - a_1) \ldots (x - a_r)$, with $c, a_i \in k$. It follows that $V(A) = \{a_1, \ldots, a_r\}$ and in this case $\mathcal{T}_{\text{Zar}}(k^1)$ is made up of all finite complement sets in k plus the empty set.

Remark 10.3 $\phi \longleftrightarrow$ constant polynomials $f = c$, $c \neq 0$; $k^1 \longleftrightarrow f = 0$.

Remark 10.4 The topology is non-Hausdorff e.g. for $k = \mathbb{C}$.

Definition 10.5 Given a topological space X, a subset Y $(\neq \phi)$ of X is irreducible if $Y = Y_1 \cup Y_2$ $(Y_1, Y_2$ closed$) \Rightarrow Y_1 = Y$ or $Y_2 = Y$.

Example 10.6 k^1 is irreducible — its only proper closed subsets are finite yet k is infinite.

Definition 10.7 If X is irreducible and $U \subset X$ is open and nonempty, then U is irreducible and dense.

Definition 10.8 If $Y \subset X$ is irreducible then so is $\bar{Y}$.

Definition 10.9 An affine (algebraic) variety is an irreducible closed subset of k^n (with the induced topology).

Definition 10.10 An open subset of an open variety is known as a quasi-affine variety.

Remark 10.11 If $\Sigma \subset k^n$ is any subset, then $V(I(\Sigma)) = \bar{\Sigma}$ in the Zariski topology.

Lemma 10.12 *If $\Sigma = V(A)$ with $A = \sqrt{A}$, then Σ is irreducible $\Leftrightarrow$ A is a prime ideal.*

Proof: $(\Rightarrow)$ If Σ is irreducible, consider an element $fg \in I(\Sigma)$. Because $\Sigma \subset V(fg) = V(f) \cup V(g)$, we have $\Sigma = (\Sigma \cap V(f)) \cup (\Sigma \cap V(g))$. Therefore, either $\Sigma = \Sigma \cap V(f)$ and $\Sigma \subseteq V(f)$, or $\Sigma = \Sigma \cap V(g)$ and $\Sigma \subseteq V(g)$. It follows that either $f \in I(\Sigma)$ or $g \in I(\Sigma)$.

$(\Leftarrow)$ If p is a prime ideal and if $V(p) = Y_1 \cup Y_2$, then $p = I(Y_1) \cap I(Y_2)$ so that $p = I(Y_1)$ or $p = I(Y_2)$, i.e. $V(p) = Y_1$ or $V(p) = Y_2$. □