PhD Seminars 2002/2003: Prof. R. Brown Semester 2

Gareth Evans

October 29, 2003

Contents

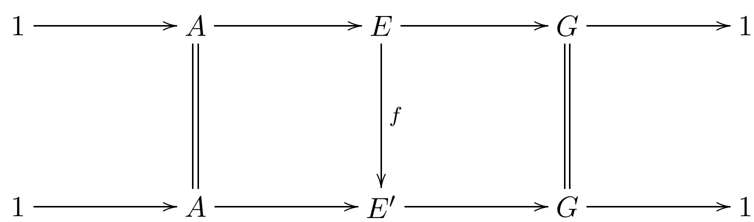
1 Seminar 1: 21st March 2003

1.1 The Schreier Theory of Non-Abelian Extensions

Consider the problem of describing all extensions

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

of A by G (where A and G are groups, i is injective, π is surjective, and $\ker \pi = \mathrm{im}\ i$) up to equivalence, i.e. are there f's making the following diagram commute? (the short S Lemma implies that f is an isomorphism):

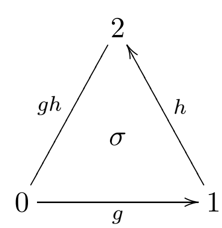


Schreier (1926) solved this problem, and this is described in Zassenhaus, Marhsall Hall. Now choose a function $s : G \to E$ such that $\pi s = 1_G$. Use this to construct a bijection $E \to G \times A$ and then try to write down an induced multiplication on $G \times A$ in the form $(g, a)(h, b) = (gh, k^2(g, h)a^{k^1(h)}b)$, where $k^1 : G \to \mathrm{Aut}A$ and $k^2 : G \times G \to A$ (satisfying conditions defining a 'factor set') come from the derivation of s being a morphism and the associativity condition.

The problems with the above include the notion that we should avoid describing a group by its multiplication table, that we want to handle the case when G is infinite, and that we often want to describe G by a presentation.

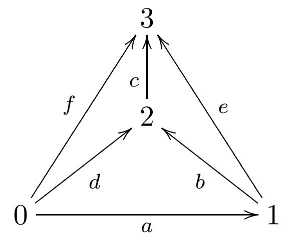
We are going to use free crossed resolutions. There are several bugbears related to this, including the choice of conventions, the Homotopy Addition Lemma (HAL) and '$\delta_3\delta_4 = 0$'. We shall now discuss some of these 'bugbears'.

Consider the following diagram:

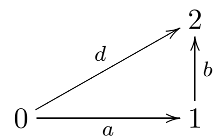


In this situation, what is δ_2? One possible choice is $\delta_2 = [g][h][gh]^{-1}$, but there are many other choices. This is one of the problems with two dimensional algebra as opposed to one dimensional algebra — we need to set an ordering. We also want to say that the boundary of a simplex is the sum of its faces (in which way!) — this is the HAL.

Let us now consider the following diagram (cf. Van Kampen diagrams):



Let us set $\delta\sigma^2 = d^{-1}ab$ (coming from σ^2 as shown below:)



We need $\delta\delta = 0$. Consider $c^{-1}d^{-1}abc.c^{-1}b^{-1}e.e^{-1}a^{-1}f.f^{-1}dc = 1$, derived from the four vertex diagram shown above. This equation implies that (when calculated in a crossed module) $\delta\sigma^3 = (\delta_3\sigma^3)^c(\delta_0\sigma^3)^{-1}(\delta_2\sigma^3)^{-1}(\delta_1\sigma^3)$. By contrast, Brown, Higgins and G. W. Whitehead use the following: For $n \geqslant 4$, $\delta\sigma^n = (\delta_0\sigma)^{-1} + \sum_{i=1}^{n}(-1)^i\delta_i\sigma$ $(a = \delta_2\delta_3\ldots\delta_n\sigma)$; for $n = 3$, $\delta\sigma = (\delta_0\sigma)^{-a} + (\delta_2\sigma) - (\delta_1\sigma) - (\delta_3\sigma)$; and for $n = 2$, $\delta\sigma = (\delta_2\sigma)(\delta_0\sigma)(\delta_1\sigma)^{-1}$ ($\delta\sigma = abd^{-1}$ comes from the three vertex ('0-1-2') diagram above). For Brown-Higgins it is easy to check that $\delta\delta = 0$, but Whitehead takes a page to check that $\delta_3\delta_4 = 0$.

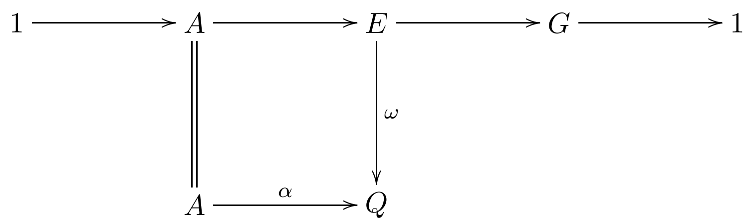
Remark 1.1 Cancellation depends on the crossed module rule. Rebracket to get a term with $\delta = 0$ which lies in the centre and therefore is able to be moved around.

Remark 1.2 Implementing the above on a computer is a rewrite problem.

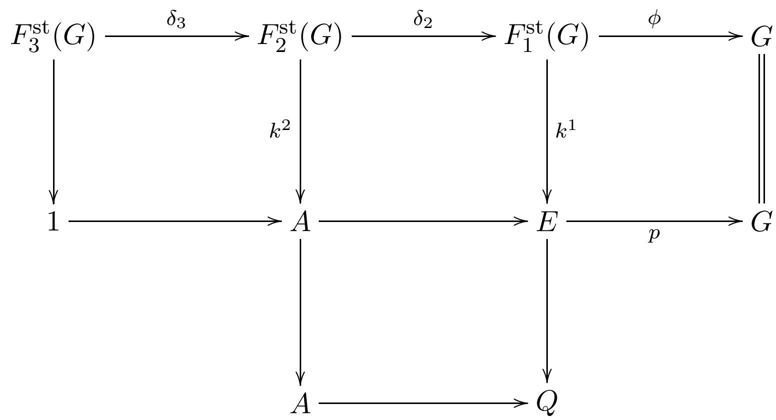
Remark 1.3 An easier method to do the above uses the Lyndon Identity Property (see page 177 of Brown-Huchesmann — needs simplicial identities and crossed modules). Alternatively, we may assume that Δ^n is contractible and work out for $\pi\Delta^n$ its generators and contracting homotopy (if G is a group, then $F^{\text{st}}(G)$ is generated in dimension n by $[g_1, \ldots, g_n]$; its universal cover has vertices $g \in G$; and the contracting homotopy of $\tilde{F}^{\text{st}}(G)$ is given essentially by $(g, [g_1, \ldots, g_n]) \mapsto (1, [g, g_1, \ldots, g_n])$).

Dedecker considered cohomology with coefficients in $A \to \text{Aut}A$. A problem however was that $\text{Aut}A$ is not functorial in A.

Observation 1: The early work concentrated on $H^2(G, A)$ (non-abelian) and on $H^2(G, A \to \text{Aut}A)$. Dedecker considered $H^2(G, A \xrightarrow{\alpha} Q)$, where $\alpha : A \to Q$ is a crossed module classifying extensions of type $\alpha: A \to Q$ (a morphism of crossed modules):

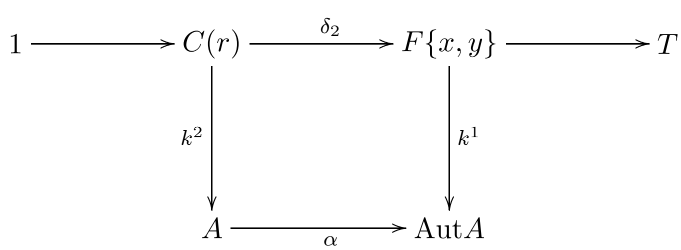


Observation 2: We move from 'choosing a section' of p to a more algebraic setting using $F^{\mathrm{st}}(G)$ — instead of choosing a section of G, we choose a k^1 such that $pk^1 = \phi$ (in the following diagram, $F_1^{\mathrm{st}}(G) =$ the free group on G):



Notice in the above diagram that $pk^1\delta_2 = \phi\delta_2 = 1$, but $A \xrightarrow{i} E \xrightarrow{p} G$ is exact. This means that for all $x \in F_2^{\mathrm{st}}(G)$ there exists an $a_x \in A$ such that $ia_x = k^1\delta_2 x$. Further, $F_2^{\mathrm{st}}(G)$ is free on $[g,h]$ so that there exists a k^2 such that $k^2[g,h] = a_{[g,h]}$; and $ik^2\delta_3 = k^1\delta_2\delta_3 = k^1(\delta_2\delta_3) = 1$. But i is injective so that $k^2\delta_3 = 0$, and so δ_3 expresses 'associativity'. This leads us to the conclusion that the algebraic difficulties are encoded in δ_3.

The final point for today's seminar is that free crossed resolutions of G are unique up to homotopy, so that we can replace $F^{\mathrm{st}}(G)$ by any other free crossed resolution. As an example, consider $T = \langle x, y \mid x^2y^{-3}\rangle$, the trefoil group. This has a free crossed resolution which is visualised as follows ($\delta_2 r = x^2y^{-3}$):



The extension $A \to E \to T$ is specified up to 'equivalence' by $a \in A$ and $\alpha_x, \alpha_y \in \mathrm{Aut}A$ such that $\alpha a = (\alpha_x)^2(\alpha y)^{-3}$.

2 Seminar 2: 28th March 2003

Remark 2.1 Two handouts were given out for this seminar, including "The Homotopy Addition Lemma: Notes by Ronnie Brown".

In the previous seminar, we saw that an extension of A by G of the type of the crossed module $\alpha:\ A \to Q$ is given by

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1, \\ & & \| & & \downarrow{\scriptstyle \omega} & & & & \\ & & A & \xrightarrow{\alpha} & Q & & & & \end{array}$$

where the top row is exact and the square is a morphism of crossed modules (E operates on A via Q).

Definition 2.2 $\pi\Delta^n_*$ is the fundamental crossed complex of the n-simplex filtered by skeletons.

Claim: $\pi\Delta^n_*$ is freely generated by elements σ^n_r corresponding to the r-faces of Δ^n for $0 \leqslant r \leqslant n$ (this requires groupoids). How we write down $\delta_n(\sigma^n_r)$ depends on the choice of orientation, but in any case (in a crossed complex) $\delta_{r-1}\delta_r = 0$.

So how do we prove that $\pi\Delta^n_*$ is a crossed complex? One way to do this is to use some basic facts on relative homotopy theory. An alternative way is to go via cubical ω-groupoids — but for this we need cubes for $\_\otimes\_$ of crossed complexes!

What about the equivalence of such extensions? For this, consider the following diagram:

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1 \qquad \alpha:\ E \to Q$$
$$1 \longrightarrow A \xrightarrow{i'} E' \xrightarrow{p'} G \longrightarrow 1 \qquad \alpha':\ E' \to Q$$

We want an isomorphism $h : E \to E'$ such that $hi = i'$, $p'g = p$ and $w'h = w$. This isomorphism follows from what we have been discussing, so we can set $\mathrm{Ext}_{(A\to Q)}(G, A)$ $(= \pi_0(\text{groupoid}))$ to be the set of classes of such extensions.

To describe this set in a more combinatorial setting, let $\mu :\ M \to P$ be a crossed module and consider the diagram below, where $k = (k^2, k^1)$ and $\ell = (\ell^2, \ell^1)$ are morphisms.

$$\begin{array}{ccc} M & \longrightarrow & P \\ {\scriptstyle k^2,\ell^2}\downarrow & \overset{k}{\swarrow} & \downarrow{\scriptstyle k^1,\ell^1} \\ A & \longrightarrow & Q \end{array}$$

The homotopy $h : k \simeq \ell$ is given by $h : P \to A$, and we have the following crossed module homotopy (CMH) axioms:

CMH1) $h(p_1 p) = (hp_1)^{\ell^1 p}(hp)$ $(= \ell^1\text{-derivation})$;
CMH2) $k^1(p) = (\ell^1 p)(\alpha h p)$;
CMH3) $k^2 m = (\ell^2 m)(h\mu m)$.

Claim: The homotopy is an equivalence relation (Proof omitted!)

Remark 2.3 Later, we'll discuss homotopies on crossed complexes as morphisms $\mathcal{I} \otimes C \to D$, where $\mathcal{I}$ is the groupoid shown below. The reason for doing this is that the crossed complex better reflects the geometry.

$$0 \rightleftarrows 1$$

Theorem 2.4 *Let μ, α and G be defined as $\mu : M \to P$, $\alpha : A \to Q$ and $G = \operatorname{coker} \mu$; and let $[M \to P,\ A \to Q]^0$ denote the homotopy classes of all morphisms $k = (k^2, k^1) : (M \to P) \longrightarrow (A \to Q)$ such that $k^2(\ker \mu) = 1$. Then there exists a natural injection $\tilde{E} : [M \to P,\ A \to Q]^0 \longrightarrow \operatorname{Ext}_{A \to Q}(A, G)$ which sends $[k]$ to ${}^{(P \ltimes A)}/_{\{(\mu m,\ (k^2 m)^{-1}),\ m \in M\}}$; P acts on A via k^1 (i.e. via Q); and $\tilde{E}$ is surjective if P is free.*

In the classical case, $M \to P$ is given by $F_2^{\text{st}}(G) \xrightarrow{\delta_2} F_1^{\text{st}}(G)$, where $F_1^{\text{st}}(G)$ is free on $[g]$ $(g \in G)$ and $F_2^{\text{st}}(G)$ is a free crossed $F_1^{\text{st}}(G)$-module on $[g, h]$, with $\delta_2[g, h] = [g][h][gh]^{-1}$ so that $k^2(\ker \delta_2) = 0 \Leftrightarrow k^2\delta_3 = 0$ (δ_3 is the map $F_3^{\text{st}}(G) \to F_2^{\text{st}}(G)$).

Now k^1 is determined by $k^1[g] = k^1(g)$, k^2 is determined by $k^2[g, h] = k^2(g, h)$, and the conditions $\alpha k^2 = k^1\delta_2$ and $k^2\delta_2 = 0$ constitute a factor set (k^2, k^1). *False Claim:* the above leads to the multiplication $E(K) = C_1(G) \ltimes A$ given by $(\delta_2(g, h), (k^2(g, h))^{-1})$, $g, h \in G$: we claim that there exists a bijection $G \times A \to E(K)$ given by $(g, a) \mapsto [\![[g], a]\!]$ so that that the following formula holds: $[\![[g], a]\!][\![[h], b]\!] = [\![[gh], k^2(g, h)a^{[h]}b]\!]$. This claim is not quite correct because it requires $\delta_2[g, h] = [gh]^{-1}[g][h]$ (we have $\delta_2[g, h] = [g][h][gh]^{-1}$).

The advantage of the general case in the theorem is that we can replace $F^{\text{st}}(G)$ by any other free crossed resolution of G. *Claim:* Any free crossed resolution of G is unique up to homotopy equivalence. In more detail, suppose that we have the following diagram, where $G = \operatorname{coker} \delta_2^C$, $H = \operatorname{coker} \delta_2^D$, and f is a morphism:

$$\begin{array}{ccccccccccccc}
\longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \xrightarrow{\phi} & G \\
 & & & & & & & & & & & \downarrow{\scriptstyle f} \\
\longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots & \longrightarrow & D_2 & \longrightarrow & D_1 & \longrightarrow & H
\end{array}$$

Let the top row be free and let the bottom row be exact. Then f lifts to a morphism $f_* : C \to D$ and any two such lifts are homotopic.

Example 2.5 The free crossed resolution of Cyc[q] is the cyclic group of order q generated by t with $t^q = 1$ (multiplicatively). Let C_1 be the free group on x_1 (multiplicatively), let $\phi(x_1) = t$, and let $\mathcal{R} = \mathbb{Z}[\text{Cyc}[q]]$ for $n \geqslant 2$ so that C_n is the free R-module on x_n for $n \geqslant 2$.

It follows that $\delta_2(x_2) = x_1^q$, $\delta_n(x_n) = x_{n-1}(1-t)$ for $n =$ odd and $n \geqslant 3$, and $\delta_n(x_n) = x_{n-1}N(t)$ for $n =$ even and $n \geqslant 3$. Therefore, $N(t) = 1 + t + \ldots + t^{q-1}$ and this is the reason why $\delta_{n-1}\delta_n = 0$ is a deep result: $(1-t)N(t) = 0$ given $t^q = 1$.

Exercise 2.6 Look at the case Cyc[q] $\to$ Cyc[qp] and calculate the lift.

Remark 2.7 To get contracting homotopies we have to go to $\tilde{G}$, the universal covering groupoid of G.

Remark 2.8 To get normal forms in $G = \text{gp}\langle X, R\rangle$, it may be easier to go to a covering groupoid and do calculations there.

Remark 2.9 Let X be the set of generators for a group G. If we consider Cayley(G, X), the Cayley graph of the group, notice that there is a map between $F(\text{Cayley}(G, X))$, the free groupoid on the graph, and $F(X)$, the covering morphism of groupoids.

3 Seminar 3: 14th April 2003

Remark 3.1 The 1994 paper *"Covering Groups of Non-Connected Topological Groups Revisited"* by Ronald Brown and Osman Mucuk was given out during this seminar.

Let $p : \tilde{X} \to X$ be a covering space, and assume that X has a universal cover ($\Leftrightarrow$ certain local conditions on X).

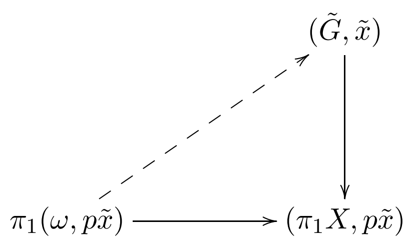
Definition 3.2 For each $x \in X$ there exists an open neighbourhood U of X such that $p^{-1}(U) =$ a disjoint union of open connected sets each mapped homeomorphically by p to U. *Examples:* $p : \mathbb{R} \to S^{-1}$ $(t \mapsto e^{2\pi it})$ and $p : S^1 \to S^1$ $(z \mapsto z^2)$.

Now let $q : \tilde{G} \to G$ be a covering morphism of groupoids (= unique path lifting = star bijective). We have $\mathrm{St}_G x = \cup_{y \in \mathrm{Ob}(G)} G(x, y)$ (star injective, star surjective). Further, $\mathrm{St}_{G}x \to \mathrm{St}_G px$ are bijections for all $x \in \mathrm{Ob}(G)$.

Example 3.3 The groupoid $\mathcal{I}$ maps to $\mathbb{Z}_2 = \{0, 1\}$ $(\iota, \iota^1 \mapsto 1$, a covering morphism).

Classification Theorem: $\mathrm{GpdCov}(G) \simeq \mathrm{Set}^G$ (the right hand side is a functor category of actions of G on sets). *Classification Theorem for Topological Covering Maps:* For 'good' X, we have $\mathrm{TopCov}(X) \to \mathrm{GpdCov}(\pi_1 X)$.

For $p : \tilde{X} \to X$, it follows that the fundamental groupoid $\pi_1 p : \pi_1 \tilde{X} \to \pi X$ is an equivalence of categories (see Galois theory). How to go backward? This is easy — let $\tilde{X}$ be equal to $\mathrm{Ob}(\tilde{G})$ as a set and the topology on $\tilde{X}$ is then obtained by local lifting — for $\tilde{x} \in \tilde{X}$ choose a 'good' neighbourhood U of $p\tilde{x}$ such that



lifts uniquely. Then $\pi_1 \tilde{X} \cong \tilde{G}$.

Now suppose that X is a topological group and consider $\mathrm{TopGpCov}(X)$. For $p : \tilde{X} \to X$, p is a topological group morphism and also a covering map. It follows that we have $\pi_1 : \mathrm{TopGpCov}(X) \to \mathrm{GpGpdCov}(\pi_1 X)$ so that if X is a topological group then $\pi_1 X$ is a group groupoid because of the natural isomorphism $\pi_1(X \times X) \cong \pi_1 X \times \pi_1 X$. Further, if G and $\tilde{G}$ are group groupoids then $p : \tilde{G} \to G$ is a covering morphism and also a group morphism. It follows that π_1 is an equivalence of categories and so the topology maps to the algebra.

But we also know that group groupoids are equivalent to crossed modules ($G \mapsto (\mathrm{St}_G e \to \mathrm{Ob}(G))$ and $(M \xrightarrow{\mu} P) \mapsto (P \ltimes M \overset{t}{\underset{s}{\rightrightarrows}} P)$ with $s(p,m) = p$ and $t(p,m) = p(\mu m)$). *Question:* Given a group groupoid G (with associated crossed module $\mathrm{St}_G e \xrightarrow{\mu} \mathrm{Ob}(G)$, cokernel $\pi_0 G =$ a group, and kernel $G(e) =$ an abelian group), does G have a universal cover (as a group groupoid), i.e. is $q : \tilde{G} \to G$ a covering morphism of group groupoids such that $\tilde{G}(\tilde{e}) = \{1\}$ and hence all $\tilde{G}(\tilde{x}) = \{1\}$?

For classic extension theory (see MacLane), we have $A \to \mathrm{Aut}(A) \to \mathrm{Out}(A) \xleftarrow{\theta} \Phi$. We must now ask whether there is an extension of the form

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \theta} & & \\
 & & A & \longrightarrow & \mathrm{Aut}A & \longrightarrow & \mathrm{Out}A & &
\end{array}$$

realising θ?

Remark 3.4 There exists a nice account of the more general crossed module case using crossed complexes in Brown-Mucak (utilising fibrations of crossed complexes).