PhD Seminars 2002/2003: Prof. T. Porter Semester 1

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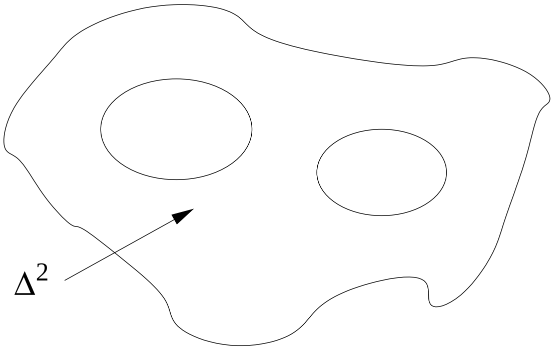
1 Seminar 1: 8th October 2002

1.1 Simplicial Sets

A simplicial set K is a sequence K_n of sets linked by face and degeneracy maps $d_k : K_n \to K_{n-1}$ and $s_k : K_n \to K_{n+1}$ (K_n is the set of simplices of dimension n). As an example, if we have a space X, then the singular complex is $\mathrm{Sing}(X)_n = \mathrm{Top}(\Delta^n, X)$, where Δ^n is the standard n-simplex, the convex hull of $\{\underline{e}_0, \ldots, \underline{e}_n\} \subset \mathbb{R}^{n+1}$.

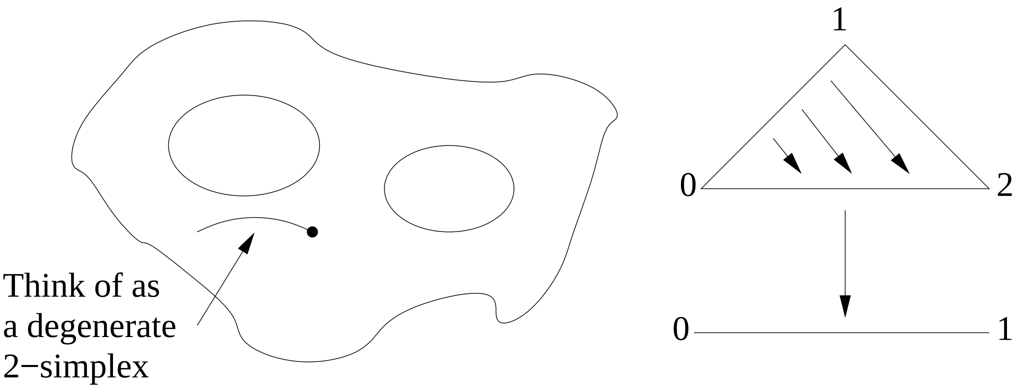
As a second example, consider the space shown below:



We have $d_k : \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1}$, $\sigma : \Delta^n \to X$, and $d_k(\sigma) = \sigma.\delta_k$. If $\underline{x} \in \Delta^n$, then $\underline{x} = \sum_{i=0}^{n} x_i \underline{e}_i$ with $\sum x_i = 1$ and $x_i \geqslant 0$. If $\underline{x} \in \delta_k(\Delta^{n-1})$, then $x_k = 0$. δ_k is defined as $\delta_k : \Delta^{n-1} \to \Delta^n$, and it is linear:

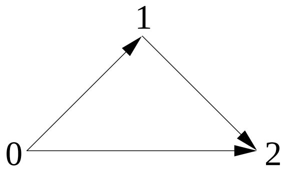
$$\delta_k(\underline{e}_i) = \begin{cases} \underline{e}_i & \text{if } i < k \\ \underline{e}_{i+1} & \text{if } i \geqslant k \end{cases}$$

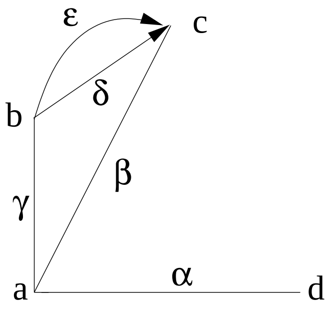
Now we also have $\sigma_k : \Delta^n \to \Delta^{n-1}$ with $\sigma_k(x_0, \ldots x_n) = (x_0, \ldots, x_k + x_{k+1}, \ldots, x_n)$. Consider the following diagrams for $\Delta^2 \xrightarrow{\sigma_0} \Delta^1$:



Theorem 1.1 *(Kan, 1958): Reduced Simplicial Sets $\xrightarrow{G}$ Simplicial Groups. Simplicial sets model all homotopy types.*

For the first diagram below, we have $d_1 d_2 = d_1 d_1$ at vertex 0, $d_0 d_2 = d_1 d_0$ at vertex 1, and $d_0 d_1 = d_0 d_0$ at vertex 2. For the second diagram below, we have $K_0 = \{a, b, c, d\}$ and $K_1 = \{\alpha, \beta, \gamma, \delta, \epsilon, s_0(a), s_0(b), s_0(c), s_0(d)\}$.



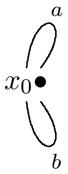


$G(K)_n$ = the free group on $K_{n+1} \setminus (s_0 K_n) = < K_{n+1} \mid s_0(K_n) >$.

2 Seminar 2: 15th October 2002

Recall that in the last seminar we looked at Kan's 1958 theorem: Reduced Simplical Sets $\xrightarrow{G}$ Simplicial Groupoids. Let us now start with K_0 reduced and $\#(K_0)$ = one vertex only. Then $(GK)_n$ = the free group on $\{\bar{x} \mid x \in K_{n+1}\}$ with $\overline{s_0 x} = e$ if $x \in K_n$. Now we have face morphisms $d_0\bar{x} = (\overline{d_1 x})(\overline{d_0 x})^{-1}$ and $d_i\bar{x} = \overline{d_{i+1} x}$ for $i > 0$; and degenerates $s_i(x) = \overline{s_{i+1} x}$. We also have $\pi_1(X) = \pi_0(\Omega X)$ and $\pi_1(\Omega^{n-1}X) \cong \pi_0(\Omega^n X)$, where Ω = loops in X.

Suppose now that we wanted to model the following:



In this situation, we have $K_0 = \{x_0\}$, $K_1 = \{a, b, s_0 x_0\}$, $K_2 = \{s_0 a, s_0 b, s_1 a, s_1 b, s_0 s_0 x_0\}$, $K_3 = \{s_0 s_0 a$, etc.$\}$, $d_0 a = d_1 a = d_0 b = d_1 b = x_0$, $d_0 s_0 = \mathrm{id}$, and $d_0 s_1 = s_1 d_0$, etc. Note that an easy way to work with the s_i's and the d_i's is to use numerical notation, e.g. $0123 \xrightarrow{s_0} 0012 \xrightarrow{d_0} 012 \ldots$

- $G(K)_0 = < \bar{a}, \bar{b} \mid \phi >$ or $G(K)_0 = < \bar{a}, \bar{b}, \overline{s_0 x_0} \mid \overline{s_0 x_0} = 1 >$.
- $G(K)_1 = < \overline{s_1 a}, \overline{s_1 b} \mid \phi >$.
- $d_1 \overline{s_1 a} = \overline{d_2 s_1 a} = \bar{a}$ and $d_1 \overline{s_1 b} = \bar{b}$.
- $d_0 \overline{s_1 a} \overset{(\mathrm{def}^{\underline{\mathrm{n}}})}{=} (\overline{d_1 s_1 a})(\overline{d_0 s_1 a})^{-1} = (\bar{a})(\overline{s_0 x_0})^{-1} = \bar{a}$.

Theorem 2.1 *(Dwyer-Kan, 1984): Simplicial Sets $\xrightarrow{G}$ Simplicial Groupoids.*

2.1 Moore Complex

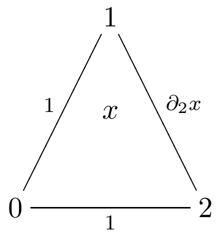
Consider the non-abelian chain complex

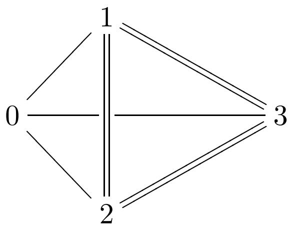
$$(NG)_n = \bigcap_{i=1}^{n} \ker d_i^n.$$

$\partial_n : NG_n \to NG_{n-1}$ is the restriction of d_0 with $\partial_{n-1}\partial_n = 1$, and $\mathrm{Im}\,\partial_{n+1} \lhd NG_n$ so that we can define homology groups of NG.

Now $g\partial x g^{-1} = g d_0 x g^{-1} = d_0(s_0 g x s_0 g^{-1})$, and

$$H_n(NG_n) = \frac{\ker \partial_n}{\mathrm{Im}\,\partial_{n+1}}.$$





In the first diagram above, we have $x \in NG_2$; and in the second diagram above, the face $1-2-3$ is the face $\partial_3 y$ (all other faces are '1's).

Now $x:\ 1 \simeq \partial_x$ so that

$$\pi_1(G) = \frac{\ker(\partial_1 :\ NG_1 \to NG_0)}{\operatorname{Im}(\partial_2 :\ NG_2 \to NG_1)} = H_1(NG).$$

In general, we have

$$\pi_n(G) \cong H_n(NG, \partial) = \frac{\ker d_0 \cap NG_n}{d_0(NG_{n+1})}$$

and we have extra structure such as Whitehead products, etc. (which can be defined in a group structure of G).

Now $N(GK)_0 = <\bar{a}, \bar{b} \mid \phi >= G(K)_0$ and $N(GK)_1 = \ker d_1 = \{1_{GK_1}\}$, etc.

$P = <a, b \mid a^3 = 1, b^2 = 1, (ab)^2 = 1>$, where r '=' a^3, s '=' b^2 and t '=' $abab$.

$K(P)$ is like K but $K(P)_2 = \{\bar{r}, \bar{s}, \bar{t}, s_0 a, s_0 b, \ldots\}$ — but not quite since we cannot define $d_1 r$, etc. However, we can build a simplicial group model of $K(P)$:

$$G(P)_0 = <\bar{a}, \bar{b} \mid \phi> \text{ and } G(P)_1 = <\bar{r}, \bar{s}, \bar{t}, \overline{s_1 a}, \overline{s_1 b} \mid \phi> \ \ (\pi_0(G(P)) = S_3,\ \pi_1(G(P)) = ?)$$

Now $d_1\bar{r} = d_1\bar{s} = s_1\bar{t} = 1$, $d_0\bar{r} = \bar{a}\bar{a}\bar{a}$, $d_0\bar{t} = \bar{a}\bar{b}\bar{a}\bar{b}$, and $d_0\bar{s} = \bar{b}\bar{b}$.

3 Seminar 3: 22nd October 2002

3.1 Crossed Modules (Left Actions!)

A crossed module is a map $\partial : C \to P$, where P acts on C on the left and we have axioms $\partial({}^{p}c) = p\,\partial c\,p^{-1}$ (CM1) and ${}^{\partial c}c' = cc'c^{-1}$ (CM2, the Peiffer identity).

Example 3.1 $N \lhd P$ (with P acting by conjugation) gives a crossed module $\partial : N \to P$. A normal subgroup inclusion gives a crossed module and a monomorphism.

Example 3.2 Take M to be a P-module. Then we have a crossed module $\partial : M \to P$ with $\partial(m) = 1_P$ for all $m \in M$. Fact: If $\partial : C \to P$ is a crossed module, then $\ker \partial$ is a P-module. A module gives a crossed module and a trivial ∂.

Example 3.3 Take G to be a group: $\alpha : G \to \mathrm{Aut}(G)$ is a crossed module, with $\ker \alpha = Z(G)$ and $\mathrm{Aut}(G)/\mathrm{Im}\,\alpha \cong \mathrm{Out}(G)$.

Example 3.4 If $F \to t \to B$ is a fibration of pointed spaces, then $\pi_1(F) \to \pi_1(E)$ is a crossed module. Better still, $F \to t \to B$ is a 'homotopy crossed module', i.e. a crossed module structure up to homotopy. There are recent results by KHK & TP, RB & GJ on related situations.

Consider the following fibration, where $p(e_0) = b_0$ and Bb_0 is a base point:

$$\begin{tikzcd}
 & F = p^{-1}(b_0) \arrow[d] \\
 & E \arrow[d, "\phi"] \\
I \arrow[r] & B
\end{tikzcd}$$

There is a unique path lifting as shown in the following diagram, where $\bar{\sigma}(0) = e$, $p(e) = b_0$, and there exists a unique $\sigma' : I \to E$ such that $\sigma'(0) = e$ and $p\sigma' = \sigma$:

$$\begin{tikzcd}
\{0\} \arrow[r, "\bar{\sigma}"] \arrow[d] & \boxed{\overset{e}{\rightsquigarrow}} & E \\
I \arrow[ru, "\sigma'"'] \arrow[r, rightsquigarrow, "\sigma"'] & {} \arrow[r, rightsquigarrow, "b_0"'] & B
\end{tikzcd}$$

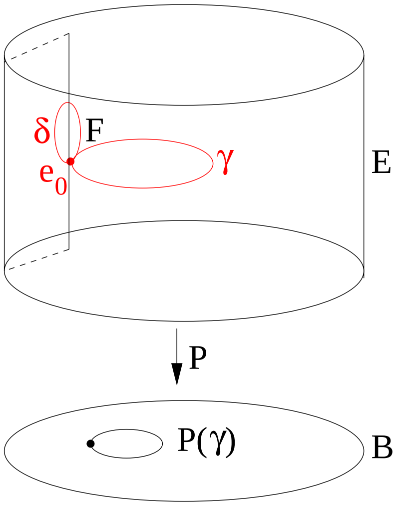
If we relax the conditions and generalise, we obtain the following:

$$\begin{tikzcd}
A \arrow[r, "f_A"] \arrow[d, leftrightarrow, "i"'] & E \arrow[d, "p"', "\text{fibration}"] \\
X \arrow[ur, "\exists\ \text{lift}"] \arrow[r, "f_X"'] & B
\end{tikzcd}$$

In the above diagram, i is a strong deformation retract and there exists an F such that $Fi = f_A$ and $pF = f_X$. Further,

$$\pi_1(E) = \frac{\text{loops in } E \text{ at } e_0}{\sim_{\text{rel end points}}}.$$

Is there an action of $\pi_1(E)$ on $\pi_1(F)$?



In the above, γ is a loop in E at e_0, δ is a loop in F at e_0, and $\gamma.\delta.\gamma^r$ is a loop in E at e_0 ($r =$ reverse). It follows that $p(\gamma.\delta.\gamma^r) = p(\gamma).c_{b_0}.p(\gamma^r) \cong_H c_{b_0}$ (c is a constant path).

Now we have $f_1 = \gamma\delta\gamma^r : I \to E$ and $f_0 = H : I^2 \to B$:

$$\begin{tikzcd}
I \arrow[r, "f_1"] \arrow[d, leftrightarrow] & E \arrow[d] \\
I^2 \arrow[ur, "\exists\ F"'] \arrow[r, "f_0"'] & B
\end{tikzcd}$$

It follows that $F : \gamma\delta\gamma^r \cong \gamma'$ in F and $[\gamma'] = {}^{[\gamma]}[\delta]$. Further, $\Omega X \sim$ loops at x_0 in X; $\Omega X \times \Omega X \to \Omega X$; $\Omega X \xrightarrow{(\ )^r} \Omega X$; and $c_{x_0} : \{x\} \to \Omega X$. Now ΩX looks like a group up to homotopy and $\Omega F \to \Omega E$ looks like a crossed module up to homotopy (see Hilton's grey book).

Example 3.5 Take (X, A, x_0) to be a pointed pair of spaces. Then $\pi_2(X, A, x_0) \xrightarrow{\partial}_{\text{b'dry}} \pi_1(A, x_0)$ is a crossed module, e.g. the pointed CW-complex $\pi_2(X, X^{(1)}, x_0) \to \pi_1(X^{(1)}, x_0)$, where $X^{(1)}$ is the free group.

3.2 First Links

Aside: Let $G_\bullet \xrightarrow{p_\bullet} L_\bullet$ be a fibration of simplicial groupoids. Then $p_\bullet$ is an epimorphism in each dimension; $p_\bullet$ is determined up to isomorphism by $\ker p_\bullet \lhd G_\bullet$; $H_\bullet \lhd G$ gives a fibration $G_\bullet \to G_\bullet/H_\bullet$; and the fibration over G/H is $H_\bullet$.

Theorem 3.6 *(Loday, 1980's): Every crossed module occurs like the following up to isomorphism: If $H_\bullet \lhd G_\bullet$, then $\pi_0(H_\bullet) \to \pi_0(G_\bullet)$ is a crossed module and*

$$\pi_0(G_\bullet) = \frac{NG_0}{\partial NG_1}.$$

π_0 destroys 'monicness' but not 'crossed-moduleness'.

Remark 3.7 If $\dfrac{NG_1}{\partial(NG_2)} \xrightarrow{\partial} NG_0$, then $\ker\partial = \pi_1(G_\bullet)$ and $\operatorname{coker}\partial \cong \dfrac{NG_0}{\partial NG_1} \cong \pi_0(G_\bullet)$.

Theorem 3.8 *(MacLane Whitehead, 1950): Crossed modules model all (connected) homotopy 2-types.*

Remark 3.9 If we have $X \xrightarrow{f} Y$ then we also have $\pi_n(X) \xrightarrow{f_X} \pi_n(Y)$.

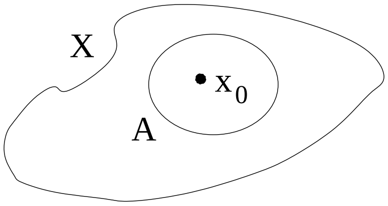
Theorem 3.10 *(Loday, 1982): 'Higher-dimensional crossed modules' model all (connected) n-types.*

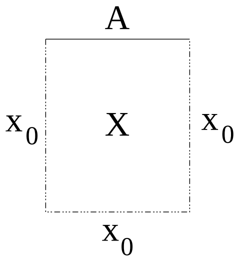
Remark 3.11 Beyond dimension 2, there are three routes to choose: crossed complexes, crossed n-cubes and hypercrossed modules/complexes e.g. 2-crossed complexes.

4 Seminar 4: 29th October 2002

4.1 Crossed Complexes

Recall that $\pi_2(X, A, x_0) = [(I^2, \partial I^2, J), (X, A, \{x_0\})]$.



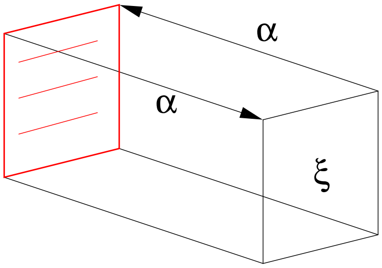


In the CW-complex $X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(n)}$, $\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{(n-1)}), (X, A, x_0)]$ is the n$^{\text{th}}$ rel homotopy group of (X, A, x_0). Consider $\pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, x_0)$. The vertical arrow in the following diagram is induced by the expression $(I^{n-1}, \partial I^{n-1}, J^{(n-2)}) \to (X^{(n-1)}, \{x_0\}, \{x_0\}) \to (X^{(n-1)}, X^{(n-2)}, \{x_0\})$:

$$\begin{tikzcd}
\pi_n(X^{(n)}, X^{(n-1)}, x_0) \arrow[r] \arrow[dr, "\partial"] & \pi_{n-1}(X^{(n-1)}, x_0) \cong \pi_n \arrow[d] \\
 & \pi_{n-1}(X^{(n-1)}, X^{(n-2)}, x_0)
\end{tikzcd}$$

Crossed complexes are a generalisation of the homotopy system, e.g. X CW-complex (pointed for simplicity). If $\pi(X)_n = \pi_n(X^{(n)}, X^{(n-1)}, x_0)$, then we have boundary maps $\pi(X)_n \to \pi(X)_{n-1} \to \ldots \to \pi(X)_1$. So

$$\ldots \to \pi(X)_n \to \pi(X)_{n-1} \to \ldots \to \pi_2(X^{(3)}, X^{(2)}) \to \pi_2(X^{(2)}, X^{(1)}) \xrightarrow{\partial} \pi_1(X^{(1)}).$$



- Modules generalise abelian groups and vector spaces;
- Chain complexes of modules over $\pi_1(X) = \pi_1(X^{(1)})/\partial\pi(X)_2$ and crossed modules are linked by actions;

- A crossed complex can be thought of as a crossed module with a ‘tail’.

Consider now a simplicial group $G_\bullet$ and let D_n be the subgroup of G_n generated by the degenerate elements.

$$\begin{tikzcd} G_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[l, bend left=20, "s_1"] \arrow[l, bend left=45, "s_0"] & G_0 \arrow[l, bend left=20, "s_0"] \end{tikzcd}$$

Let $z = s_0(g)s_1(g')s_0(g)^{-1}(s_1(gg'g^{-1}))^{-1} \in D_2 \subseteq G_2$ $(g, g' \in NG_1)$.
Then $d_2(z) = d_2s_0(g)d_2s_1(g')d_2s_0(g)^{-1}d_2(s_1(gg'^{-1}g^{-1}) = s_0d_1(g)g's_0d_1(g)^{-1}(gg'^{-1}g^{-1})$; $d_1(z) = 1$; and $d_0(z) = gs_0d_0(g')g^{-1}s_0d_0(gg'g^{-1})^{-1}$.

Exercise 4.1 Find an element z such that $z \in NG_2 \cap D_2$ (A Peiffer commutator).

Theorem 4.2 *(Ashley, Gerrasco-Cegarra): $NG_\bullet$ carries a crossed complex structure $\longleftrightarrow$ for $n \geqslant 2$, $NG_n \cap D_n$ is trivial.*

Now if $G_\bullet$ is a simplicial group,

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)\partial(NG_{n+1} \cap D_{n+1})};$$

$\rightarrow C(G)_{n+1} \xrightarrow{\partial_-} C(G)n \rightarrow \ldots$ is a crossed complex; and $C(G)_n$ ‘=’ $\pi_{n-1}(G^{(n)}, G^{(n-1)})$.

Any crossed complex is a Moore complex of a simplicial group(oid) (up to isomorphism) giving a full faithful functor

$$\mathrm{CrsComp} \rightarrow \mathrm{Simp.\,Gpd}$$

of which C is a left adjoint, *i.e.* CrsComp *is a ‘variety’ in the category of simplicial groupoids.*

Remark 4.3 There is an extensive homotopy theory of crossed complexes, including classifying spaces, vKT, equivariant theory and links with (non-abelian) cohomology. Prof. R. Brown and others including Prof. T Porter work in this area.

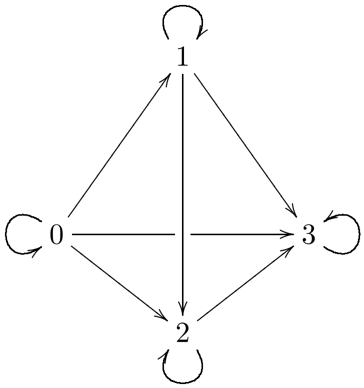
5 Seminar 5: 5th November 2002

5.1 Small Categories

Let $\mathcal{C}$ be a small category with objects C_0 and arrows C_1. If s = source = tail = domain and if t = target = head = codomain, consider the following directed graph and algebraic structure (partial composition ${}_t\times_s$):

$$\begin{tikzcd} C_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & C_0 \arrow[l, bend left, "e"] \end{tikzcd}$$

Example 5.1 The small category [3] with $C_0 = \{0, 1, 2, 3\}$ and $C_1 = \{(i, j) \mid i \leqslant j\}$ can be visualised as follows:



In this example, $s(i, j) = i$, $t(i, j) = j$, $e(i) = (i, i)$ and $se = te = \mathrm{id}_{c_0}$. In general, we have $C_1\, {}_t\times_s C_1 \to C_1$, where $C_1\, {}_t\times_s C_1 = \{(\alpha, \beta) \mid t(\alpha) = s(\beta)\}$. In this example, $C_1\, {}_t\times_s C_1 = \{(i, j)(j, k) \mid i, j, k, \text{ etc.}\}$, with $(i, j) \circ (j, k) \mapsto (i, k)$.

Example 5.2 Suppose that $\partial : C \to P$ is a crossed module. Consider the following diagrams, where $s(c, p) = p$, $t(c, p) = (\partial c)p$, and $e(p) = (1_c, p)$:

$$\begin{tikzcd} C \rtimes P \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & C_0 \arrow[l, bend left, "e"] \end{tikzcd} \quad ; \quad \begin{tikzcd} \underset{p}{\bullet} \arrow[r, "{(c,p)}"] & \underset{\partial c.p}{\bullet} \arrow[r, "{(c',\partial c.p)}"] & \underset{\partial(cc')p}{\bullet} \end{tikzcd} \quad = \quad \begin{tikzcd} \underset{p}{\bullet} \arrow[r, "{(c'c,p)}"] & \underset{\partial(cc')p}{\bullet} \end{tikzcd}$$

Exercise 5.3 Prove that the above composition is an internal one. *Hint:* Write down the group structure on $(C \rtimes P)\,{}_t\times_s (C \rtimes P)$ and then check it is a homomorphism.

5.2 The Nerve of a Small Category

Given any small category $\mathcal{X}$ we can form a simplicial $\mathbb{S}$et $\mathrm{Ner}(\mathcal{X})$, the nerve of $\mathcal{X}$.

$$\mathrm{Ner}(\mathcal{X})_n = \mathrm{Cat}([n], \mathcal{X}).$$

Consider $x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} x_n$. If $d_i(u_1, \dots, u_n) = (u_1, \dots, u_i u_{i+1}, u_{i+2}, \dots, u_n)$ for $0 < i < n$, then $d_0(u_1, \dots, u_n) = (u_2, \dots, u_n)$, $d_n(u_1, \dots, u_n) = (u_1, \dots, u_{n-1})$, and $s_i(u_1, \dots, u_n)$ = insert an identity at x_i.

Let us now abbreviate $C_1 \, {}_t\!\times_s C_1$ to $C_1 \overset{\times}{{}_{C_0}} C_1$ so that $\mathrm{Ner}(\mathcal{X})_n = X_1 \overset{\times}{{}_{X_0}} X_1 \overset{\times}{{}_{X_0}} \dots \overset{\times}{{}_{X_0}} X_1$ (n factors) and $d_i : \mathrm{Ner}(\mathcal{X})_n \to \mathrm{Ner}(\mathcal{X})_{n-1}$. The following illustrates the case $n = 3$, and this can be duplicated for internal categories in Groups:

$$X_1 \overset{\times}{{}_{X_0}} X_1 \overset{\times}{{}_{X_0}} X_1 \xrightarrow{d_1} X_1 \overset{\times}{{}_{X_0}} X_1.$$

Applying this to
$$\begin{tikzcd} C \rtimes P \arrow[r, shift left] \arrow[r, shift right] & P \arrow[l, bend right] \end{tikzcd}$$
gives a simplicial group $\mathrm{Ner}(\mathcal{C})$.

Fact: $N(\mathrm{Ner}(\mathcal{C})) : \ \dots \to 1 \to C \xrightarrow{\partial} P$. *Check:* Let $\mathcal{G} = \mathrm{Ner}(\mathcal{C})$. Then we have the following table:

$\mathcal{G}_2$	$\mathcal{G}_1$	$\mathcal{G}_0$
$$\begin{tikzcd} & \partial c.p \arrow[dr, "(c',\partial cp)"] & \\ p \arrow[ur, "(c,p)"] \arrow[rr, "(cc',p)"'] & \sigma & \partial(cc')p \end{tikzcd}$$ $(C \times C) \rtimes P$	$$\begin{tikzcd} \bullet \arrow[r, "(c,p)"'] & \bullet \end{tikzcd}$$ $C \rtimes P$	P

Notes: $NG_1 \cong C$; $\ker d_1 = \{(c,p) \mid d_1(c,p) = 1_P\}$; $d_1(c,p) = p$; and $NG_0 = P$. Further, for $\mathcal{G}_2$ above, we have $d_1(\sigma) = (c'c, p) = (1_C, 1_P)$ if $p = 1_P$ and $c' = c^{-1}$; $d_2(\sigma) = (c,p) = 1$ if $c = 1_C$ and $p = 1_P$; and $\sigma \in \ker d_1 \cap \ker d_2 \Rightarrow \sigma = (1_C, 1_C, 1_P)$. Finally, if G has $NG =$ a crossed module $\mathcal{C}$, then $G \cong \mathrm{Ner}(\mathcal{C})$.

5.3 Crossed n-cubes

A crossed n-cube is an n-fold crossed module (Loday, 1984). As an example, consider the case $n = 2$. Let G, N_1 and N_2 be normal subgroups of a group G. We have a h-map $h : N_1 \times N_2 \to N_1 \cap N_2$ ($(n_1, n_2) \mapsto [n_1, n_2]$) and the following diagram:

$$\begin{tikzcd} N_1 \cap N_2 \arrow[r, "\triangleleft"] \arrow[d, "\triangle"'] & N_2 \arrow[d, "\triangle"] \\ N_1 \arrow[r, "\triangleleft"'] & G \end{tikzcd}$$

Generally, a crossed square has the form

$$\begin{tikzcd} L \arrow[r, "\lambda'"] \arrow[d, "\lambda"'] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd}$$

where μ, λ, ν and λ' are crossed modules and we have a map $h : M \times N \to L$.

A better formulation (Ellis-Steiner) is (sketch) to let $< 2 > = \{1, 2\}$, $A \subseteq < 2 >$, $i \in < 2 >$ and $\mu_i : M_A \to M_{A \setminus \{i\}}$. Then $h : M_A \times M_B \to M_{A \cup B}$ gives actions ${}^a b = h(a, b)b$ and we have 11 actions similar to commutator rules. In the following diagram, the h-map is $h : M_\phi \times M_{\{1\}} \to M_{\{1\}}$:

$$\begin{tikzcd}
M_{<2>} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & M_{<1>} \arrow[d, "\mu_1"] \\
M_{\{2\}} \arrow[r, "\mu_2"'] & M_\phi
\end{tikzcd}$$

For a crossed n-cube, replace $< 2 >$ by $< n >$ — and we still have 11 axioms! Now $(G, N_1, \ldots, N_n)$ gives a crossed n-cube, $(G_\bullet, N_{1\bullet}, \ldots, N_{n\bullet})$ is a simplicial groupoid and n normal subgroups, and $\pi_0(G_\bullet)$, etc. gives a crossed n-cube — and all are like this!

Consider $n = 3$: (G, N_1, N_2, N_3) and $M_A = \cap\{N_i \mid i \in A\}$:

$$\begin{tikzcd}
 & & N_1 \cap N_2 \cap N_3 \arrow[rr] \arrow[dll] \arrow[dd] & & N_2 \cap N_3 \arrow[dd] \\
N_1 \cap N_3 \arrow[rrr] \arrow[dd] & & & N_3 \arrow[ur] \arrow[dd] & \\
 & & N_1 \cap N_2 \arrow[rr] \arrow[dll] & & N_2 \arrow[dl] \\
N_1 \arrow[rrr] & & & G &
\end{tikzcd}$$

A fibrant n-cube of spaces is as follows, where all f's are fibrations, $E \to E_1 \underset{B}{\times} E_2$ is a fibration, and π_1(Square of fibres) is a crossed square:

$$\begin{tikzcd}
E \arrow[r, "f_1'"] \arrow[d, "f_2'"'] & E_2 \arrow[d, "f_2"] \\
E_1 \arrow[r, "f_1"'] & B
\end{tikzcd}$$

$M(G,2)$ is the following crossed square which generalises to $M(G,n)$ (the $M(G,n)$'s are linked via extensions and correspond to a variety of simplicial groupoids):

$$\begin{tikzcd}
\dfrac{NG_2}{\partial(NG_3)} \arrow[r] \arrow[d] & \ker d_1 \arrow[d] \\
\ker d_0 \arrow[r] & G_1
\end{tikzcd}$$

6 Seminar 6: 6th December 2002

6.1 Chain Complexes

Take R to be a ring. Let us consider some examples of R-modules.

- $R = F$, a field: we have vector spaces.
- $R = \mathbb{Z}$: we have abelian groups.
- $R = F[x]$: we have vector spaces with an endomorphism $T : V \to V$. Also, if we have $v \in V$ then $p(x) = \sum_{n=0}^{N} a_n x^n$ and $p(x).V = \sum_{n=0}^{N} a_n T^n(v)$.
- $R = k(G)$, where k is a commutative ring: we have a group algebra. Now assume that G is finite with $\#(G)$ elements. Then $k(G) \sim k^{\#(G)}$, the free k-module, with basis $\{\underline{e}_g : g \in G\}$. So if $\underline{x} \in k^{\#(G)}$, i.e. $\underline{x} = \sum x_g \underline{e}_g$, then define multiplication $k^{\#(G)} \times k^{\#(G)} \to k^{\#(G)}$ by $\underline{e}_g.\underline{e}_h = \underline{e}_{gh}$ and extending linearly. Further, if $g \to G\ell(n, k)$ with $n = \#(G)$, then $\rho(g)(e_h) = \underline{e}_{gh}$ is the left regular representation.

A right $k(G)$ module M is a right G-module M_k and also a homomorphism $G \to k$ Linear $\mathrm{Aut}(M_k)$, where M_k is the underlying (free) k-module. Further, if G is a small category $\mathbb{G}$ with one object, then Mod-$k(G) \approx (\text{Mod } k)^{\mathbb{G}}$.

Definition 6.1 A chain complex is a sequence

$$(\mathcal{C}, d) = \ldots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \ldots$$

of R-modules and R-mod morphisms such that $d_{n-1} d_n = 0$.

$(\mathcal{C}, d)$ is positive if $C_n = 0$ for all $n \leqslant 0$; negative if $C_n = 0$ for all $n \geqslant 0$; and bounded if there exists an $N \in \mathbb{N}$ such that $C_n = 0$ if $|n| > N$. The morphisms of chain complexes $\underline{f} = f_n$ such that $d_n^0 f_n = f_{n-1} d_n^C$ can be visualised as follows:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
C_n \arrow[r, "f_n"] \arrow[d, "d_n^C"'] & D_n \arrow[d, "d_n^D"] \\
C_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & D_{n-1} \arrow[d] \\
{} \arrow[d, dashed, no head] & {} \arrow[d, dashed, no head] \\
{} & {}
\end{tikzcd}$$

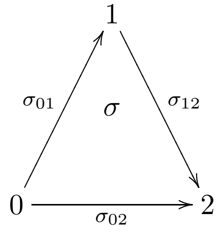
ChMod_R is the category of chain complexes in Mod-R, used in suspension and desuspension / looping and 1st cylinders and cocylinders.

Let us now take $\mathcal{C}$ to be a chain complex and let us form a new chain complex $(I\otimes C)_n = C_n \oplus C_n \oplus C_{n-1}$ by defining $d(x,y,z) = (dx - z, dy + z, -dz)$.

Exercise 6.2 Show that $d^2 = 0$, where $d^2 = (d(dx-z) - dz, d(dy+z) - dz, -d(-dz))$.

6.2 An Aside on some Topological Examples

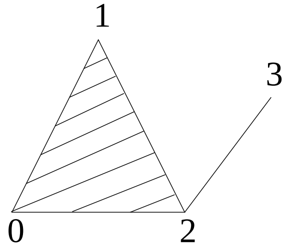
Example 6.3 Consider a simplicial set K, where K is a commutative ring. Then $C(K)_n = K^{K_n} =$ functions (K_n, K) of finite support, and $\underline{x} = \sum_{\sigma\in K_n} x_\sigma \sigma$, where $\{x_0\}$ is of finite support, i.e. $\{\sigma \,|\, x_\sigma \neq 0\}$ is finite. Further, $d\underline{x} = \sum x_0 d(\sigma)$, where $d(\sigma) = \sum_{i=0}^{n}(-1)^i d_i(\sigma)$, e.g. the diagram shown below, where $\sigma_{02} = d_1(\sigma)$, and $d(\sigma) = d_0(\sigma) - d_1(\sigma) + d_2(\sigma) = \sigma_{12} - \sigma_{02} + \sigma_{01}$:



Example 6.4 Simplicial Complexes. Consider a set $V(K)$ of vertices and a family

$$S \subset \mathcal{P}_{\text{finite}}(V(K)) \setminus \{\phi\}$$

closed under 'subset of': finite $\sigma \subset V(K)$, $\sigma \in S$, $\sigma \neq \phi$ and $\tau \neq \phi$; $\tau \subset \sigma \Rightarrow \tau \in S$. In the following diagram, we have $V(K) = \{0,1,2,3\}$ and $S = \{\{0,1,2\},\{0,1\},\{1,2\},\{0,2\},\{2,3\},\{0\},\{1\},\{2\},\{3\}\}$:



To get from K to a simplicial set, pick a total order on $V(K)$, e.g. $2 < 1 < 3 < 0$. Then K_S is as follows: $(K_S)_0 = \{[0],[1],[2],[3]\}$, $(K_S)_1 = \{[2,1],[2,3],[1,0],[2,0],[0,0],[1,1],[2,2],[3,3]\}$, $(K_S)_2 = \{[2,1,0],[2,2,1],[2,1,1], \text{etc.}\}$, $\ldots$, $(K_S)_n = \{[a_0,\ldots,a_n] \,|\, \{a_0,\ldots,a_n\}$ (as set) is in S and $a_0 \leqslant \ldots \leqslant a_n\}$. Now $d_i : [a_0,\ldots,a_n] = [a_0,\ldots,\hat{a}_i,\ldots,a_n]$ (omit the ith position), and we can define a chain complex $C(K)$ via $C(K_S)$ or directly.

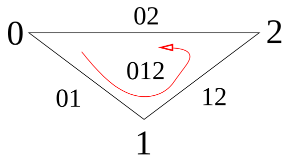
Proposition 6.5 *Given any chain complex* $(C_\bullet, d_\bullet)$,

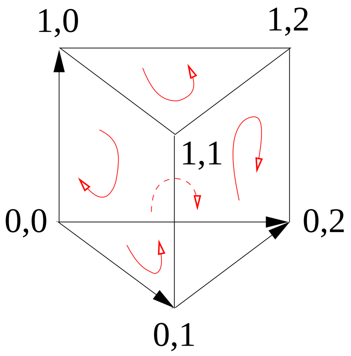
$$H_n(C_\bullet) = \frac{\ker d_n}{\operatorname{Im} d_{n+1}} \sim \text{ n}^{\text{th}} \text{ homology group.}$$

$C(K)_n$ is the free K-module on n simplices, i.e. the subsets in S with $n+1$ elements. We need to put a local order on the simplicies to obtain d, e.g. put a total order on $V(K)$ and show that $H_n(C(K))$ does not depend on the choice of order. *Fact:* Let $\chi_i(K)$ be the number of i simplices in K and let $\chi(K) = \sum(-1)^i\chi(K)$ be the Euler-Poincaré characteristic. Over $\mathbb{Z}$, $rk(H_n(C(K)) \sim$ nth Betti number $\beta_n(K)$, and $\sum(-1)^n\beta_n(K) = \chi(K)$.

7 Seminar 7: 17th December 2002

Recall the formula $(I \otimes C)_n = C_n \oplus C_n \oplus C_{n-1}$ and consider the diagrams below:





For the first diagram above, we have $C_0 = < e_0, e_1, e_2 \mid \phi >_{\text{Ab}}$, $C_1 = < e_{01}, e_{12}, e_{02} \mid \phi >_{\text{Ab}}$ (with $\partial(e_{ij}) = e_j - e_i$), and $C_2 = < e_{012} \mid \phi >$ (with $\partial(e_{012}) = e_{01} + e_{12} - e_{02}$).

For the second diagram above, we have $(I \otimes C)_0 = \underset{(0)}{C_0} \oplus \underset{(1)}{C_0}$ and $(I \otimes C)_1 = \underset{(0)}{C_1} \oplus \underset{(1)}{C_1} \oplus \underset{(01)}{C_0}$. Now $\underset{(0)}{C_1} = < e_{0,01}, e_{0,12}, e_{0,02} \mid \phi >_{\text{Ab}}$, etc., and $\underset{(01)}{C_0} = < e_{01,0}, e_{01,1}, e_{00,2} \mid \phi >_{\text{Ab}}$. Further, $(I \otimes C)_2 = \underset{(0)}{C_2} \oplus \underset{(1)}{C_2} \oplus \underset{(01)}{C_1}$ and $(I \otimes C)_3 = C_3 \oplus C_3 \oplus C_2$. So we have (for example) $\partial(e_{0,ij}) = e_{0,j} - e_{0,i}$, etc.; $\partial(e_{0,012}, 0, 0) = (e_{0,01} + e_{0,12} - e_{0,12}, 0, 0)$; and $\partial(0, 0, e_{01,ij}) = (-e_{01,j}, e_{1,ij}, e_{01,i} - e_{01,j})$.

Example 7.1 Recall that $\partial(x, y, z) = (\partial x - z, \partial y + z, -\partial z)$. Consider $I = 0\bullet \text{———} \bullet 1$ and let R be a commutative ring. Then $I_0 = Re_0^n \oplus Re_1^n$ (where the superscript of the e indicates the dimension); $I_1 = Re^1$; and $\partial(e^1) = e_1^0 - e_0^0$.

7.1 Tensor Products of Chain Complexes

Consider

$$\text{Set}(X \times Y, Z) \cong \text{Set}(X, Y \to Z),$$

where $f(x, y) \in Z$, the right hand side is cartesian closed, and $Y \to Z$ can be written as Z^Y or as $\text{Set}(Y, Z)$. What should we have for the question mark in

$$\text{VSpaces}\,(\mathbb{R}^\ell \;?\; \mathbb{R}^m, \mathbb{R}^n) \cong \text{VSpaces}(\mathbb{R}^\ell, \underline{\text{Lin}}(\mathbb{R}^m, \mathbb{R}^n)),$$

where the right hand side has dimension $\ell(mn)$? We cannot have '$? = \oplus$' as the left hand side would therefore have dimension $(\ell + m)n$. So we rather form $\mathbb{R}^\ell \otimes \mathbb{R}^m$ with basis $\underline{e}_i^\ell \otimes \underline{e}_j^m$ — and the conclusion is that VSpaces is not cartesian closed but is monoidal closed.

7.2 Back to the Aside!

Consider $e_0 : C \to I \otimes C$ $(x \mapsto (x, 0, 0))$; $(I \otimes C) \to C$ $((x, y, z) \mapsto x + y)$; and $e_1(y) = (0, y, 0)$. Now if we have $f_0, f_1 : C \to D$ let us consider the homotopy $h : f_0 \approx f_1$ with $he_0 = f_0$ and $he_1 = f_1$ defined by

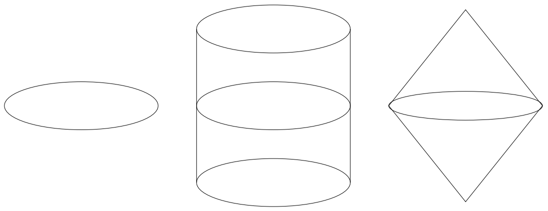
$$h(x, y, z) = h(x, 0, 0) + h(0, y, 0) + h(0, 0, z) = f_0(x) + f_1(y) + h'(z).$$

Now $d^D h = h d^{I \otimes C}$ since h is a chain map; $h(d^C x - z, d^C y + z, -d^C z) = f_0(d^C x) + f_1(d^C y) - f_0(z) + f_1(z) - h'(d^C z)$; and $d^D h(x, y, z) = d^D f_0(x) + d^D f_1(y) + d^D h'(z)$.

Let us now consider the chain homotopy $h' : C_{n-1} \to D_n$ of degree 1, where $d^D h' + h' d^C = f_1 - f_0$ so that $f_1 = f_0 + d^D h' + h' d^C$. The following diagram shows that chain homotopies are degree 1 maps and also shows that the homotopy h' is really a pair (f_0, h') from which f_1 and hence h can be retrieved:

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] \\
C_{n+1} \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, "d^C"'] & D_{n+1} \arrow[d, "d^D"] \\
C_n \arrow[ur, "h'"] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d] & D_n \arrow[d] \\
C_{n-1} \arrow[ur, "h'"] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, dashed] & D_{n-1} \arrow[d, dashed] \\
{} & {}
\end{tikzcd}$$

7.3 Suspension and Desuspension



First diagram: Space X (C). Second diagram: $I \times X$ $(I \otimes C)$. Third diagram: ΣX, suspension (ΣC, $(\Sigma C)_n = C_{n-1}$, $d^{\Sigma C} z = -d^C z$).