PhD Seminars 2002/2003: Dr. C. D. Wensley Semester 1

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October 29, 2003

Contents

1 Seminar 1: October 11 2002

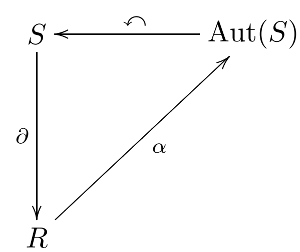
1.1 Crossed Modules

Crossed modules generalise abelian modules, normal subgroups, permutations of sets, etc. We use the notation $\mathcal{X} = (\partial : S \to R)$ to denote a crossed module.

The ingredients of a crossed module are as follows:

- A group homomorphism (boundary) ∂
- Source and range groups S, R
- An action of R on S, i.e. a group homomorphism
  $\alpha : R \to Aut(S)$,
  $r \mapsto \alpha_r : S \to S \ (s \mapsto s^r)$,
  so that $(s_1 s_2)^r = s_1^r s_2^r$, $s^{(r_1 r_2)} = (s^{r_1})^{r_2}$ and $\alpha_{1_R} = \mathrm{id}_S$.
  The situation is summarised by the following diagram:

$\mathcal{X} \quad =$



- Axioms
  $CM1$: $\partial(s^r) = r^{-1}(\partial s)r$
  $CM2$: $s_0^{\partial s} = s^{-1} s_0 s \ (= s_0^s)$

1.2 Examples

Example 1.1 $S \trianglelefteq R$. In this example, ∂ is a subgroup inclusion and $s^r = r^{-1}sr$.

Properties:

(a) $im\ \partial \ \trianglelefteq \ R$ (from $CM1$)

(b) $ker\ \partial$ is central in S (from $CM2$)

(c) $ker\ \partial$ is a module over $R/im\ \partial$, i.e. $im\ \partial$ acts trivially on $ker\ \partial$.

Proof: $k^{-1}k^{\partial s} = k^{-1}s^{-1}ks = 1$ ($ker\ \partial$ is central)
$= (s^{-1})^{\partial k} s = s^{-1}s = 1_s$ (proves (b) and (c)). □

Example 1.2 S is abelian. In this example, $\partial = 0 : S \to R$ ($s \mapsto 1_R$), i.e. S is an R-module.

Lemma 1.3 *$\partial = 0 \Rightarrow S$ is abelian.*

Example 1.4 $G \xrightarrow{\iota} Aut(G)$. Here, ι is the inner automorphism map, i.e. $\iota g : G \to G,\ h \mapsto g^{-1}hg$.

Example 1.5 Central Extensions: $S \xrightarrow{\partial} R$, where ∂ is surjective and ker ∂ is central in S.

For each $r \in R$, choose a pre-image $p_r \in \partial^{-1}r$, i.e. choose a transversal in the cosets $S/\ker\partial$.
Define $s^r = s^{p_r} = p_r^{-1}sp_r$.
We need to check that $s^{r_1r_2} = p_{r_1r_2}^{-1}sp_{r_1r_2}$.
Now $p_{r_1r_2} = kp_{r_1}p_{r_2}$ since they have the same image under ∂.
Since $k^{-1}sk = s$, $s^{(r_1r_2)} = (s^{r_1})^{r_2}$.

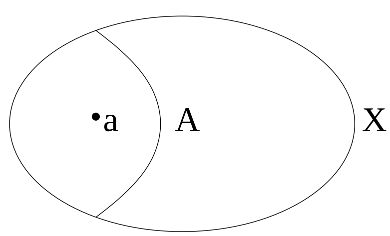
Exercise 1.6 Check $CM1$ and $CM2$ for Example 1.5.

$$\begin{aligned} CM1:\ LHS &= \partial(s^r) \\ &= \partial(p_r^{-1}sp_r) \\ &= (\partial p_r^{-1})(\partial s)(\partial p_r) \qquad (\partial \text{ is a group homomorphism}) \\ &= \partial(\partial^{-1}(r^{-1}))\partial(s)\partial(\partial^{-1}(r)) \\ &= r^{-1}(\partial s)r = RHS. \ \square \end{aligned}$$

$$\begin{aligned} CM2:\ LHS &= s_0^{\partial s} \\ &= p_{\partial s}^{-1}s_0p_{\partial s} \\ &= (\partial^{-1}(\partial s))^{-1}s_0\partial^{-1}(\partial s) \\ &= s^{-1}s_0s = RHS. \ \square \end{aligned}$$

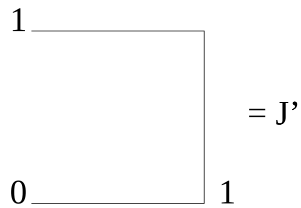
Note: In the above, we should replace any $\partial^{-1}(x)$'s with y's, where $y \in \partial^{-1}(x)$ — but it is obvious that $\partial(y) = x$ so the notation $\partial^{-1}(\partial(x)) = x$ shouldn't be ambiguous.

Example 1.7 A Topological Example. Let (X, A, a) be a based pair of spaces as shown in the diagram below ($a \in A \subseteq X$).



The second relative homotopy group $\Pi_2(X, A, a)$ consists of homotopy classes rel J' of continuous maps $\alpha : (I^2, \dot{I}^2, J') \to (X, A, a)$, where $I = [0, 1]$, $I^2 = I \times I$, $\dot{I}^2 =$ the boundary and

$J' = (I \times \{0,1\}) \cup (\{1\} \times I) \subset I^2$ as shown in the diagram. It follows that $I^2 \to X$, boundary $\to A$ and $J' \to a$.



Whitehead's Crossed Module is $\Pi_2(X, A, a) \xrightarrow{\partial} \Pi_1(A, a)$, i.e. $\alpha \mapsto \beta = \alpha|_{\{0\}\times I}$ and there is an action of $\beta_2 \in \Pi_1(A, a)$ on $\alpha_1 \in \Pi_2(X, A, a)$.

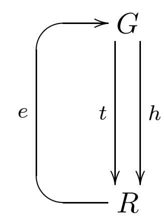
2 Seminar 2: October 18 2002

2.1 Cat^1-groups

Cat^1-groups are made up of the following ingredients:

- Groups G and R
- Group homomorphisms t, h and e (t = tail = source, h = head = target, e = embedding)
- Axioms:
  CAT1) $ter = her = r$
  CAT2) $[\ker\ t, \ker\ h] = \{1_G\}$

The structure of a Cat^1-group can be visualised as follows:



Remark 2.1 We make the following observations about a Cat^1-group:

(a) $\ker t\ \cap\ \mathrm{im}\ e = 1$.

(b) $G\ \cong\ \mathrm{im}\ e \ltimes \ker t$, where the isomorphism is specified by $g \mapsto (etg, (etg)^{-1}g)$ with inverse map $(er, k) \mapsto (er)k$. Check: $t((etg)^{-1}g) = ((te)(tg)^{-1}(tg)) = 1$.

(c) $\ker t = \{(1, s) \mid s \in S\}$ and $\ker h = \{((\partial s)^{-1}, s) \mid s \in S\}$.

(d)

$$\begin{aligned}
[(1,s_1),((\partial s_2)^{-1},s_2] &= (1,s_1^{-1})(\partial s_2,(s_2^{-1}))(1,s_1)((\partial s_2)^{-1},s_2) \\
&= ((\partial s_2)(\partial s_2)^{-1},(s_1^{-1})^{\partial s_2(\partial s_2)^{-1}}(s_2^{-1})^{(\partial s_2)^{-1}}s_1^{(\partial s_2)^{-1}}s_2) \\
&= (1,s_1^{-1}s_2^{-1}s_2s_1s_2^{-1}s_2) \\
&= (1,1).
\end{aligned}$$

(e) From the above, we can deduce that the axiom CAT2) can be rewritten as follows:
CAT2) $[(etg_1)^{-1}g_1, (ehg_2)^{-1}g_2] = 1$.
Notice that this is a completely algebraic formula.

We can construct a Cat^1-group from a Crossed Module and vice-versa. To see this, we must first recall the definitions of a semidirect product and a wreath product.

2.2 Semidirect Products

Definition 2.2 Given a group R acting on a group S, we define the semidirect product $R \ltimes S$ to be the set of pairs (r, s) with product $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1^{r_2} s_2)$ (left action gives $S \rtimes R$).

The identity is $(1_R, 1_S)$, so to find the inverse we require $1 = r_1 r_2$ (so that $r_2 = r_1^{-1}$) and $s_1^{r_2} s_2 = 1$ (so that $s_2 = (s_1^{r_2})^{-1} = (s_1^{-1})^{r_1^{-1}}$), i.e. $(r, s)^{-1} = (r^{-1}, (s^{-1})^{r^{-1}})$.

Example 2.3 Consider the semidirect product $C_2 \ltimes C_3$, where C_2 is generated by b and C_3 is generated by a, and we have action $a^b = a^{-1}$. In the full multiplication table for this example, one entry is $(b, 1)^{(1,a)} = (1, a)^{-1}(b, 1)(1, a) = (1, a^{-1})(b, a) = (b, (a^{-1})^b a) = (b, a^2) = (b, a^{-1})$.

Remark 2.4 The general product for a semidirect product can be written as follows:
$(r_1, s_1)(r_2, s_2) \ldots (r_k, s_k) = (r_1 r_2 \ldots r_k,\ s_1^{r_2 \ldots r_k} s_2^{r_3 \ldots r_k} \ldots s_{k-1}^{r_k} s_k)$.

2.3 Wreath Products

Definition 2.5 Let H be a group, let $S = H^n$, and let R be a subgroup of S_n. The Wreath Product of R and H is defined to be $R \wr H = R \ltimes S$, with the action being the permutation of factors.

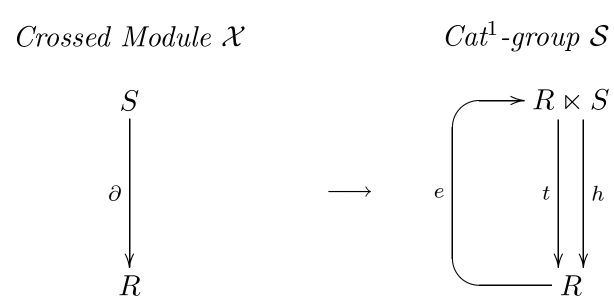
Example 2.6 Consider $C_2 \wr C_2$. When H is a permutation group of degree m, we usually write down an $n \times m$ matrix $\{1, \ldots, mn\}$, and the i-th row permutes row i. In this example, $H \times H = \{(), (1, 2), (3, 4), (1, 2)(3, 4)\}$, R permutes the whole rows generated by $(1, 3)(2, 4)$, and we obtain the Dihedral group D_8 of size 8.

Exercise 2.7 Investigate $C_2 \wr C_3$ and $C_3 \wr C_2$.

Remark 2.8 $|R \wr H| = |R|\,|H|^n$.

2.4 Constructing a Cat^1-group from a Crossed Module

We have now reached the stage where we can construct a Cat^1-group from a Crossed Module:

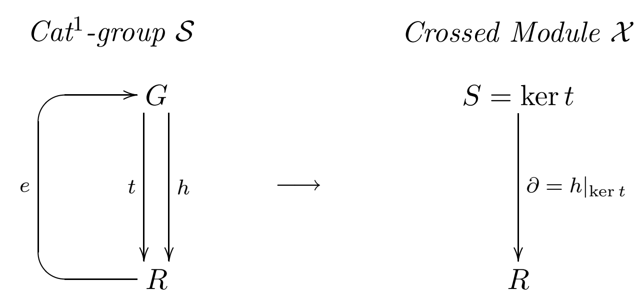


The homomorphisms e, t and h are specified as follows:
$er = (r, 1)$,
$t(r, s) = r$,
$h(r, s) = r(\partial s)$

Let us now check that h really is a homomorphism:

$$\begin{aligned} h(r_1 r_2, s_1^{r_2} s_2) &= r_1 r_2 \partial(s_1^{r_2}) \partial(s_2) \\ &= r_1 r_2 (r_2^{-1} \partial(s_1) r_2) \partial(s_2) \\ &= r_1 \partial(s_1) r_2 \partial(s_2) \\ &= h(r_1, s_1) h(r_2, s_2) \end{aligned}$$

2.5 Constructing a Crossed Module from a Cat^1-group



3 Seminar 3: October 25 2002

3.1 Semidirect Products: An Example

Definition 3.1 Before we start, recall that in a semidirect product $R \ltimes S$ the multiplication is defined as

$$(r,s)(r',s') = (rr', s^{r'}s').$$

Example 3.2 Consider the semidirect product given by S_3 acting on S_3, so that $b^a = a^{-1}ba = a^{-1}a^2b = ab$. The multiplication table looks as follows:

S_3		Regular Representation	$\wedge a$	$\wedge a^2$	$\wedge b$	$\wedge ab$	$\wedge a^2b$
e	1	()	(4 5 6)	(4 6 5)	(2 3)(5 6)	(2 3)(4 6)	(2 3)(4 5)
a	2	(1 2 3)(4 6 5)					
a^2	3	(1 3 2)(4 5 6)					
b	4	(1 4)(2 5)(3 6)			etc.		
ab	5	(1 5)(2 6)(3 4)					
a^2b	6	(1 6)(2 4)(3 5)					

$R \ltimes S$ is generated by $(a,1),(b,1),(1,a)$ and $(1,b)$, (i.e. (456), (23)(56), (123)(456) and (14)(25)(36)). It follows (for example) that $(1,b)(a,1) = (a,ab)$ (i.e. (14)(25)(36)(456) = (152634)).

Definition 3.3 The *holomorph* of a group G is given by

$$Aut(G) \ltimes G \geqslant Inn(G) \ltimes G.$$

3.2 Presentations for $R \ltimes S$

If we have presentations $R = < X \mid V >$ and $S = < Y \mid W >$, the presentation for $R \times S$ is given by

$$R \times S = < X \sqcup Y \mid V \cup W \cup \{[x,y] \mid x \in X, y \in Y\} > .$$

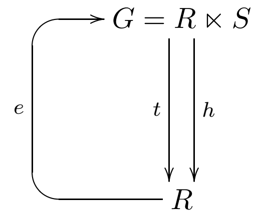
If we let $X' = \{(x,1) \mid x \in X\}$ and $Y' = \{(1,y) \mid y \in Y\}$, then the presentation for $R \ltimes S$ is given by

$$R \ltimes S = < X' \sqcup Y' \mid V' \cup W' \cup \{(x^{-1},1)(1,y)(x,1) = (1,y^x)\} >,$$

remembering to expand all the x^{-1} and all the y^x into generators.

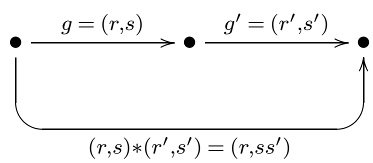
3.3 Groupoids

Consider the following Cat^1-group, where $t(r,s) = r$, $h(r,s) = r(\partial s)$, and $er = (r,1)$:

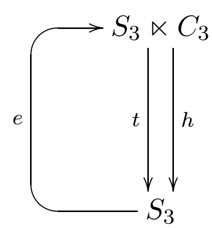


Consider that we construct a groupoid $\mathcal{G}$ with objects the elements of R and arrows the elements of G. We observe the following in this situation:

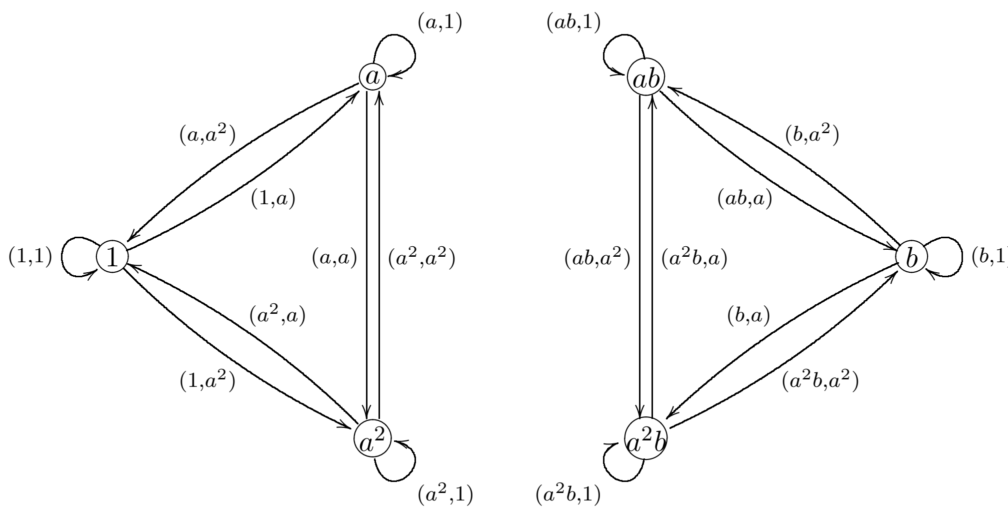
- If we have an arrow $tg \xrightarrow{g} hg$, and if $g = (r,s)$, then we must have $tg = r$ and $hg = r(\partial s)$.
- The inverse of $g = (r,s)$ is given by $\widetilde{(r,s)} = (r(\partial s), s^{-1})$.
- $\ker t = \{(1,s) \mid s \in S\}$ and $\ker h = \{((\partial s)^{-1}, s) \mid s \in S\}$.
- $1_r = er$.
- Composition is summarised by the following diagram, noting that $tg = r$, $r(\partial s) = hg = tg' = r'$, and $hg' = r'(\partial s')$. Observe further that we can also write $g * g' = g(etg'^{-1})g' = g(ehg^{-1})g'$.



Let us now consider Example 3.2 again.



We can deduce that $\ker t = \{(1,1),(1,a),(1,a^2)\}$ (i.e. $\ker t = \{$arrows which start at the identity$\}$), and $\ker h = \{(1,1),(a^2,a),(a,a^2)\}$ (i.e. $\ker h = \{$arrows which end at the identity$\}$. The groupoid can be visualised as follows:



Remark 3.4 We have just constructed a *group groupoid*.

Remark 3.5 A groupoid can be thought of as a disjoint union of graphs. The set of components form a group isomorphic to the cokernel of ∂, and the objects in each component form a coset of $R/im\ \partial$.

Exercise 3.6 Show that $\{(g_1, g_2) \mid hg_1 = tg_2\} \xrightarrow{*} G$ is a group homomorphism.

4 Seminar 4: November 1 2002

4.1 The Peiffer Subgroup of a Pre Crossed Module

Definition 4.1 A Pre Crossed Module $\delta : Q \to R$ is the same as a Crossed Module (see Seminar 1) but without the requirement that the axiom CM2 be satisfied.

Definition 4.2 The Peiffer elements of a Pre Crossed Module $\delta : Q \to R$ are the elements

$$< q_1, q_2 > = q_1^{-1} q_2^{-1} q_1 q_2^{\delta q_1},$$

i.e. $< q_1, q_2 > = 1 \iff q_2^{\delta q_1} = q_1^{-1} q_2 q_1$.

We make the following observations about the Peiffer elements:

(i) $\delta(< q_1, q_2 >) = 1_R$ (by CM1);

(ii) $q_3^{-1} < q_1, q_2 > q_3 = < q_1 q_3, q_2 >< q_3, q_2^{\delta q_1} >^{-1}$;

(iii) $< q_1^r, q_2^r > = < q_1, q_2 >^r$;

(iv) $[q_1 q_3, q_2] = q_3^{-1} q_1^{-1} q_2^{-1} q_1 q_3 q_2 = q_3^{-1}(q_1^{-1} q_2^{-1} q_1 q_2) q_3 q_3^{-1} q_2^{-1} q_3 q_2 = [q_1, q_2]^{q_3} [q_3, q_2]$.

Let P be the subgroup of Q generated by the Peiffer elements. It can be shown that P is a subgroup of $\ker \delta$, that P is normal in Q, and that P is R-invariant.

Proposition 4.3 *If $S = Q/P$, $\partial(Pq) = \delta q$, and the action is given by $(Pq)^r = P(q^r)$, then $\mathcal{X} = (\partial : S \to R)$ is a crossed module and $(\nu, 1)$ is a pre crossed module morphism.*

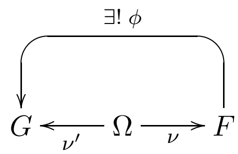
$$\begin{array}{ccccc} S' & \xleftarrow{\nu'} & Q & \xrightarrow{\nu} & S = (Q/P) \\ \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \partial} \\ R & \xleftarrow[1]{} & R & \xrightarrow[1]{} & R \end{array}$$

Furthermore, if $\mathcal{X}' = (\partial' : S' \to R)$ is another crossed module and if $(\nu, 1) : (\delta : Q \to R) \longrightarrow \mathcal{X}'$ is a pre crossed module morphism, then there exists a unique crossed module morphism $(\mu, 1) : \mathcal{X} \to \mathcal{X}'$ such that $(\mu, 1) \circ (\nu, 1) = (\nu', 1)$.

Remark 4.4 If $(\theta, \phi) : P_1 \to P_2$ is a pre crossed module morphism, the Peiffer subgroup in P_1 maps into the Peiffer subgroup in P_2.

4.2 Free Groups

If Ω is a set, the Free Group on Ω is a group F and a map $\nu:\ \Omega \to F$ such that if G is a group $\nu':\ \Omega \to G$ then there exists a unique group homomorphism $\phi:\ F \to G$ such that $\phi\nu = \nu'$.



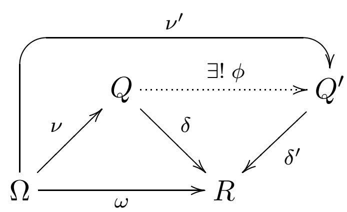
Existence: Take $F = F(\Omega)$ to be all words in Ω and Ω^{-1}. Any $\phi:\ F \to G$ is determined by the images of the generators. For example, if $\varrho \in \Omega$ then $\nu\varrho$ is the letter ϱ considered as a word, and if $u = \varrho_1^{\pm}\varrho_2^{\pm}\ldots\varrho_n^{\pm}$, then $\phi u = (\nu'\varrho_1)^{\pm}\ldots(\nu'\varrho_n)^{\pm}$.

4.3 Free Pre-Crossed Modules

Let Ω be a set, R a group and $\omega : \Omega \to R$ a function. Then $(\delta : Q \to R)$ is the free pre-crossed module on ω if the following holds: If $(\delta' : Q' \to R)$ is another pre-crossed module with map $\nu' : \Omega \to Q'$ and $\delta\nu = \delta'\nu' = \omega$, then there exists an unique pre-crossed module morphism

$$(\phi, 1):\ (\delta:\ Q \to R) \longrightarrow (\delta':\ Q' \to R').$$

The situation is summarised by the following diagram:



Construction:

- Let $Q = F(\Omega \times R)$, the elements of which are words in the $(\varrho_i, r_i)^{\pm}$.
- $\delta(\varrho, r) = (\omega\varrho)^r = r^{-1}(\omega\varrho)r.$
- The action of R on Q is given by $(\varrho, r)^{r'} = (\varrho, rr')$.
- Define $\nu:\ \Omega \to Q$ $(\varrho \mapsto (\varrho, 1))$, noting that $(\varrho, r) = (\varrho, 1)^r$ and $\delta\nu\varrho = \delta(\varrho, 1) = (\omega\varrho)' = \omega\varrho$ (so that $\delta\nu = \omega$).

- Check CM1 (i.e. $\delta(s^r) = r^{-1}(\delta s)r$):

$$\begin{aligned} LHS &= \delta((\varrho, r)^{r'}) \\ &= \delta(\varrho, rr') \\ &= (\omega\varrho)^{rr'} \\ &= r'^{-1}r^{-1}(\omega\varrho)rr' \\ &= r'^{-1}(r^{-1}(\omega\varrho)r)r' \\ &= r'^{-1}(\omega\varrho)^r r' \\ &= r'^{-1}\delta(\varrho, r)r' \\ &= RHS. \ \Box \end{aligned}$$

- For the universal property, we need a morphism $\phi : \ Q \to Q'$ ($(\varrho, r) \mapsto ?$) such that $\phi\nu\varrho = \nu'\varrho$. In particular, $\phi(\varrho, 1) = \nu'\varrho$ so that $\phi(\varrho, r) = (\nu'\varrho)^r$ (ϕ preserves action).

  It follows that $\delta'\phi(\varrho, r) = \delta'((\nu'\varrho)^r) = r^{-1}(\partial'\nu'\varrho)r = r^{-1}(\omega\varrho)r = \delta(\varrho, r)$.

For a crossed module, we take $S = Q/P$, where P is the Peiffer subgroup with elements

$$\begin{aligned} <(\varrho_1, r_1), (\varrho_2, r_2)> &= (\varrho_1, r_1)^{-1}(\varrho_2, r_2)^{-1}(\varrho_1, r_1)(\varrho_2, r_2)^{\delta(\varrho_1, r_1)} \\ &= (\varrho_1, r_1)^{-1}(\varrho_2, r_2)^{-1}(\varrho_1, r_1)(\varrho_2, r_2 r_1^{-1}(\omega\varrho_1)r_1). \end{aligned}$$

This gives $\mathcal{F}_\omega = (\partial : \ S \to R)$, the free crossed module on ω.

5 Seminar 5: 8th November 2002

5.1 Presentations of Groups

Recall that a presentation $\mathcal{P}$ of a group G can be written as $\mathcal{P} = grp(X, \omega : \Omega \to F(X))$, where $\omega\varrho$ is a word in the generators.

Now $H = \Omega \times F(X)$ has elements of the form

$$((\varrho_1)^{u_1})^{\pm 1}((\varrho_2)^{u_2})^{\pm 1} \ldots ((\varrho_n)^{u_n})^{\pm 1},$$

whereas the Peiffer elements have the form

$$((\varrho_1)^{u_1})^{-1}((\varrho_2)^{u_2})((\varrho_1)^{u_1}) = ((\varrho_2)^{u_2})^{\delta(\varrho_1^{u_1})},$$

where $\delta(\varrho_1^{u_1}) = u_1^{-1}(\omega\varrho_1)u_1$.

$$\begin{array}{ccccc} & & H & \qquad & H/P \\ & & \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\partial} \\ \Omega & \xrightarrow{\omega} & F(X) & & F(X) \end{array}$$

$$\ker \partial = \Pi_2(\mathcal{P}) = \mathbb{Z}G, \text{ the module of identities among relations.}$$

Example 5.1 Consider the following presentation for Q_8: $X = \{a, b\}$; $\Omega = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, where ω is as follows: $\varrho_1 \mapsto a^4$, $\varrho_2 \mapsto b^4$, $\varrho_3 \mapsto abab^{-1}$, and $\varrho_4 \mapsto a^2b^2$. Consider

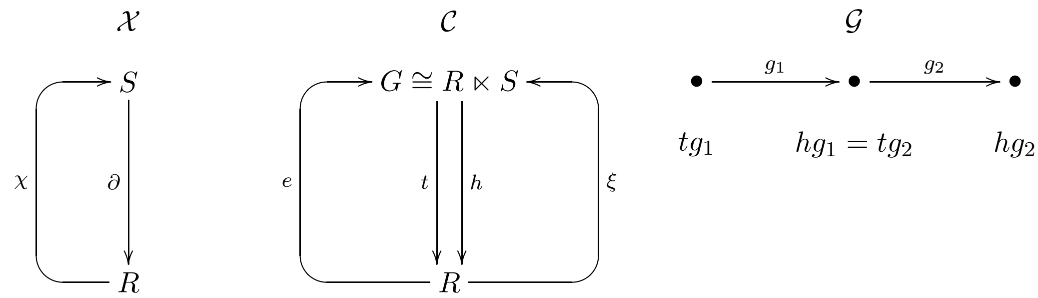
$$\iota = (\varrho_4)^{-1}(\varrho_1)^{a^2}((\varrho_4)^{a^2})^{-1}(\varrho_2).$$

Applying δ, we get $b^{-2}a^{-2}.a^{-2}a^4a^2.a^{-2}b^{-2}a^{-2}a^2.b^4 = 1$, and so ι is an identity among relations.

How do we generate $\ker \partial$? Brown/Rayak (LMS JCM 1) gives a set of generators, but we must still decide which are redundant. Heyworth/Wensley does some computations and Heyworth/Reinet talks about Gröbner bases over $\mathbb{Z}G$.

6 Seminar 6: 15th November 2002

6.1 Derivations and Sections



In the above diagram, χ is a derivation, ξ is a section ($t\xi = 1_R$), and $g_1 * g_2 = g_1(ehg_1^{-1})g_2 = g_1(etg_2^{-1})g_2$. The section ξ is regular if $h\xi \in \mathrm{Aut}(R)$.

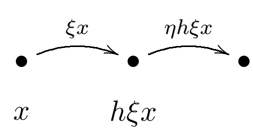
Each ξ (or χ) determines automorphisms:
$\rho : \ R \to R, \ r \mapsto r(\partial\chi r) = h\xi r$;
$\sigma : \ S \to S, \ s \mapsto s(\chi\partial s) = s(e\partial s^{-1})(\xi\partial s)$;
$\gamma : \ G \to G, \ g \mapsto (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg)$.

Check:

$$\begin{aligned}
t(\gamma g) &= (h\xi tg)(tg^{-1})(tg)(hg^{-1})(hg) \\
&= h\xi tg; \\
h(\gamma g) &= (h\xi tg)(h\xi tg^{-1})(hg)(hg^{-1})(h\xi hg) \\
&= h\xi(hg).
\end{aligned}$$

6.1.1 Sections

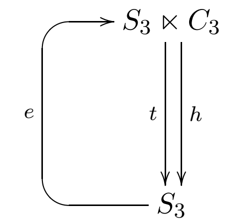
If G is a groupoid, $X = Ob(G)$ and $\circ$ denotes composition in G, consider the following diagram:



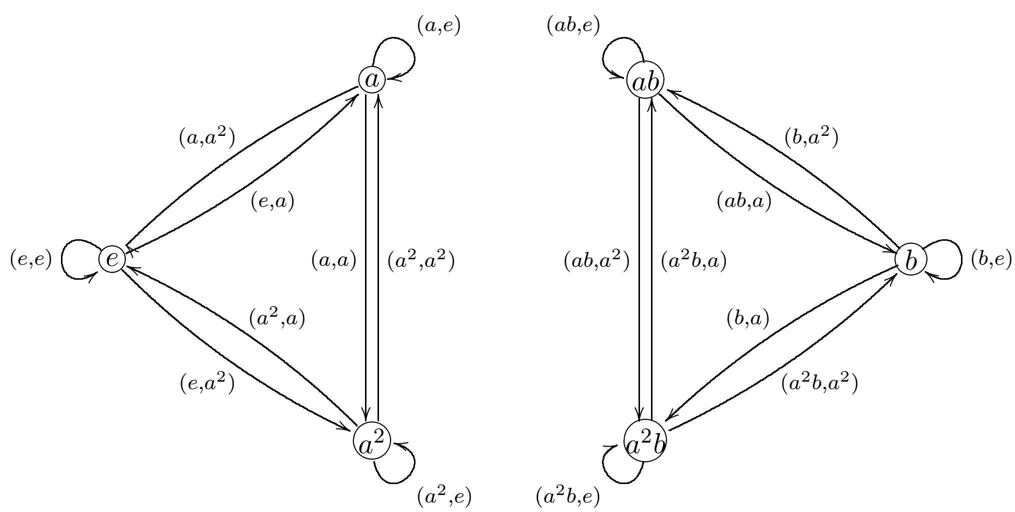
$$\begin{aligned}
(\xi * \eta)(x) &= (\xi x) \circ (\eta h\xi x); \\
(\xi * \eta)(r) &= (\xi r) * (\eta h\xi r) \\
&= (\xi r)(eh\xi r^{-1})(\eta h\xi r) \\
&= (\xi r)(et\eta r^{-1})(\eta h\xi r).
\end{aligned}$$

- We use Whitehead multiplication on sections;
- The set of regular sections form a group;
- The inverse of g is $\tilde{g} = (ehg)g^{-1}(etg)$.

Example 6.1 Consider the Cat^1-group shown below, where $er = (r, e)$ (beware: two 'e's); $t(r, s) = r$; $h(r, s) = r(\partial s) = rs$; and we think of the group S_3 as $S_3 = \{e, a, a^2, b, ab, a^2b\}$ (with $ba = a^2b$ and $ba^2 = ab$):



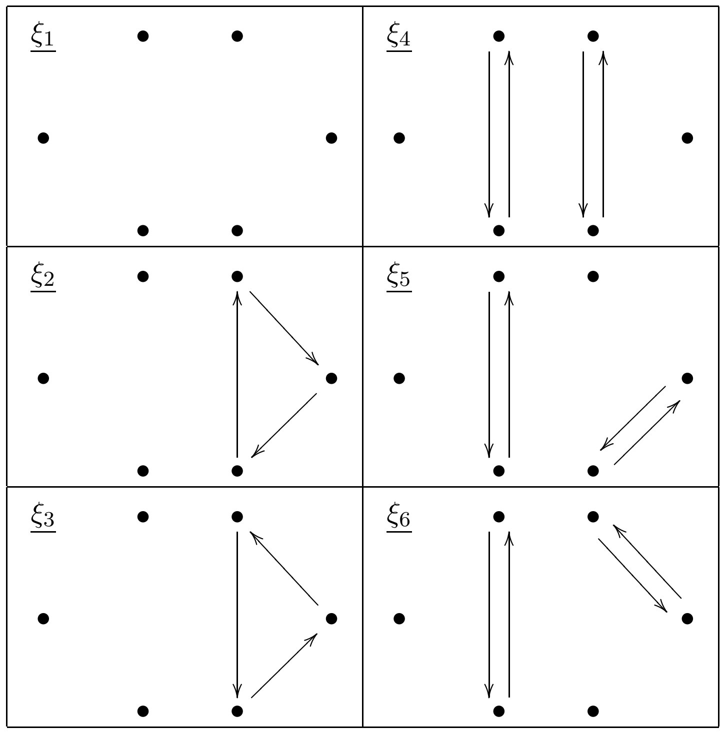
The group groupoid can be visualised as follows:



Here are the regular sections $\xi : R \to G$ (where $t\xi r = r \quad \forall\, r \in R$):

	e	a	a^2	b	ab	a^2b
ξ_1	(e,e)	(a,e)	(a^2,e)	(b,e)	(ab,e)	(a^2b,e)
ξ_2	(e,e)	(a,e)	(a^2,e)	(b,a)	(ab,a)	(a^2b,a)
ξ_3	(e,e)	(a,e)	(a^2,e)	(b,a^2)	(ab,a^2)	(a^2b,a^2)
ξ_4	(e,e)	(a,a)	(a^2,a^2)	(b,e)	(ab,a^2)	(a^2b,a)
ξ_5	(e,e)	(a,a)	(a^2,a^2)	(b,a)	(ab,e)	(a^2b,a^2)
ξ_6	(e,e)	(a,a)	(a^2,a^2)	(b,a^2)	(ab,a)	(a^2b,e)

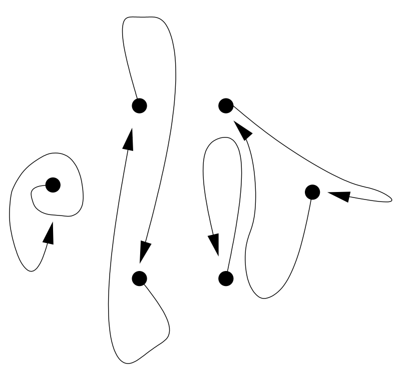
Each section can be visualised as follows (there should be an identity arrow at each isolated vertex):



To compose two sections we use Whitehead composition to give the following $*$-table:

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6
ξ_1	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6
ξ_2	ξ_2	ξ_3	ξ_1	ξ_6	ξ_4	ξ_5
ξ_3	ξ_3	ξ_1	ξ_2	ξ_5	ξ_6	ξ_4
ξ_4	ξ_4	ξ_5	ξ_6	ξ_1	ξ_2	ξ_3
ξ_5	ξ_5	ξ_6	ξ_4	ξ_3	ξ_1	ξ_2
ξ_6	ξ_6	ξ_4	ξ_5	ξ_2	ξ_3	ξ_1

To see how the table is constructed, consider the entry $\xi_2 * \xi_4 = \xi_6$:



6.1.2 Derivations

A derivation $R \xrightarrow{\xi} R \ltimes S$ $(r \mapsto (r, \chi r))$ determines a map $R \to S$ $(r \mapsto \chi r)$. If ξ is a homomorphism then $\xi(r_1 r_2) = (\xi r_1)(\xi r_2)$. It follows that $(r_1 r_2, \chi(r_1 r_2)) = (r_1, \chi r_1)(r_2, \chi r_2) = (r_1 r_2, (\chi r_1)^{r_2}(\chi r_2))$ so that $\chi(r_1 r_2) = (\chi r_1)^{r_2}(\chi r_2)$.

6.1.3 Composition of Derivations

$$\begin{aligned}
\xi_1 \circ \xi_2(r) &= (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2 h \xi_1 r) \\
&= (r_1, \chi_1 r).eh(r^{-1}, \chi_1 r^{-1}).\xi_2 h(r, \chi_1 r) \\
&= (r_1, \chi_1 r)(r^{-1}(\partial \chi_1 r^{-1}), e).\xi_2(r(\partial \chi_1 r)) \\
&= (\partial \chi_1 r^{-1}, (\chi_1 r)^{r^{-1}(\partial \chi_1 r^{-1})}(r(\partial \chi_1 r), \chi_2(r(\partial \chi_1 r))) \\
&= (r.\partial(\chi r)^{-1} r^{-1} r(\partial \chi_1 r), (\chi_1 r)^{r^{-1} r \partial(\chi r)^{-1} r^{-1} r(\partial \chi_1 r)}.(\chi_2 r)^{\partial(\chi_1 r)}.(\chi_2 \partial \chi_1 r)).
\end{aligned}$$

Conclusion: $\xi_1 * \xi_2(r) = (r, \chi_1 * \chi_2(r))$, where

$$\begin{aligned}
\chi_1 * \chi_2(r) &= (\chi_1 r).(\chi_1 r)^{-1}(\chi_2 r)(\chi_1 r).(\chi_2 \partial \chi_1 r) \\
&= (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r) \\
&= (\chi_2 \circ \chi_1)(r).
\end{aligned}$$

Further, $\chi(1) = \chi(1^2) = (\chi 1)^1 = (\chi 1) = (\chi 1)^2 \Rightarrow \chi 1_R = 1_S$.

As we have seen, $\chi : R \to S$ defines $(e, \sigma) : \mathcal{X} \to \mathcal{X}$.
X is regular iff (e, σ) is an automorphism.

Let W be the set of all derivations so that $(W, *)$ is a monoid with identity $O : R \to S$ $(r \mapsto 1)$.
W is known as the Whitehead group.

7 Seminar 7: 22nd November 2002

7.1 The Actor Crossed Module of a Crossed Module

$$\begin{array}{ccc} \mathcal{X} & & \mathcal{A} \\ S & \xrightarrow{\ddot{\iota}} & W(\mathcal{X}) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\Delta} \\ R & \xrightarrow[\iota]{} & \mathrm{Aut}(\mathcal{X}) \end{array}$$

In the above diagram, W is the Whitehead group of derivations $\chi :\ R \to S$.

- $\chi(rs) = (\chi r)^s(\chi s)$;
- $\chi_1 * \chi_2(r) = (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r)$ (on the right);
- $\mathrm{Aut}(\mathcal{X}) = \{\beta = (\ddot{\beta}, \dot{\beta}) :\ \mathcal{X} \to \mathcal{X}$ (invertible)$\}$, summarised by the following diagram (where $\partial\ddot{\beta} = \dot{\beta}\partial$, $\ddot{\beta}(s^r) = (\ddot{\beta}s)^{\dot{\beta}r}$, and $\dot{\beta}(q^r) = (\dot{\beta}q)^{\dot{\beta}r}$):

$$\begin{array}{ccc} S & \xrightarrow{\ddot{\beta}} & S \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ R & \xrightarrow[\dot{\beta}]{} & R \end{array}$$

- Δ is defined by $\Delta\chi = \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$, where $\ddot{\beta}_\chi(s) = s(\chi\partial s)$ and $\dot{\beta}_\chi(r) = r(\partial\chi r)$.

Definition 7.1 The action of $\mathrm{Aut}(\mathcal{X})$ on W is defined as follows: $\chi^\beta(r) = \ddot{\beta}\chi\dot{\beta}^{-1}(r)$.

Example 7.2 Consider the morphism $\iota :\ \mathcal{X} \to \mathcal{A}(\mathcal{X})$, specified as follows:

$\iota :\ R \to \mathcal{A}$, $r \mapsto \beta_r = (\ddot{\beta}_r, \dot{\beta}_r)$ (where $\ddot{\beta}_r(s) = s^r$ and $\dot{\beta}_r(q) = q^r$ is the standard r-action);
$\ddot{\iota} :\ S \to W$, $s \mapsto \chi_s$, where $\chi_s(r) = (s^{-1})^r s$ is a regular derivation known as a *principal derivation*.

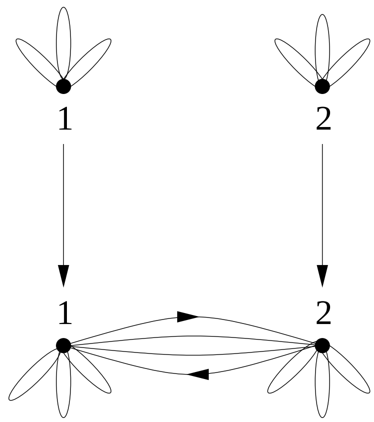
Check: $\chi_s(qr) = (s^{-1})^{qr}s$;
$(\chi_s q)^r(\chi s r) = \{(s^{-1})^q s\}^r\{(s^{-1})^r s\} = (s^{-1})^{qr}s^r(s^{-1})^r s$.

Finally, for a crossed square structure, we need $h :\ R \times W \to S$, $(r, \chi) \mapsto \chi r$.

7.2 Crossed Modules over/of Groupoids

Consider the crossed module $\mathcal{S} \xrightarrow{\partial} \mathcal{R}$, where $\mathcal{S} = S \times \{1, 2, \ldots, n\}$ (a discrete groupoid); $\mathcal{R} = R \times \mathcal{I}_n$; $\mathcal{I}_n$ is the tree groupoid on $\{1, \ldots, n\}$; and $R \times \mathcal{I}_n = \{(r, i, j) \mid r \in R, 1 \le i, j \le n\}$, with underlying graph K_n (with loops) and composition given by $(r, i, j)(q, j, k) = (rq, i, k)$.

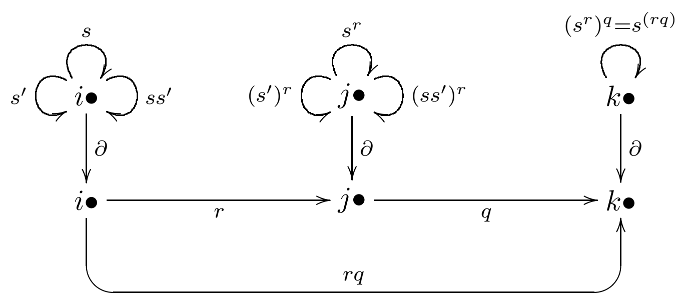
Example 7.3 Take $S = C_3$, $R = S_3$ and $n = 2$:



In the groupoid case, a crossed module consists of

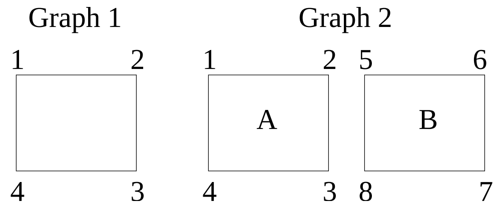
- a range groupoid $\mathcal{R}$;
- a totally disconnected source groupoid $\mathcal{S}$ with the same object set;
- a morphism $\partial : \mathcal{S} \to \mathcal{R}$ which fixes all objects;
- a right action of $\mathcal{S}$ on $\mathcal{R}$; and
- axioms CM1 ($\partial(s^r) = r^{-1}(\partial s)r$) and CM2 ($s_1^{\partial s_2} = s_2^{-1}s_1s_2$).

An action of $\mathcal{R}$ on $\mathcal{S}$ is as follows: if $r \in \mathcal{R}(i,j)$ and s is a loop at i, then s^r is a loop at j with the usual axioms: $(s^r)^q = s^{(rq)}$, $s^1 = s$, and $(s_1s_2)^r = s_1^r s_2^r$ (whenever these are defined).



7.3 Automorphisms of Combinatorial Structures

Consider the following graphs:



For Graph 1, the automorphism 'thing' is D_8. But what about Graph 2? Do we have $\mathrm{Aut}\, G2 = S_2 \wr D_8 = S_2 \ltimes (D_8 \times D_8)$ (size 128), or do we have $\mathrm{Aut}\, G2 = D_8 \times \mathcal{I}_2$ (size 32)?

If σ identifies vertices, then U_σ is $S_2 \wr D_8$. Following this approach, $\mathrm{Aut}(R \times \{1, \ldots, n\})$ should be $(\mathrm{Aut}\, R) \times \mathcal{I}_n$. In our example, $\mathrm{Aut}(C_3) \cong C_2$, with $C_3 = \{e, a, a^2\}$ and $C_2 = \{\epsilon, \tau\}$, say (with $\tau a = a^2$).

An action of $S_3 \times \mathcal{I}_2$ should be a groupoid homomorphism $\alpha :\ S_3 \times \mathcal{I}_2 \to C_2 \times \mathcal{I}_2$. Here is one such α (with $S_3 = \{e, a, a^2, b, ab, a^2b\}$):

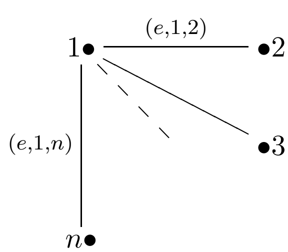
r	αr
$(e,1,1)\ (a,1,1)\ (a^2,1,1)$	$(\epsilon,1,1)$
$(b,1,1)\ (ab,1,1)\ (a^2b,1,1)$	$(\tau,1,1)$
$(e,2,2)\ (a,2,2)\ (a^2,2,2)$	$(\epsilon,2,2)$
$(b,2,2)\ (ab,2,2)\ (a^2b,2,2)$	$(\tau,2,2)$
$(e,1,2)\ (a,1,2)\ (a^2,1,2)$	$(\epsilon,1,2)$
etc.	

This α determines an action of $\mathcal{S}$ on $\mathcal{R}$, e.g.

$$\begin{aligned}(a,1,1)^{(b,1,2)} &= (\alpha(b,1,2))(a,1,1)\\ &= (\tau,1,2)(a,1,1)\\ &= (\tau a,2,2)\\ &= (a^2,2,2)\end{aligned}$$

What is $\mathrm{Aut}(G \times \mathcal{I}_n)$? Here is an outline of a method to find it:

Step A: Choose a tree such as



Step B: Pick automorphisms fixing objects:

- Choose an $\alpha \in \mathrm{Aut}(G)$ at 1 $\bullet$;
- For each $(e, 1, j)$, choose a $(g_j, 1, j)$ (there are $|G|^{n-1}$ choices).

Step C: Apply any $\pi \in S_n$. Conclusion: $|\text{invertible functors}| = |\mathrm{Aut}(G)|\,|G|^{n-1}.n!$.

Step D: Find the natural transformations. As an example, if we consider $C_3 \times \mathcal{I}_2$ $(2 \times 3^1 \times 2 = 12)$, then we might have the following (note that the first three rows of the first column correspond to the identity automorphism, and also note that we have a choice in the fourth row of the table):

(g, i, j)	$\alpha(g, i, j)$
$(e, 1, 1)$	$(e, 1, 1)$
$(a, 1, 1)$	$(a, 1, 1)$
$(a^2, 1, 1)$	$(a^2, 1, 1)$
$(e, 1, 2)$	$(a, 1, 2)$
$(a, 1, 2)$	$(a^2, 1, 2)$
$(a^2, 1, 2)$	$(e, 1, 2)$
$(e, 2, 1)$	$(a^2, 2, 1)$
$(a, 2, 1)$	$(e, 2, 1)$
$(a^2, 2, 1)$	$(a, 2, 1)$
$(e, 2, 2)$	$(e, 2, 2)$
$(a, 2, 2)$	$(a, 2, 2)$
$(a^2, 2, 2)$	$(a^2, 2, 2)$

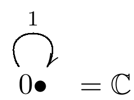
Exercise 7.4 What is the automorphism groupoid for the example that we have been considering in this seminar?

8 Seminar 8: 3rd December 2002

8.1 Morphism Digraphs

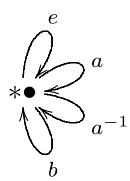
The functor category $\mathbb{S}\text{et}^{\mathbb{C}^{\text{op}}}$ is a category with objects that are functors $\mathbb{C}^{\text{op}} \to \mathbb{S}\text{et}$ and morphisms that are natural transformations. Let us now consider some examples of these functor categories.

Example 8.1



Here, for $\alpha : \mathbb{C}^{\text{op}} \to \mathbb{S}\text{et}$, we have $0 \mapsto \alpha(0)$, a set; and $1 \mapsto \text{id}_{\alpha(0)}$. It follows that α picks out a set, and so $\mathbb{S}\text{et}^{\mathbb{C}^{\text{op}}} \cong \mathbb{S}\text{et}$.

Example 8.2

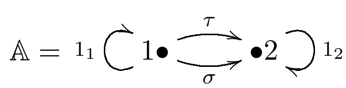


etc.

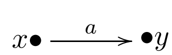
In this example, take $\mathbb{C}$ to be a group. We have $\alpha(*) =$ a set S; $\alpha(e) = \text{id}_S$; $\alpha(a) =$ an endomorphism of S; $\alpha(a^{-1}) =$ the inverse of $\alpha(a)$; arrows $\mapsto$ bijections $S \to S$; and $\alpha(ab) = (\alpha b) * (\alpha a) = (\alpha a) \circ (\alpha b)$. Conclusion: $\mathbb{S}\text{et}^{\mathbb{C}^{\text{op}}} =$ a category of $\mathbb{C}$-sets.

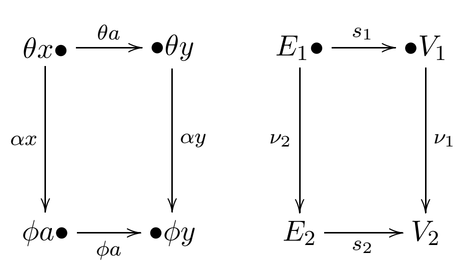
Example 8.3 $\mathbb{C} =$ a groupoid — left as an exercise.

Example 8.4

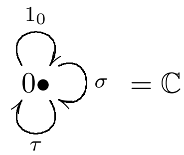


Here, $\alpha 1$ is a set V, $\alpha 2$ is a set E, $\alpha\sigma$ is a function $s : E \to V$, and $\alpha\tau$ is a function $t : E \to V$. A natural transformation $\nu : \alpha_1 \to \alpha_2$, $(V_1, E_1, s_1, t_1) \mapsto (V_2, E_2, s_2, t_2)$ gives maps $\nu_1 : V_1 \to V_2$ and $\nu_2 : E_1 \to E_2$ such that $\nu_1 \circ s_1 = s_2 \circ \nu_2$ and $\nu_1 \circ t_1 = t_2 \circ \nu_2$. It follows that $\mathbb{S}\text{et}^{\mathbb{A}^{\text{op}}}$ is a category of digraphs.





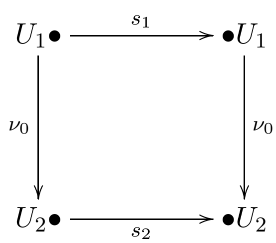
Example 8.5



In this example, we have $\sigma\tau = \sigma$ and $\tau\sigma = \tau$. It follows that because $(\sigma\tau)\sigma = \sigma\sigma$ and $\sigma(\tau\sigma) = \sigma\tau = \sigma$, then we have $\sigma^2 = \sigma$. Similarly, $\tau^2 = \tau$.

Let α be a functor $\alpha : \mathbb{C}^{\text{op}} \to \mathbb{S}\text{et}$ where $\alpha 0 =$ a set U, $\alpha\sigma$ is a map $s : U \to U$ and $\alpha\tau$ is a map $t : U \to U$ (so that $ts = s$ and $st = t$). It can be shown that $\operatorname{Im} s = \operatorname{Im} t$ (if $su = x$ then $t(su) = x$, etc.) Denote this common image by V and $U \setminus V$ by E — we again obtain a digraph with vertices V and arcs E — and now $sv = tv = v$ for each $v \in V$.

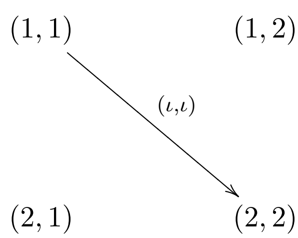
For a natural transformation, we need a function $\nu_0 : U_1 \to U_2$ such that $\nu_0 s_1 = s_2 \nu_0$ and $\nu_0 t_1 = t_2 \nu_0$. This allows ν_0 to map arcs to vertices, and therefore the two digraph categories have the same objects (digraphs) but differ in their digraph morphisms (the second has more morphisms).



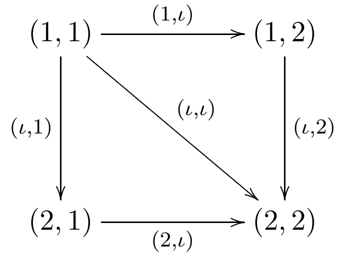
8.2 Products

Let $I = 1\bullet \xrightarrow{1} \bullet 2$. What is $I \times I$ (and more generally $\alpha_1 \times \alpha_2$)?

The $\mathbb{A}$ *case*: The vertices are $V_1 \times V_2 = \{1, 2\} \times \{1, 2\}$ and the edges are $E_1 \times E_2 = \{\iota\} \times \{\iota\}$:



The $\mathbb{C}$ *case*: The elements are $U_1 \times U_2 = \{1, 2, \iota\} \times \{1, 2, \iota\}$ and the vertices are $V_1 \times V_2$:



8.3 Internal HOMs

Consider the $\mathbb{C}$ case. $D^C \equiv \mathrm{DGPH}(C, D)$ (for digraphs C and D) is a digraph defined by the following condition: For all digraphs B there is an isomorphism $\mathrm{Dgph}(B \times C, D) \cong \mathrm{Dgph}(B, \mathrm{DGPH}(C, D))$. In the case of $\mathbb{S}$ets, we want a set $\mathrm{SET}(C, D)$ and we want functions $(B \times C, D) \cong$ functions $(B, \mathrm{SET}(C, D))$. Note that the left hand side of the isomorphism corresponds to functions of two variables, $(b, c) \mapsto f(b, c)$; and that the right hand side of the isomorphism corresponds to variable functions of 1 variable, $b \mapsto (c \mapsto f(b, c))$. It follows that $\mathrm{SET}(C, D)$ is just (functions $C \to D$).

9 Seminar 9: 10th December 2002

9.1 Graphs of Morphisms of Graphs

Suppose $f : D^C \to E$ is an isomorphism. Consider two functors $\mathcal{F}_1, \mathcal{F}_2 :$ Dgph $\to \mathbb{S}$et, where $\mathcal{F}_1 B = \mathrm{Dgph}(B, D^C)$ and $\mathcal{F}_2 B = \mathrm{Dgph}(B, E)$.

9.1.1 Diagram Form

Identify the vertices of D^C with the morphisms $T \to D^C$, where T is the terminal digraph $\bullet$; and identify loops and arcs with morphisms $I \to D^C$, where $I = 1\bullet \xrightarrow{\iota} \bullet 2$. Thus the elements $\mathrm{DGPH}(C, D)(\nu_1, \nu_2)$ are morphisms of the form $\psi' : I \times C \to D$, where $(1, a) \mapsto \nu_1 a$, $(2, a) \mapsto \nu_2 a$, and $(\iota, a) \mapsto \psi a \in D(\nu_1 sa, \nu_2 ta)$.

In $I \times C$, from $1\bullet \xrightarrow{\iota} \bullet 2$ and $x\bullet \xrightarrow{a} \bullet y$ we get

$$\begin{array}{ccc} (1,x) & \xrightarrow{(1,a)} & (1,y) \\ {\scriptstyle(\iota,x)}\downarrow & \searrow^{(\iota,a)} & \downarrow{\scriptstyle(\iota,y)} \\ (2,x) & \xrightarrow[(2,a)]{} & (2,y) \end{array}$$

while in ψ', we have

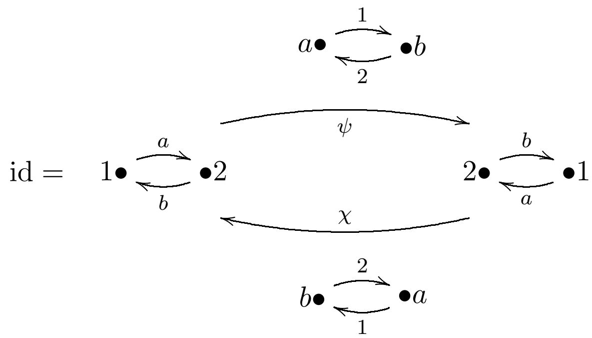
$$\begin{array}{ccc} \nu_1 x & \xrightarrow{\nu_1 a} & \nu_1 y \\ {\scriptstyle\psi x}\downarrow & \searrow^{\psi a} & \downarrow{\scriptstyle\psi y} \\ \nu_2 x & \xrightarrow[\nu_2 a]{} & \nu_1 y \end{array}$$

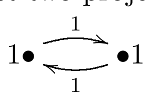
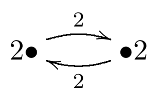
For the diagram form, we identify ψ' with $\psi : U(C) \to U(D)$, $(a : x \to y) \mapsto (\psi a : \nu_1 x \to \nu_2 y)$.

9.1.2 Examples

Example 9.1 Consider $C = 1\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet 2$. Our task is to determine $\mathrm{Aut}(C) \subseteq C^C$.

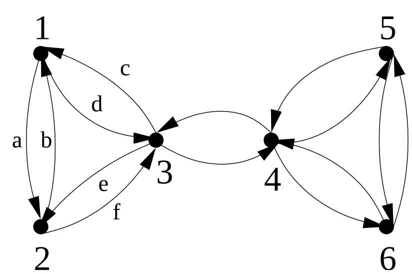
The digraph isomorphisms $C \to C$ consist of id_C and $\rho:\ 1 \leftrightarrow 2,\ a \leftrightarrow b$:



- $\psi c:$ image(source c) $\to$ image(target c).
- We get a permutation group on $U(C)$: $\{(\,), (1,2)(a,b), (1,a)(2,b), (1,b)(2,a)\}$.
  This is the automorphism digraph of C.
- $\mathrm{END}(C) \equiv C^C$.
- We get two projections onto vertices:
  $\pi_1 =$  and $\pi_2 =$  .

Example 9.2 If C has n vertices and no arcs, then $\mathrm{Aut}(C)$ has $n!$ vertices and no arcs (S_n). If C is the complete digraph on n vertices, then $\mathrm{Aut}(C)$ is the complete digraph on $n!$ vertices.

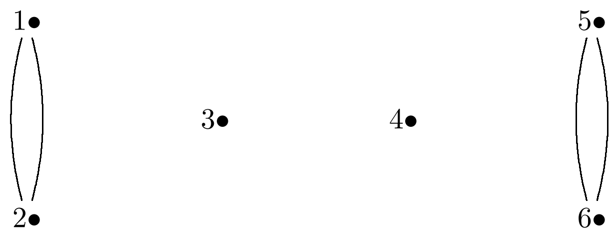
Example 9.3 Consider D =



- $\mathrm{Aut}(D) \cong D_8 =$
  $\{(\,), (1,2), (5,6), (1,2)(5,6), (1,5)(2,6)(3,4), (1,5,2,6)(3,4), (1,6,2,5)(3,4), (1,6)(2,5)(3,4)\}$.
- $\mathrm{AUT}(D) \cong$ 2 copies of the complete digraph on 4 vertices (top row / bottom 2 rows).

Definition 9.4 A subgraph S of D is *symmetrically embedded* if it is a complete subgraph and if any permutation of S extends to an automorphism of D fixing all vertices not in S.

In Example 9.3, the four maximal symmetrically embedded subgraphs are as follows ($(1,2)(a,b)$ extends to $(1,2)(a,b)(c,e)(d,f)$):



The inner automorphism group of D is the product of symmetric groups, one for each maximally symmetrically embedded subgraph.

Theorem 9.5 *(Shrimpton):* $\mathrm{AUT}(D) = \bigsqcup_{\kappa}$ Complete Graphs*, where* $\kappa =$ *'cosets of inner automorphism group'.*