PhD Seminars 2002/2003: Prof. R. Brown Semester 1

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Contents

1 Seminar 1: 12th November 2002

1.1 The Van Kampen Theorem

1.1.1 History

Local to global theorems / problems such as integration and the Euler characteristic are very important in mathematics. One such case that we will be looking at in these seminars will be the application and uses of the fundamental group $\pi_1(X, x)$ of a space X at a point x.

The fundamental group was first mentioned by Poincarè in 1895 in his monodromy in complex variable theory. A problem is that the fundamental group depends on the base point x, and in general is not commutative – which is important in applications in geometry and analysis.

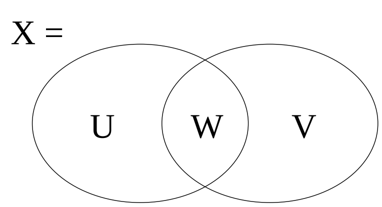
So how do we calculate $\pi_1(X, x)$? Early on, the 'edge path group' $\pi_1(K, x)$ was calculated, where K is a simplicial complex. In this situation, the loops in the 1-skeleton give generators and triangles give relations.

Seifert looked at the case where $K_1 \cup K_2 = K$, $K_1 \cap K_2 = K_0 \ni X$, and K_0 is connected. He deduced from the edge path groupoid that $\pi_1(K, x) = \pi_1(K_1, x) *_{\pi_1(K_0,x)} \pi_1(K_2, x)$.

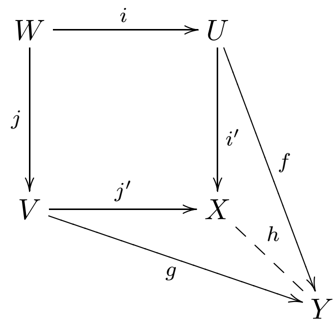
Van Kampen was asked for the fundamental group of the complement of a complex algebraic curve $f(z, w) = 0$ in $\mathbb{C}^2$. Van Kampen wrote down some formulae for $\pi_1(U \cup_W V, x)$ but the 'proof' was incomprehensible! However, he 'did' do the case where W is not necessarily connected, a case which is needed for applications. In this proof, a nice concept was introduced of a *pushout*, which is now relevant to local to global problems and gluing.

1.1.2 Pushouts

Consider the space X shown below:

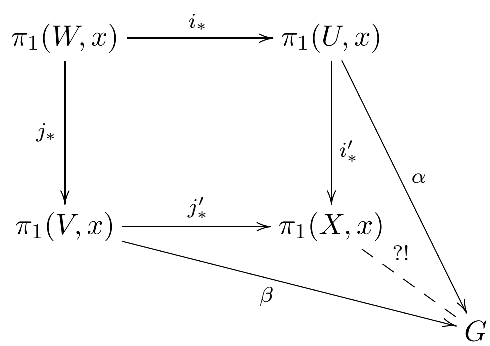


If U and V are open then this is a pushout: Given f and g such that $fi = gj$ then there exists a unique $h:\ X \to Y$ such that $hi' = f$ and $hj' = g$:



In sets, we need only have $W = U \cap V$. In topology, we need an extra condition such as $\operatorname{Int} U \cup \operatorname{Int} V = X$ (We are OK if U and V are closed).

Question: Is the following a pushout of groups?



Answer (Crowell, 1960): Yes, if U and V are open and W is path connected. Note: this is a non-commutative local to global theorem.

1.1.3 More History

The topologists of the early 20$^{\text{th}}$ century knew that $\pi_1(X, x)^{\text{ab}} \cong H_1(X)$ if X was commutative and that $H_n(X)$ existed for all $n \geqslant 0$. The next task was to generalise $\pi_1(X, x)$ (the non-commutative case) to all dimensions.

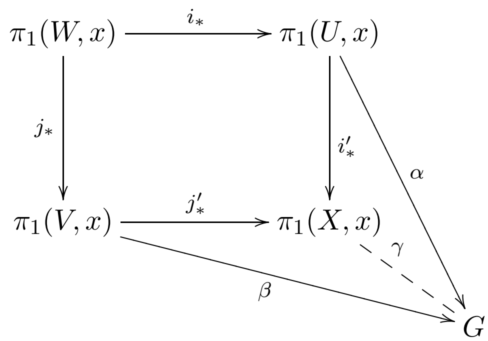
Cech wrote a paper in 1932 on $\pi_n(X, x)$ for $n \geqslant 2$. Alexandroff and Hopf proved abelian for $n \geqslant 2$, and in 1935 Hurewitz wrote the first of four notes on $\pi_n(X, x)$. The work of JHC Whitehead during 1941-1949 on crossed modules was also very important.

1.1.4 An Application

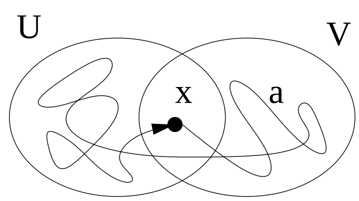
Consider $a:\ I^n \to X$, where I^n is metric and compact (but note that X is not necessarily metric or compact). Let $\mathcal{U}$ be an open cover of X, and let us look at $a^{-1}(\mathcal{U})$ = an open cover of I^n. Conclusion:

we can subdivide I^n into little cubes so that a(a little cube) $\subseteq$ some set of $\mathcal{U}$.

Let us consider the following diagram again:

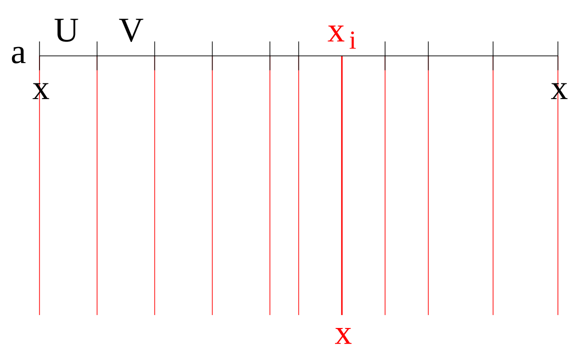


If we have $a : (I, \dot{I}) \to (X, x)$, how do we get $\gamma[a] \in G$? The solution lies in using the Lebesque Covering Lemma to write $a = a_n + \ldots + a_1$ such that each a_i has image in either U or V.

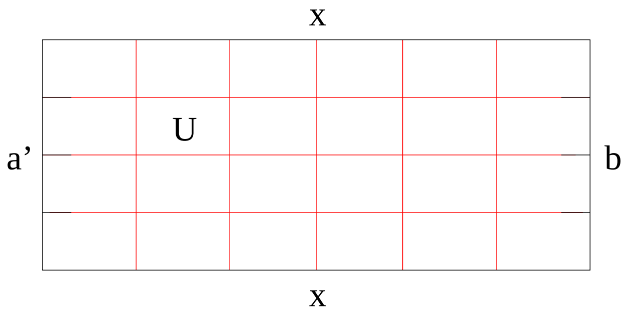


Remark 1.1 An annoyance in using this method is that a_i does not have end points at x. We could use $\pi_1(X)$ = the fundamental groupoid, and we could easily get the result this way, but we don't get an answer for $\pi_1(S^1, 1)$ when using this method!

For a point x_i in the path a, join x_i to the point x by a path in W, U and V according to whether the point x_i is in U or in V.

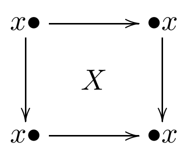


Fill to get $a \simeq a' = a'_n + \ldots + a'_1$, and then 'lift' $[a'_i]_U \subset \pi_1(U, x)$ or $[a'_i]_V \subset \pi_1(V, x)$. Map over to G, and then use $\alpha_{i_*} = \beta_{j_*}$ to show that the images in G may be composed to give $b = b_n + \ldots + b_1 \in G$. This 'defines' $\gamma[a]$, but we have lots of choices to make!

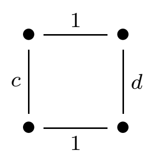


For $h:\ I^2 \to X$, subdivide refining subdivisions at each end. Deform h into $h' = [h'_{ij}]$ and then look at $h' = [h'_{ij}]$: h'_{ij} is in U or V with vertices at x.

Basic Fact: If we have $I^2 \xrightarrow{c} X$ with vertices at x, we get a commutative square in $\pi_1(X, x)$:



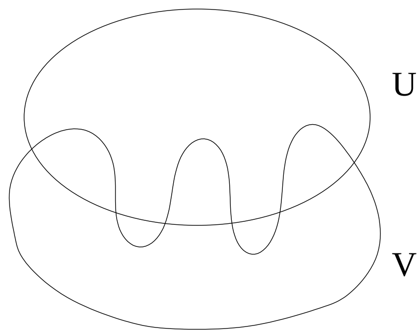
In particular, we get commutative squares in $\pi_1(U, x)$ or $\pi_1(V, x)$ and so in G. But any composite of commutative squares in a group is commutative. In particular, we have what is shown below so that $c = d$ in G.



2 Seminar 2: 19th November 2002

2.1 Using Groupoids

In the last seminar, we looked at a space $X = U \cup V$ where U and V were open and $W = U \cap V$ was path connected. But if W is path connected, we can't calculate $\pi_1(S^1, 1)$. Standard treatments go into covering spaces but what if we wanted to consider the following space?



Hu used $\pi_1 X$ = the fundamental groupoid = homotopy classes rel end points of maps $I \to X$. In 1963 Higgins wrote about presentations of groupoids with applications to groups and defined free products with amalgamation of groupoids. For example, the diagram below is a pushout of groupoids, and the proof is beautiful — we just subdivide, with no deformation arguments needed.

$$\begin{array}{ccc} \pi_1 W & \longrightarrow & \pi_1 U \\ \downarrow & & \downarrow \\ \pi_1 V & \longrightarrow & \pi_1 X \end{array}$$

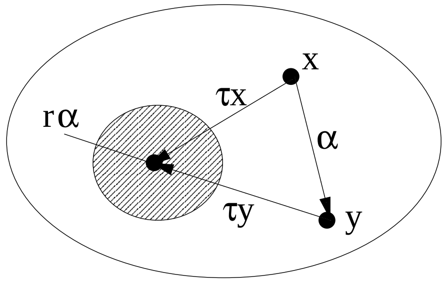
But the problem of calculating the $\pi_1(S^1, 1)$-group still exists. The crucial point is that instead of considering a single base point x, we need to consider a set of base points A.

$$\begin{array}{ccc} \pi_1(W, A) & \longrightarrow & \pi_1(U, A) \\ \downarrow & & \downarrow \\ \pi_1(V, A) & \longrightarrow & \pi_1(X, A) \end{array}$$

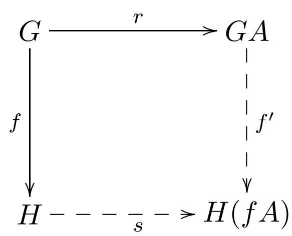
We need U and V open, etc. as before; A meets each path component of W (usually $A \subseteq W$); and there is 'no' change in the proof.

Trick: In groupoids, we use retractions. Let $G = \mathrm{Ob}(G)$. Then $A \subseteq \mathrm{Ob}(G)$ meets each component of G; GA is the full subgroupoid of G on A; and $GA \underset{r}{\overset{i}{\rightleftarrows}} G$ is a strong deformation retract: for each

$x \in \mathrm{Ob}(G)$, choose a $\tau x : x \mapsto rx$ in G such that $rx \in A$ (and if $x \in A$ then $\tau x = 1_x$). In the diagram below, $r\alpha = (\tau x)^{-1}\alpha(\tau y)$.

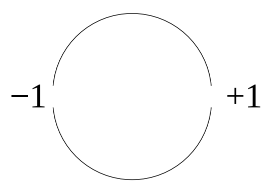


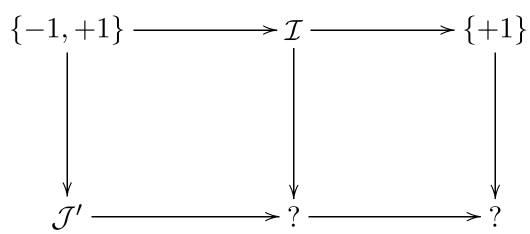
Lemma 2.1 *Consider the diagram shown below:*



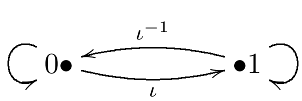
If Ob(f) *is bijective then we can complete the diagram with the dotted lines* s *and* $f' = f|_{GA}$ *as shown and this is a pushout in* Gpd*, the category of groupoids.*

Let us now consider $S' = E'_+ \cup E'_-$:

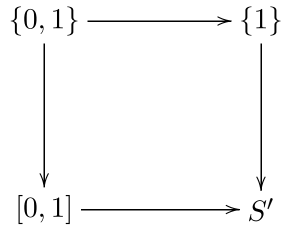




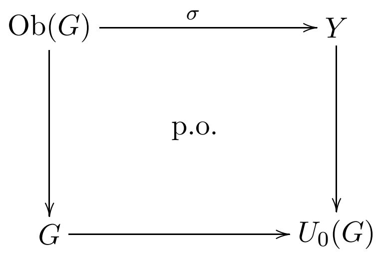
$D(X)$ is the discrete groupoid, only containing identities. $I(X)$ is the indiscrete groupoid, where $\mathrm{Ob}(I(X)) = X$, $\mathrm{Arr}(I(X)) = X \times X$, and we have composition $(x, y)(y, z) = (x, z)$. In the above diagram, $I(\{0, 1\}) = \mathcal{I}$ is the 'transition groupoid', shown below. The conclusion for all of the above is that we get a homotopy theory for groupoids and the '?' in the bottom right hand corner of the above diagram must be $\mathbb{Z}$ by the universal property.



$\mathcal{I}$ does for groupoids what $\mathbb{Z}$ does for groups – compare with



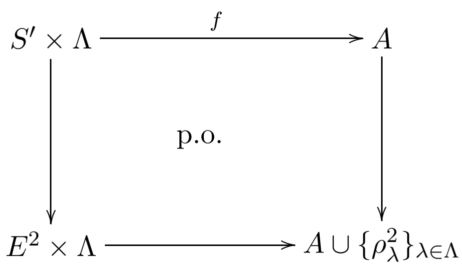
A nice general construction is the following, where the $U_0(G)$ in the diagram is interchangeable with $\sigma_*(G)$:



The basic pictures are as follows: In dimension 1, we have intervals and composing commuting squares; and in dimension 2, we have squares and composing commutative cubes.

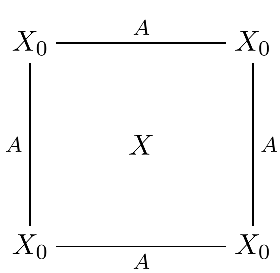
2.2 Whitehead's Theorem

Theorem 2.2 *$\pi_2(A \cup \{\rho^2_\lambda\}, A, x) \to \pi_1(A, x)$ is the free crossed module on "2-cells", with $\lambda \in \Lambda$ (Λ is the discrete topology).*



$$(E^2, S^1) \xrightarrow{f_\lambda} (A \cup \{\rho^2_\lambda\}_\lambda, A)$$

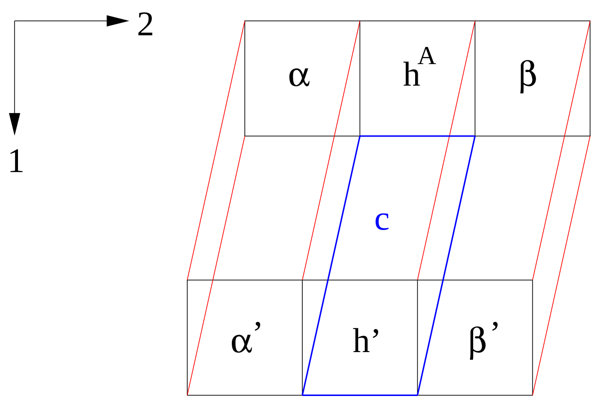
Whitehead's theorem is about the second relative homotopy group. It is a universal result in two-dimensional homotopy theory, and our proposed theory ought to recover the theorem. We'd better then have a functor from pairs (X, A) of spaces, where $A \subseteq X$. Take $X_0 \subseteq A$, a set of base points.



This gives the following:

$$\begin{array}{c} \rho_2(X, A, X_0) \\ \downarrow \ \downarrow \ \downarrow \ \downarrow \\ \pi_1(A, X_0) \\ \downarrow \ \downarrow \\ X_0 \end{array}$$

Let $\alpha \in \rho_2(X, A, X_0)$. Consider the diagram shown below:

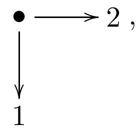


If $h: \partial_2^+ \alpha \simeq \partial_2^- \beta$ rel vertices, if the bottom is filled by a constant c, and if the top face is in A, then we get a homotopy $\alpha \circ_2 h \circ_2 \beta \cong \alpha' \circ_2 h \circ_2 \beta'$; $[\alpha] \circ_2 [\beta] =! \; [\alpha \circ_2 h \circ_2 \beta]$. It is not hard now to show that ρ_2 is a double groupoid.

3 Seminar 3: 26th November 2002

3.1 The Interchange Law

If we have $\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$ and the directions



the interchange law is as follows (whenever these compositions are defined):

$$(\alpha \circ_1 \gamma) \circ_2 (\beta \circ_1 \delta) = (\alpha \circ_2 \beta) \circ_1 (\gamma \circ_2 \delta).$$

A famous result is that if $\circ_1$ and $\circ_2$ are monoid structures then $\circ_1 = \circ_2$ and they are commutative. Does this have any connection with the fundamental group? Note that if $\circ_1$ is a group and $\circ_2$ is a groupoid then this is equivalent to a crossed module.

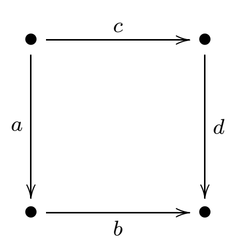
We need an interchange law for $\rho_2(X_*)$. Let $\alpha, \beta, \gamma, \delta \in \rho_2(X_*)$, let double composition be defined, and let $a \in \alpha$, $b \in \beta$, $c \in \gamma$, $d \in \delta$.

$$\begin{bmatrix} a & h & b \\ k & \square & k' \\ c & h' & d \end{bmatrix}$$

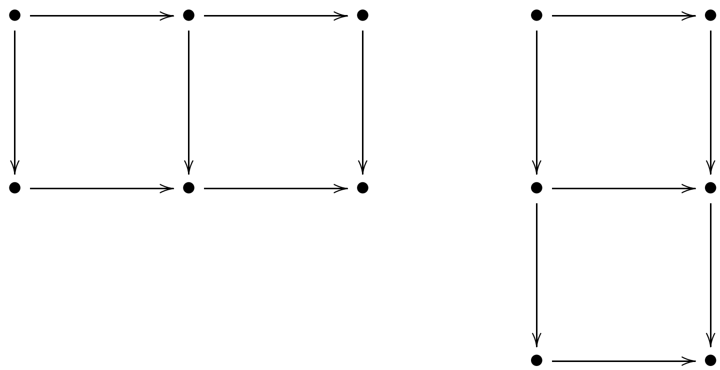
Let h, h', k and k' be homotopies rel end points of paths, e.g. $h : \partial_2^+ a \cong \partial_2^- b$ and $k : \partial_1^+ a \cong \partial_1^- c$. Fill in the hole and then read off the interchange law: $(\alpha \circ_2 \beta) \circ_1 (\gamma \circ_2 \delta) = [a \circ_2 h \circ_2 b] \circ_1 [c \circ_2 h' \circ_2 d] =$ the matrix shown above.

3.2 Double Categories

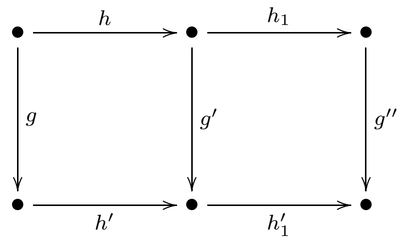
Ehresmann defined double categories in the book 'Categories and Structures' (1964). As an example, consider $\square\, \mathbb{C}$, the double category of commuting squares in $\mathbb{C}$, i.e. we have $ab = cd$ in the following diagram:



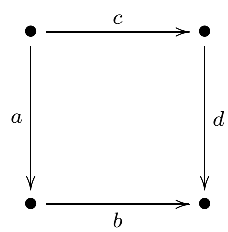
In this case, it is easy to check commutativity:



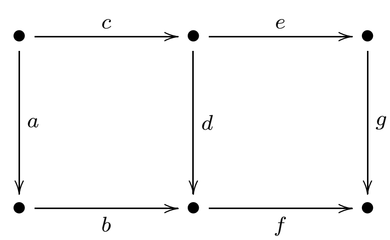
In particular, $\mathbb{C}$ could be a group — take, for example, groups G and H and construct $G\bar{\times}H$ (the compositions $\circ_1$ and $\circ_2$ are obvious but we need to check the interchange law):



Example 3.1 Let G be a group and consider a subgroup $N \leqslant G$.



In the above diagram, consider that $b^{-1}a^{-1}cd \in N$.



Now if we have $b^{-1}a^{-1}cd \in N$ and $f^{-1}d^{-1}eg \in N$ above, we would like to have $f^{-1}b^{-1}a^{-1}ceg = f^{-1}(b^{-1}a^{-1}cd)f \in N$ and $f^{-1}d^{-1}eg \in N$. For this, we need N to be normal in G. But an element of N is determined by the boundary, and so we can replace $N \lhd G$ by $\partial : C \to G$. It follows that for the square

$$\left(n : \ a \ \begin{matrix} c \\ b \end{matrix} \ d \right)$$

shown above, we have $\partial n = b^{-1}a^{-1}cd$. For the composition

$$\left(n:\ a \genfrac{}{}{0pt}{}{c}{b} d\right) \circ_2 \left(m:\ d \genfrac{}{}{0pt}{}{e}{f} g\right),$$

we need that G operates on C to give

$$\left(n^f m:\ a \genfrac{}{}{0pt}{}{ce}{bf} g\right).$$

Further, $\partial(n^f) = f^{-1}(\partial\, n)f$ (but we need a pre-crossed module).

We can now go on to work out $\circ_1$ and can therefore deduce that the interchange law corresponds to CM2 for a crossed module.

3.3 Double Groupoids

In general, a double groupoid can be visualised as follows, where S = Squares, H = Horizontal Edges, V = Vertical Edges and P = Points:

$$\begin{array}{ccc} S & \rightrightarrows & H \\ \downdownarrows & & \downdownarrows \\ V & \rightrightarrows & P \end{array}$$

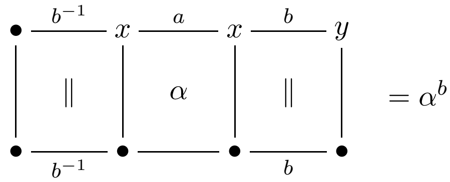
The double category is edge symmetric if $H = V$, and we then have the notion of elements of S with commuting boundaries.

$\lambda(C \to G)$ has special commuting squares.

A double groupoid morphism has the form $\Theta : \square G \longrightarrow \lambda(C \xrightarrow{\partial} G)$. If $\mathbb{D}$ is an edge symmetric double groupoid and if $\mathbb{D}_1 = G$ is a horizontal (vertical) edge groupoid, then $\Theta : \square \mathbb{D}_1 \to \mathbb{D}$ respects $\circ_1$ and $\circ_2$ and is the identity on $\mathbb{D}_1$. Note that we could also drop the edge symmetric condition and take $\Theta : \square G \to \mathbb{D}$.

Theorem 3.2 *λ defines an equivalence of categories, Crossed Modules of Groupoids $\xrightarrow{\lambda}$ Edge Symmetric Double Groupoids with Thin Structure (Brown & Spencer, 1972).*

Now if $\mathbb{D} = \begin{array}{ccc} S & \rightrightarrows & H \\ \downdownarrows & & \downdownarrows \\ V & \rightrightarrows & P \end{array}$, we should have H operating on $\gamma_H(\mathbb{D})$:



Now $\alpha^b = [(\epsilon, b)^{-1}, \alpha, \epsilon_1 b]_2$. Clearly $\partial(\alpha^b) = b^{-1}(\partial\alpha)b$; and also it is an operation $(\alpha\beta)^b = \alpha^b\beta^b$.

$$\begin{bmatrix} (\epsilon, b)^{-1} & \alpha & \epsilon_1 b \\ \beta^{-2} & \square & \beta \end{bmatrix} \quad \begin{bmatrix} H & \alpha & 1 \\ \beta^{-1_2} & \square & \beta \end{bmatrix}$$

The interchange law implies that $\beta^{-2}\alpha\beta = \alpha^{\partial\beta}$. In particular, $\gamma_H(\rho_2(X_*))$ is a crossed module of groupoids.

Remark 3.3 There exists a map $\pi_2(X, X_1, x) \to \rho_2(X, X_1, X_0)$.

Exercise 3.4 Show that $\gamma\lambda(C \to G) \cong (C \to G)$.

Exercise 3.5 (Hard!) Show that there exists an isomorphism $\lambda\gamma(\mathbb{D}) \cong \mathbb{D}$ so that (in some way!) $\gamma(\mathbb{D}) \subseteq \mathbb{D}$ and $\mathbb{D}$ can be rebuilt from $\gamma(\mathbb{D})$. General Method: we need a $\mathbb{D}_2 \xrightarrow{\phi} \lambda\gamma(\mathbb{D})_2$ and a $\lambda\gamma(\mathbb{D})_2 \xrightarrow{\psi} \mathbb{D}_2$ so that $\phi\psi = 1$ and $\psi\phi = 1$. Write down $\lambda\gamma(\mathbb{D})_2$ to see what is involved.

4 Seminar 4: 13th December 2002

In the last seminar, we looked at γ_H and γ_V, we looked at the theorem stating that there is a λ: crossed module $\to$ edge symmetric double groupoids with thin structure, and we stated the adjoint equivalence of the categories. Today, we shall look at some of the features of the proof.

The easy part of the proof is to check that $\gamma_H \lambda \simeq 1$; and the difficult part is to prove that $\lambda\gamma \simeq 1$, which says that these double groupoids can be reconstructed from the crossed module they contain. But before we go any further, we need to mention connections.

4.1 Connections

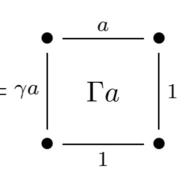
In a double groupoid (category) with thin structure we have thin elements:

$$\begin{pmatrix} & a & \\ 1 & & 1 \\ & a & \end{pmatrix} = (\,\|\,),\ \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} = (\,=\,),\ \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix} = (\,\ulcorner\,),\ \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} = (\,\square\,),\ \text{and}\ \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} = (\,\lrcorner\,).$$

Proposition 4.1 *We have the following simplifications (the final two simplifications are known as the transport laws):*

$$\begin{bmatrix} \ulcorner \\ \lrcorner \end{bmatrix} = (\,a = a\,);\ \begin{bmatrix} \ulcorner & \lrcorner \end{bmatrix} = (\,\|\,);\ \begin{bmatrix} \ulcorner & = \\ \| & \ulcorner \end{bmatrix} = (\,\ulcorner\,);\ \text{and}\ \begin{bmatrix} \lrcorner & \| \\ = & \lrcorner \end{bmatrix} = (\,\lrcorner\,).$$

In a double category, a general connection has the form $\Gamma : H \to S$ (horizontal edges to squares).

$\Gamma(a) =$ 

$\gamma : H \to V$ is the 'holonomy' of Γ and the transport law follows from this.

Remark 4.2 The main fact to note is that connections are equivalent to thin structure for edge symmetric double categories (Masa's PhD Thesis).

Suppose we are now given the following such that $ab = cd$:

$$\begin{pmatrix} & a & \\ c & & b \\ & d & \end{pmatrix}.$$

How do we build the following in S?

$$\begin{pmatrix} & a & \\ c & & b \\ & d & \end{pmatrix}$$

There are two ways to achieve this, Θ_1 and Θ_2, and both use only connections (see Brown and Mosa). The general idea is to show that Θ_1 preserves $\circ_1$, that Θ_2 preserves $\circ_2$, and that $\Theta_1 = \Theta_2$, so that we can reduce a big square in two different ways.

4.2 Commutative Cubes

Consider the face of an arbitrary cube. How do we express this face as a consequence of all the other faces? The answer lies in using rotation and folding operations to obtain something like

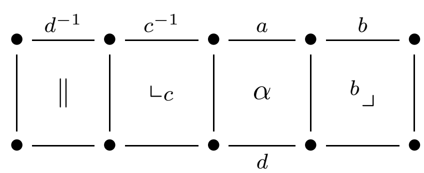
$$\alpha_3^- = \begin{bmatrix} \ulcorner & \alpha_1^- & \urcorner \\ \alpha_2^- & \alpha_3^+ & \alpha_2^+ \\ \llcorner & \alpha_1^+ & \lrcorner \end{bmatrix}.$$

Rotation: For the following matrix, we are required to prove that $\rho^4(\alpha) = \alpha$. To do this, introduce τ and prove that $\rho\tau = 1$, $\tau\rho = 1$, $\rho(\alpha \circ \beta) = \tau(\alpha \circ \beta)$, and $\rho(\alpha) = \alpha^{-1^{-2}}$.

$$\rho(\alpha) = \begin{bmatrix} \| & \ulcorner & = \\ \llcorner & \alpha & \urcorner \\ = & \lrcorner & \| \end{bmatrix}$$

Folding: For the following diagram, check $\Phi(\alpha \circ_1 \beta)$ and $\Phi(\alpha \circ_2 \beta)$ and note that α is determined by the expression $\Phi(\alpha)$. $\begin{pmatrix} & a & \\ c & & b \\ & d & \end{pmatrix}$:

$\Phi(\alpha) =$



4.3 Back to the Van Kampen Theorem

Let $\rho_2(X_0 \subseteq X_1 \subseteq X_2) \in$ a double groupoid with thin structure. Suppose that $X_2 = \bigcup_{\lambda \in \Lambda} U^\lambda$, where the U^λ are open and $U_i^\lambda = U^\lambda \cap X_i$ for $i = 0, 1, 2$. Then we have the following:

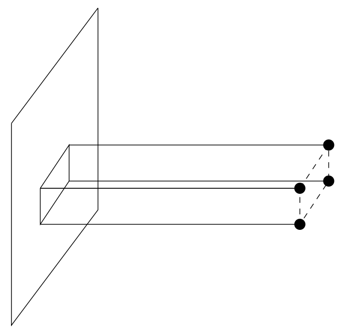
$$\bigsqcup_{\nu \in \Lambda^2} \rho_2(U_*^\nu) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \rho_2(U_*^\lambda) \xrightarrow{c} \rho_2(X_*);$$

$$U^\nu = U^\lambda \cap U^\mu \underset{b}{\overset{a}{\rightrightarrows}} U^\lambda \sqcup U^\mu.$$

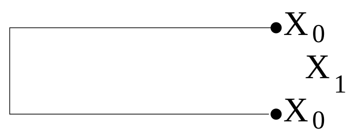
Claim: This is a coequaliser diagram if U^ν is connected for each $\nu = (\nu_1, \ldots, \nu_n)$ and X_* is connected.

Sketch Proof: Given an $f : \bigsqcup_\lambda \rho_2 \left(U_*^\lambda\right) \to G$ such that $fa = fb$ (G is a double groupoid with thin structure), we need to construct an $f' : \rho_2(X_*) \to G$ such that $f'c = f$ and prove f' is unique with this property.

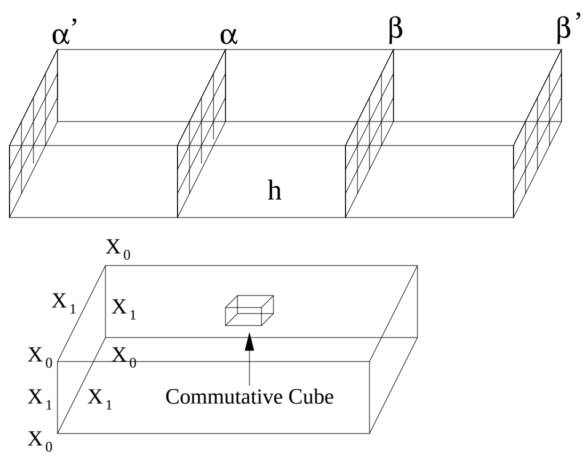
Let $\bar{\alpha} \in \rho_2(X_*)$. The Lebesque Covering Lemma implies that $\alpha = [\alpha_{ij}]$ such that α_{ij} lies in $U^{\lambda_{ij}}$.



We need a condition that vertices "of α_{ij}" can be joined to X_0 by a path in a four or less-fold intersection of the U^λ, and we also need all edges to lie in X_1.



Consider that an edge with vertices in X_0 and some edges in multiple intersections of the U^λ can be deformed in the intersection to give an edge in X_1. Push in to get $\alpha'_{ij} \in \rho_2(U_*^\lambda)$. Working by induction on the skeleton of the subdivision of the square gives $\bar{\alpha}'_{ij} \in \rho_2(U_*^{\lambda_{ij}})$. Take the $f\,\bar{\alpha}'_{ij} \in G$ which are composable — use $fa = fb$ — to get $[f'\bar{\alpha}_{ij}] \in G$. Note the independence of choice here but also note that we really want a cubical argument corresponding to the diagrams below (where $I^3 \to X_2$ with vertices in X_0 and edges in X_1):



Claim: Any composition of commutative cubes is commutative (in G).