Pedwar: Manual

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Contents

1 Introduction

Pedwar is a collection of four programs for computing Gröbner or Involutive Bases for polynomial ideals over commutative and noncommutative polynomial rings. The programs run under FreeBSD [6] and are intended for informational or educational purposes only; no warranty of any kind is provided.

2 Installation

Unpack the zip file into a suitable directory.

- The four programs `grob`, `inv`, `ncgrob` and `ncinv` appear in the top directory, together with their corresponding `README` files.
- The directory `examples` contains many examples for use with the programs.
- The directory `matrices` contains matrices for certain commutative monomial orderings.
- The directory `source` contains the ANSI C source code for the programs. Note however that in order to re-compile the programs, the AlgLib libraries from MSSRC [11] are required (see `www.mssrc.com` for more information).

3 Key Functionality

3.1 Commutative Gröbner Bases (grob)

- Implements Buchberger's Algorithm [2] for computing commutative Gröbner Bases.
- Implements the Gröbner Walk [4] algorithm for converting a Gröbner Basis with respect to one monomial ordering to a Gröbner Basis with respect to another monomial ordering.
- Monomial orderings implemented: Lex, DegLex, DegRevLex, arbitrary matrix orderings.
- Allows the computation of Logged Gröbner Bases.
- Different selection strategies can be used, such as the sugar and the double sugar strategies [8].
- How Buchberger's criteria [1] are applied can be controlled.
- **See the file `README-grob` in the distribution for further details.**

3.2 Commutative Involutive Bases (inv)

- Implements the Involutive Basis algorithm for computing Pommaret or Janet Involutive Bases.
- Three different algorithms for computing Involutive Bases are provided (as presented in [12], [3] and [7]).
- Monomial orderings implemented: Lex, DegLex, DegRevLex, arbitrary matrix orderings.
- Options for controlling how the basis is sorted during computation.
- **See the file `README-inv` in the distribution for further details.**

3.3 Noncommutative Gröbner Bases (ncgrob)

- Implements Mora's Algorithm [10] for computing noncommutative Gröbner Bases.
- Monomial orderings implemented: Lex, DegLex, DegRevLex, Wreath Product.
- Allows the computation of Logged Gröbner Bases.
- Different selection strategies can be used, such as the sugar and the double sugar strategies.
- How Buchberger's second criterion is applied can be controlled.
- **See the file `README-ncgrob` in the distribution for further details.**

3.4 Noncommutative Involutive Bases (ncinv)

- Implements the Involutive Basis algorithm for computing noncommutative Involutive Bases for 12 different divisions (see [5] for a description of these divisions).
- Three different algorithms for computing Involutive Bases are provided.
- Monomial orderings implemented: Lex, DegLex, DegRevLex, Wreath Product.
- Options for controlling how the basis is sorted during computation.
- **See the file `README-ncinv` in the distribution for further details.**

4 Examples

4.1 Commutative Gröbner Bases

- **Task:** Compute a DegLex Gröbner Basis for the ideal generated by the set of polynomials $F = \{x^2 - 2xy + 3,\ 2xy + y^2 + 5\}$ over the polynomial ring $\mathbb{Q}[x, y]$.
- **Input File:**

```
> more examples/test/test2.in
x; y;
- 2*x*y + x*x + 3;
2*x*y + y*y + 5;
>
```

- **Session Output:**

```
> grob -d examples/test/test2.in

*** GROEBNER BASIS PROGRAM ***

Using the DegLex Ordering with dictionary y < x

Polynomials in the input basis:
x^2 - 2 x y + 3,
2 x y + y^2 + 5,
[2 Polynomials]

Computing Groebner Basis...
Added Polynomial #3 to Basis...
...Groebner Basis Computed.

Here is the Groebner Basis:
x^2 - 2 x y + 3,
2 x y + y^2 + 5,
5 y^3 - 10 x + 37 y,
[3 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.
```

```
Here is the Reduced Groebner Basis:
x^2 + y^2 + 8,
2 x y + y^2 + 5,
5 y^3 - 10 x + 37 y,
[3 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.

>
```

- **Output File:**

```
> more examples/test/test2.deg
x; y;
x^2 + y^2 + 8;
2 * x * y + y^2 + 5;
5 * y^3 - 10 * x + 37 * y;
>
```

4.2 Commutative Gröbner Walk

- **Task:** Using the Gröbner Walk [4], compute a Lex Gröbner Basis for the ideal generated by the set of polynomials $F = \{y(z-t)-x+a,\ z(t-x)-y+a,\ t(x-y)-z+a,\ x(y-z)-t+a\}$ over the polynomial ring $\mathbb{Q}[x, y, z, t, a]$. Start the walk with a DegRevLex Gröbner Basis for F.

- **Input File:**

```
> more examples/posso/posso07.in
x; y; z; t; a;
y*(z - t) - x + a;
z*(t - x) - y + a;
t*(x - y) - z + a;
x*(y - z) - t + a;
>
```

- **Session Output:**

```
> grob -wrl examples/posso/posso07.in
```

```
*** GROEBNER WALK PROGRAM ***

Using the dictionary a < t < z < y < x

*** PART A: COMPUTE A DEGREVLEX GROEBNER BASIS ***

Polynomials in the input basis:
y z - y t - x + a,
-1 x z + z t - y + a,
x t - y t - z + a,
x y - x z - t + a,
[4 Polynomials]

Computing Groebner Basis...
Added Polynomial #5 to Basis...
Added Polynomial #6 to Basis...
Added Polynomial #7 to Basis...
Added Polynomial #8 to Basis...
Added Polynomial #9 to Basis...
Added Polynomial #10 to Basis...
Added Polynomial #11 to Basis...
Added Polynomial #12 to Basis...
...Groebner Basis Computed.

Here is the Groebner Basis:
y z - y t - x + a,
-1 x z + z t - y + a,
x t - y t - z + a,
x y - x z - t + a,
y t^2 - z t^2 + z^2 + 2 y t - z a - 2 t a + z - a,
y^2 t - z t^2 + 2 y t - t^2 - y a + x - a,
x^2 + y^2 + z^2 + t^2 - x a - y a - z a - t a,
z^2 t - z t^2 + y^2 + z^2 + z t - y a - z a - t a - x + a,
3 z t^2 + t^3 - 2 y t a - z t a - t^2 a - y^2 - z^2 - 3 y t + 2 t^2 - x a + 2 y
a + t a + a^2 - y + a,
z^3 - t^3 - z^2 a + y t a - z t a + t^2 a + y^2 + z^2 - t^2 + x a - y a - a^2 -
x + y,
y^3 - t^3 - y^2 a + t^2 a + y^2 - z^2 - 2 t^2 - y a + z a + 2 t a + y - t,
12 t^4 - 12 t^3 a + 20 t^3 - 9 y^2 a + 6 z^2 a + 14 y t a - 14 z t a - 11 t^2 a
+ 9 y a^2 - 6 z a^2 - 9 t a^2 - 11 y^2 - 2 z^2 + 6 y t - 6 z t + 7 t^2 - 5 x a +
```

```
 4 y a + 9 z a - 4 t a + 2 a^2 + 9 x - 5 y + 9 z + 9 t - 22 a,
[12 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
y z - y t - x + a,
x z - z t + y - a,
x t - y t - z + a,
x y - z t + y - t,
3 y t^2 + t^3 - 2 y t a - z t a - t^2 a - y^2 + 2 z^2 + 3 y t + 2 t^2 - x a + 2
y a - 3 z a - 5 t a + a^2 - y + 3 z - 2 a,
3 y^2 t + t^3 - 2 y t a - z t a - t^2 a - y^2 - z^2 + 3 y t - t^2 - x a - y a +
t a + a^2 + 3 x - y - 2 a,
x^2 + y^2 + z^2 + t^2 - x a - y a - z a - t a,
3 z^2 t + t^3 - 2 y t a - z t a - t^2 a + 2 y^2 + 2 z^2 - 3 y t + 3 z t + 2 t^2
- x a - y a - 3 z a - 2 t a + a^2 - 3 x - y + 4 a,
3 z t^2 + t^3 - 2 y t a - z t a - t^2 a - y^2 - z^2 - 3 y t + 2 t^2 - x a + 2 y
a + t a + a^2 - y + a,
z^3 - t^3 - z^2 a + y t a - z t a + t^2 a + y^2 + z^2 - t^2 + x a - y a - a^2 -
x + y,
y^3 - t^3 - y^2 a + t^2 a + y^2 - z^2 - 2 t^2 - y a + z a + 2 t a + y - t,
12 t^4 - 12 t^3 a + 20 t^3 - 9 y^2 a + 6 z^2 a + 14 y t a - 14 z t a - 11 t^2 a
+ 9 y a^2 - 6 z a^2 - 9 t a^2 - 11 y^2 - 2 z^2 + 6 y t - 6 z t + 7 t^2 - 5 x a +
 4 y a + 9 z a - 4 t a + 2 a^2 + 9 x - 5 y + 9 z + 9 t - 22 a,
[12 Polynomials]

*** PART B: GO ON THE WALK TO FIND A LEX GROEBNER BASIS ***

Here are the polynomials in the Source Groebner Basis:
y z - y t - x + a,
x z - z t + y - a,
x t - y t - z + a,
x y - z t + y - t,
3 y t^2 - 2 y t a - z t a + t^3 - t^2 a - x a - y^2 + 3 y t + 2 y a + 2 z^2 - 3
z a + 2 t^2 - 5 t a + a^2 - y + 3 z - 2 a,
3 y^2 t - 2 y t a - z t a + t^3 - t^2 a - x a - y^2 + 3 y t - y a - z^2 - t^2 +
t a + a^2 + 3 x - y - 2 a,
x^2 - x a + y^2 - y a + z^2 - z a + t^2 - t a,
```

```
-2 y t a + 3 z^2 t - z t a + t^3 - t^2 a - x a + 2 y^2 - 3 y t - y a + 2 z^2 + 3
 z t - 3 z a + 2 t^2 - 2 t a + a^2 - 3 x - y + 4 a,
-2 y t a + 3 z t^2 - z t a + t^3 - t^2 a - x a - y^2 - 3 y t + 2 y a - z^2 + 2 t
^2 + t a + a^2 - y + a,
y t a + z^3 - z^2 a - z t a - t^3 + t^2 a + x a + y^2 - y a + z^2 - t^2 - a^2 -
x + y,
y^3 - y^2 a - t^3 + t^2 a + y^2 - y a - z^2 + z a - 2 t^2 + 2 t a + y - t,
12 t^4 - 12 t^3 a - 9 y^2 a + 14 y t a + 9 y a^2 + 6 z^2 a - 14 z t a - 6 z a^2
+ 20 t^3 - 11 t^2 a - 9 t a^2 - 5 x a - 11 y^2 + 6 y t + 4 y a - 2 z^2 - 6 z t +
 9 z a + 7 t^2 - 4 t a + 2 a^2 + 9 x - 5 y + 9 z + 9 t - 22 a,
[12 Polynomials]

*** PASS 1 ***

Step 1: Calculating Initials...
........Initials Calculated.

Step 2: Computing Groebner Basis of Initials...
Added Polynomial #13 to Basis...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
........Groebner Basis of Initials Computed.

Step 3: Obtaining Lifted Groebner Basis...
........Obtained Lifted Groebner Basis of Size 13.

Step 4: Finding Next Step on the Walk...
........Next step on Walk: 1, 1/2, 1/2, 1/2, 1/2, (t = 1/2).

*** PASS 2 ***

Step 1: Calculating Initials...
........Initials Calculated.

Step 2: Computing Groebner Basis of Initials...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
Added Polynomial #16 to Basis...
Added Polynomial #17 to Basis...
Added Polynomial #18 to Basis...
```

```
Added Polynomial #19 to Basis...
........Groebner Basis of Initials Computed.

Step 3: Obtaining Lifted Groebner Basis...
........Obtained Lifted Groebner Basis of Size 13.

Step 4: Finding Next Step on the Walk...
........Next step on Walk: 1, 0, 0, 0, 0, (t = 1).

*** PASS 3 ***

Step 1: Calculating Initials...
........Initials Calculated.

Step 2: Computing Groebner Basis of Initials...
........Groebner Basis of Initials Computed.

Step 3: Obtaining Lifted Groebner Basis...
........Obtained Lifted Groebner Basis of Size 4.

Step 4: Finding Next Step on the Walk...
........No more steps.

Here are the polynomials in the Target Groebner Basis:
x + y t a + 3 y t - y a - z^3 + z^2 a - 3 z t^2 + 2 z t a - t^2 - t a - a,
y t^2 + 2 y t + z^2 - z t^2 - z a + z - 2 t a - a,
y^2 + y t a + 3 y t - 2 y a - z^3 + z^2 t + z^2 a + z^2 - 4 z t^2 + 2 z t a + z
t - z a - t^2 - 2 t a,
y z + y t a + 2 y t - y a - z^3 + z^2 a - 3 z t^2 + 2 z t a - t^2 - t a,
[4 Polynomials]

Writing Target Groebner Basis to Disk... Done.

>
```

- **Output File:**

```
> more examples/posso/posso07.lex
x; y; z; t; a;
x + y * t * a + 3 * y * t - y * a - z^3 + z^2 * a - 3 * z * t^2 + 2 * z * t * a
- t^2 - t * a - a;
```

```
y * t^2 + 2 * y * t + z^2 - z * t^2 - z * a + z - 2 * t * a - a;
y^2 + y * t * a + 3 * y * t - 2 * y * a - z^3 + z^2 * t + z^2 * a + z^2 - 4 * z
* t^2 + 2 * z * t * a + z * t - z * a - t^2 - 2 * t * a;
y * z + y * t * a + 2 * y * t - y * a - z^3 + z^2 * a - 3 * z * t^2 + 2 * z * t
* a - t^2 - t * a;
>
```

4.3 Commutative Janet Bases

- **Task:** Compute a DegRevLex Janet Basis for the ideal generated by the set of polynomials $F = \{xy - z,\ 2x + yz + z\}$ over the polynomial ring $\mathbb{Q}[x, y, z]$.
- **Input File:**

```
> more examples/test/test1.in
x; y; z;
x*y - z;
2*x + y*z + z;
>
```

- **Session Output:**

```
> inv examples/test/test1.in

*** JANET BASIS PROGRAM ***

Using the DegRevLex Ordering with dictionary z < y < x

Polynomials in the input basis:
x y - z,
y z + 2 x + z,
[2 Polynomials]

Computing an Involutive Basis...
Added a first polynomial to G...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)..
Added a polynomial to G       (4)..
...Involutive Basis Computed.

Here is the Involutive Basis (Multiplicative Variables in brackets):
```

```
y z + 2 x + z (x y z),
2 x^2 z + x z^2 + z^3 (x z),
x y - z (x y),
2 x^2 + x z + z^2 (x),
[4 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
y z + 2 x + z,
x y - z,
2 x^2 + x z + z^2,
[3 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

>
```

- **Output Files:**

```
> more examples/test/test1.drl.inv
x; y; z;
y * z + 2 * x + z; (x y z);
2 * x^2 * z + x * z^2 + z^3; (x z);
x * y - z; (x y);
2 * x^2 + x * z + z^2; (x);
> more examples/test/test1.drl
x; y; z;
y * z + 2 * x + z;
x * y - z;
2 * x^2 + x * z + z^2;
>
```

4.4 Commutative Pommaret Bases

- **Task:** Compute a DegRevLex Pommaret Basis for the ideal generated by the set of polynomials $F = \{-x^3 + x^2y - x^2 - xy^2 + x + y^3 - y^2 - y + 1,\ x^2 - 2xy + 3y^2 - 2y - 1\}$ over the polynomial ring $\mathbb{Q}[x, y]$.

- **Input File:**

```
> more examples/posso/posso12.in
x; y;
-x^3 + x^2*y - x^2 - x*y^2 + x + y^3 - y^2 - y + 1;
x^2 - 2*x*y + 3*y^2 - 2*y - 1;
>
```

- **Session Output:**

```
> inv -s2 examples/posso/posso12.in

*** POMMARET BASIS PROGRAM ***

Using the DegRevLex Ordering with dictionary y < x

Polynomials in the input basis:
-1 x^3 + x^2 y - x y^2 + y^3 - x^2 - y^2 + x - y + 1,
x^2 - 2 x y + 3 y^2 - 2 y - 1,
[2 Polynomials]

Computing an Involutive Basis...
Added a first polynomial to G...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)..
Discarded a polynomial from G (2)...
Added a polynomial to G       (3)...
Added a polynomial to G       (4)..
...Involutive Basis Computed.

Here is the Involutive Basis:
y^3 - x y - y,
x^2 y^2 - x y^2 + 2 x y - 2 y^2 + 2 y,
x^2 y - 2 x y^2 + 3 x y - 2 y^2 + 2 y,
x^2 - 2 x y + 3 y^2 - 2 y - 1,
[4 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
```

```
y^3 - x y - y,
x^2 - 2 x y + 3 y^2 - 2 y - 1,
[2 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

>
```

- **Output Files:**

```
> more examples/posso/posso12.drl.inv
x; y;
y^3 - x * y - y;
x^2 * y^2 - x * y^2 + 2 * x * y - 2 * y^2 + 2 * y;
x^2 * y - 2 * x * y^2 + 3 * x * y - 2 * y^2 + 2 * y;
x^2 - 2 * x * y + 3 * y^2 - 2 * y - 1;
> more examples/posso/posso12.drl
x; y;
y^3 - x * y - y;
x^2 - 2 * x * y + 3 * y^2 - 2 * y - 1;
>
```

4.5 Noncommutative Gröbner Bases

- **Task:** Compute a noncommutative DegLex Gröbner Basis for the ideal generated by the set of polynomials $F = \{Aa - 1,\ aA - 1,\ Bb - 1,\ bB - 1,\ a^3 - 1,\ b^2 - 1,\ (ab)^2 - 1\}$ over the polynomial ring $\mathbb{Q}\langle B, A, b, a\rangle$. Note: this corresponds to the computation of a complete rewrite system for the group S_3 using the Knuth-Bendix critical pairs completion algorithm [9] (the input set of polynomials corresponds to a monoid rewrite system for the group presentation $S_3 = \langle b, a \mid a^3, b^2, (ab)^2\rangle$).

- **Input File:**

```
> more examples/rws/s3.in
B; A; b; a;
A*a - 1;
a*A - 1;
B*b - 1;
b*B - 1;
```

```
a^3 - 1;
b^2 - 1;
(a*b)^2 - 1;
>
```

- **Session Output:**

```
> ncgrob -d examples/rws/s3.in

*** NONCOMMUTATIVE GROEBNER BASIS PROGRAM ***

Using the DegLex Ordering with ordering a < b < A < B

Polynomials in the input basis:
A a - 1,
a A - 1,
B b - 1,
b B - 1,
a^3 - 1,
b^2 - 1,
a b a b - 1,
[7 Polynomials]

Computing Groebner Basis...
Added Polynomial #8 to Basis...
Added Polynomial #9 to Basis...
Added Polynomial #10 to Basis...
Added Polynomial #11 to Basis...
Added Polynomial #12 to Basis...
Added Polynomial #13 to Basis...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
Added Polynomial #16 to Basis...
...Groebner Basis Computed.

Here is the Groebner Basis:
A a - 1,
a A - 1,
B b - 1,
b B - 1,
a^3 - 1,
```

```
b^2 - 1,
a b a b - 1,
B - b,
a^2 - A,
A^2 - a,
a b a - b,
b a b - A,
A b a - a b,
a b A - b a,
b A - a b,
A b - b a,
[16 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
A a - 1,
a A - 1,
b^2 - 1,
B - b,
a^2 - A,
A^2 - a,
a b a - b,
b a b - A,
b A - a b,
A b - b a,
[10 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.

>
```

- **Output File:**

```
> more examples/rws/s3.deg
B; A; b; a;
A*a - 1;
a*A - 1;
b^2 - 1;
B - b;
```

```
a^2 - A;
A^2 - a;
a*b*a - b;
b*a*b - A;
b*A - a*b;
A*b - b*a;
>
```

4.6 Logged Noncommutative Gröbner Bases

- **Task:** Compute a logged noncommutative Wreath Product Gröbner Basis for the ideal generated by the set of polynomials $F = \{aA - 1,\ Aa - 1,\ bB - 1,\ Bb - 1,\ a^3 - b^2\}$ over the polynomial ring $\mathbb{Q}\langle A, a, B, b\rangle$.
- **Input File:**

```
> more examples/wreath/wreath2.in
A; a; B; b;
a*A-1;
A*a-1;
b*B-1;
B*b-1;
a^3 - b^2;
>
```

- **Session Output:**

```
> ncgrob -q -w examples/wreath/wreath2.in

*** NONCOMMUTATIVE GROEBNER BASIS PROGRAM ***

Using the Wreath Product Ordering with ordering b < B < a < A

Polynomials in the input basis:
a A - 1,
A a - 1,
b B - 1,
B b - 1,
a^3 - b^2,
[5 Polynomials]
```

```
Computing Groebner Basis...
Added Polynomial #6 to Basis...
Added Polynomial #7 to Basis...
Added Polynomial #8 to Basis...
Added Polynomial #9 to Basis...
Added Polynomial #10 to Basis...
Added Polynomial #11 to Basis...
Added Polynomial #12 to Basis...
Added Polynomial #13 to Basis...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
Added Polynomial #16 to Basis...
Added Polynomial #17 to Basis...
Added Polynomial #18 to Basis...
...Groebner Basis Computed.

Here is the Groebner Basis:
a A - 1,
A a - 1,
b B - 1,
B b - 1,
a^3 - b^2,
b^2 a - a b^2,
A b^2 - a^2,
b^2 A - a^2,
B a b^2 - b a,
A b - a^2 B,
b A - B a^2,
A - a^2 B^2,
B a^2 b - b a^2 B,
B^2 a^2 - a^2 B^2,
B a b - b a B,
a^2 B^2 a - 1,
B a^2 - b a^2 B^2,
B a - b a B^2,
[18 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.
```

```
Here is the Reduced Groebner Basis:
b B - 1,
B b - 1,
a^3 - b^2,
b^2 a - a b^2,
A - a^2 B^2,
B a - b a B^2,
[6 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Quotients to Disk... Done.

>
```

- **Output Files:**

```
> more examples/wreath/wreath2.wp
A; a; B; b;
b*B - 1;
B*b - 1;
a^3 - b^2;
b^2*a - a*b^2;
A - a^2*B^2;
B*a - b*a*B^2;
> more examples/wreath/wreath2.wp.q
F1; F2; F3; F4; F5; A; a; B; b;
F3;
F4;
F5;
-F5*a + a*F5;
-A*F5*B^2 - A*b*F3*B - A*F3 + F2*a^2*B^2;
B*F5*a*B^2 - B*a*F5*B^2 - B*a*b*F3*B - B*a*F3 + F4*b*a*B^2;
>
```

4.7 Noncommutative Involutive Bases 1

- **Task:** Using the DegLex monomial ordering and the Left involutive division [5], compute a noncommutative Involutive Basis for the ideal generated by the set of polynomials $F = \{x^2 - 2xy + 3,\ 2xy + y^2 + 5\}$ over the polynomial ring $\mathbb{Q}\langle x, y\rangle$.

- **Input File:**

```
> more examples/test/test2.in
x; y;
- 2*x*y + x*x + 3;
2*x*y + y*y + 5;
>
```

- **Session Output:**

```
> ncinv -d examples/test/test2.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) ***

Using the DegLex Ordering with ordering y < x

Polynomials in the input basis:
x^2 - 2 x y + 3,
2 x y + y^2 + 5,
[2 Polynomials]

Computing an Involutive Basis...
Added a first polynomial to G...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
Added a polynomial to G       (4)...
Added a polynomial to G       (5)...
Discarded a polynomial from G (4)...
Discarded a polynomial from G (3)...
Added a polynomial to G       (4)...
Added a polynomial to G       (5)...
Added a polynomial to G       (6)...
Discarded a polynomial from G (5)...
Discarded a polynomial from G (4)...
Added a polynomial to G       (5)...
Added a polynomial to G       (6)...
Added a polynomial to G       (7)...
Discarded a polynomial from G (6)...
Discarded a polynomial from G (5)...
Discarded a polynomial from G (4)...
Discarded a polynomial from G (3)...
```

```
Discarded a polynomial from G (2)...
Discarded a polynomial from G (1)...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
Added a polynomial to G       (4)...
Added a polynomial to G       (5)...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
2 y x + y^2 + 5, (y x, 1),
2 x y + y^2 + 5, (y x, 1),
x^2 + y^2 + 8, (y x, 1),
5 y^3 - 10 x + 37 y, (y x, 1),
5 x y^2 + 5 x - 6 y, (y x, 1),
[5 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
2 y x + y^2 + 5,
2 x y + y^2 + 5,
x^2 + y^2 + 8,
5 y^3 - 10 x + 37 y,
[4 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

>
```

- **Output Files:**

```
> more examples/test/test2.deg.inv
x; y;
2*y*x + y^2 + 5; (y x, 1);
2*x*y + y^2 + 5; (y x, 1);
x^2 + y^2 + 8; (y x, 1);
5*y^3 - 10*x + 37*y; (y x, 1);
5*x*y^2 + 5*x - 6*y; (y x, 1);
```

```
> more examples/test/test2.deg
x; y;
2*y*x + y^2 + 5;
2*x*y + y^2 + 5;
x^2 + y^2 + 8;
5*y^3 - 10*x + 37*y;
>
```

4.8 Noncommutative Involutive Bases 2

- **Task:** Using the DegRevLex monomial ordering and the Left overlap involutive division [5], compute a noncommutative Involutive Basis for the ideal generated by the set of polynomials $F = \{x^3 + 3xy + y^3,\ x + y^2\}$ over the polynomial ring $\mathbb{Q}\langle x, y\rangle$.
- **Input File:**

```
> more examples/posso/posso14.in
x; y;
x^3 + 3*x*y + y^3;
x + y^2;
>
```

- **Session Output:**

```
> ncinv -s1 examples/posso/posso14.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (LOCAL DIVISION) ***

Using the DegRevLex Ordering with ordering y < x

Polynomials in the input basis:
y^3 + x^3 + 3 x y,
y^2 + x,
[2 Polynomials]

Computing an Involutive Basis...
Added a first polynomial to G...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
Discarded a polynomial from G (2)...
```

```
Discarded a polynomial from G (1)...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
x^3 + 2 y x, (y x, 1),
y^2 + x, (y x, x),
x y - y x, (y x, x),
[3 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
x^3 + 2 y x,
y^2 + x,
x y - y x,
[3 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

>
```

- **Output Files:**

```
> more examples/posso/posso14.drl.inv
x; y;
x^3 + 2*y*x; (y x, 1);
y^2 + x; (y x, x);
x*y - y*x; (y x, x);
> more examples/posso/posso14.drl
x; y;
x^3 + 2*y*x;
y^2 + x;
x*y - y*x;
>
```

4.9 Noncommutative Involutive Bases 3

- **Task:** Using the Wreath Product monomial ordering; the Strong Right Overlap involutive division [5]; and thick divisors [5], compute a noncommutative Involutive Basis for the ideal generated by the set of polynomials $F = \{xy - z,\ 2x + yz + z,\ x + yz\}$ over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$.

- **Input File:**

```
> more examples/test/test3.in
x; y; z;
x*y - z;
2*x + y*z + z;
x + y*z;
>
```

- **Session Output:**

```
> ncinv -s2 -e2 -m2 -w examples/test/test3.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (LOCAL DIVISION) ***

Using the Wreath Product Ordering with ordering z < y < x

Polynomials in the input basis:
x y - z,
2 x + y z + z,
x + y z,
[3 Polynomials]

Computing an Involutive Basis...
Added a first polynomial to G...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
Discarded a polynomial from G (2)...
Discarded a polynomial from G (1)...
Added a polynomial to G       (2)...
Added a polynomial to G       (3)...
Added a polynomial to G       (4)...
Added a polynomial to G       (5)...
...Involutive Basis Computed.
```

```
Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
z^2, (1, z y x),
y z - z, (y, z y x),
z y + z, (1, z y x),
z x, (1, z y x),
x + z, (y, z y x),
[5 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
z^2,
y z - z,
z y + z,
x + z,
[4 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

>
```

- **Output Files:**

```
> more examples/test/test3.wp.inv
x; y; z;
z^2; (1, z y x);
y*z - z; (y, z y x);
z*y + z; (1, z y x);
z*x; (1, z y x);
x + z; (y, z y x);
> more examples/test/test3.wp
x; y; z;
z^2;
y*z - z;
z*y + z;
x + z;
>
```

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