Thesis Corrections

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Contents

Notes:

- All page numbers refer to the 'old' version of the thesis.
- Changes appear in red throughout.
- I would like to thank both referees for providing the majority of these corrections.

1 Chapter 1

- Page 10, Definition 1.1.6: Extra definition added (1.1.7) to define noncommutative ring.

> **Definition 1.1** A ring R is *commutative* if multiplication (as well as addition) is commutative, that is $r_1 \times r_2 = r_2 \times r_1$ for all $r_1, r_2 \in R$.
>
> **Definition 1.2** A ring R is *noncommutative* if $r_1 \times r_2 \neq r_2 \times r_1$ for some $r_1, r_2 \in R$.

- Page 11: change made 'coefficients' → 'images'.
- Page 11, Definition 1.1.11: Amended to include the definition of how to add and multiply functions in $R[x_1, \ldots, x_n]$. Also Remark 1.1.13 removed to save repeating some similar information.
- Page 12, Definition 1.1.15: Amended as above to include the definition of how to add and multiply functions in $R\langle x_1, \ldots, x_n\rangle$.
- Page 17: Definition 1.2.10 changed to Remark.

2 Chapter 2

- Page 24: Reference added to the translation lemma which allows us to deduce that f_1 and f_2 reduce to the same polynomial by showing that $f_1 - f_2$ reduces to 0. Change:

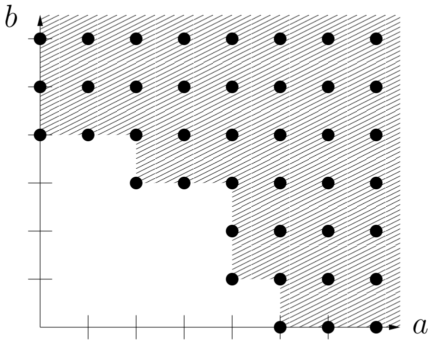
> (⇒) By Newman's Lemma (cf. [7], page 176), showing that the remainder of the division of p by G is unique is equivalent to showing that the division process is *locally confluent*, that is if there are polynomials f, f_1, $f_2 \in \mathcal{R}$ with $f_1 = f - t_1g_1$ and $f_2 = f - t_2g_2$ for terms t_1, t_2 and $g_1, g_2 \in G$, then there exists a polynomial $f_3 \in \mathcal{R}$ such that both f_1 and f_2 reduce to f_3. By the translation lemma (cf. [7], page 200), this in turn is equivalent to showing that the polynomial $f_2 - f_1 = t_1g_1 - t_2g_2$ reduces to zero, which is what we shall now do.

- Page 26, Dickson's Lemma: Indicated where the proof has been modified from:

  Proof (cf. [22], Page 47): Let J be a monomial ideal over $\mathcal{R}$ generated by a set of monomials S. ...

- Page 27: Added an example of Dickson's Lemma:

  Example 2.1 Let $S = \{y^4,\ xy^4,\ x^2y^3,\ x^3y^3,\ x^4y,\ x^k\}$ be an infinite set of monomials generating an ideal J over the polynomial ring $\mathbb{Q}[x, y]$, where k is an integer such that $k \geqslant 5$. We can visualise J by using the following monomial lattice, where a point (a, b) in the lattice (for non-negative integers a, b) corresponds to the monomial x^ay^b, and the shaded region contains all monomials which are reducible by some member of S (and hence belong to J).



To show that J can be finitely generated, we need to construct the set T as described in the proof of Dickson's Lemma. The first step in doing this is to construct the sequence of sets $S_j = \left\{\frac{s}{\gcd(s,\, y^j)} \mid s \in S\right\}$ for all $j \geqslant 0$.

$$\begin{aligned}
S_0 &= \{y^4,\ xy^4,\ x^2y^3,\ x^3y^3,\ x^4y,\ x^k\} = S \\
S_1 &= \{y^3,\ xy^3,\ x^2y^2,\ x^3y^2,\ x^4,\ x^k\} \\
S_2 &= \{y^2,\ xy^2,\ x^2y,\ x^3y,\ x^4,\ x^k\} \\
S_3 &= \{y,\ xy,\ x^2,\ x^3,\ x^4,\ x^k\} \\
S_4 &= \{y^0 = 1,\ x,\ x^2,\ x^3,\ x^4,\ x^k\} \\
S_{j+1} &= S_j \quad \text{for all } j+1 \geqslant 5.
\end{aligned}$$

Each set S_j gives rise to an ideal C_j consisting of all monomials $m \in \langle S_j \rangle$ of the form $m = x^i$ for some $i \geqslant 0$. Because each of these ideals is an ideal over the polynomial ring $\mathbb{Q}[x]$, we can use an inductive hypothesis to give us a finite generating set T_j for each C_j. In this case, the first paragraph of the proof of Dickson's Lemma tells us how to apply the inductive hypothesis — each set T_j is formed by choosing the monomial $m \in S_j$ of lowest degree such that $m = x^i$ for some $i \geqslant 0$.

$$\begin{aligned} T_0 &= \{x^5\} \\ T_1 &= \{x^4\} \\ T_2 &= \{x^4\} \\ T_3 &= \{x^2\} \\ T_4 &= \{x^0 = 1\} \\ T_{j+1} &= T_j \text{ for all } k+1 \geqslant 5. \end{aligned}$$

We can now deduce that

$$T = \{x^5\} \cup \{x^4y\} \cup \{x^4y^2\} \cup \{x^2y^3\} \cup \{y^4\} \cup \{y^5\} \cup \cdots$$

is a generating set for J. Further, because $T_{j+1} = T_j$ for all $k+1 \geqslant 5$, we can also deduce that the set

$$T' = \{x^5,\ x^4y,\ x^4y^2,\ x^2y^3,\ y^4\}$$

is a finite generating set for J (a fact that can be verified by drawing a monomial lattice for T' and comparing it with the above monomial lattice for the set S).

- Page 28, Correctness and Termination of Buchberger's Algorithm: Indicated where the proof has been modified from:

  Proof (cf. [7], Page 213): *Correctness.* If the algorithm terminates, it does do with a set of polynomials G with the property that

- Page 28: As written, $J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$ do generate the same ideal. I forgot that the condition should apply to the corresponding sets of lead monomials, not to the sets of polynomials; I have changed the termination paragraph as follows:

  Termination. If the algorithm does not terminate, then an endless sequence of polynomials must be added to the set G so that the set A

never becomes empty. Let $G_0 \subset G_1 \subset G_2 \subset \cdots$ be the successive values of G. If we consider the corresponding sequence $\mathrm{LM}(G_0) \subset \mathrm{LM}(G_1) \subset \mathrm{LM}(G_2) \subset \cdots$ of lead monomials, we note that these sets generate an ascending chain of ideals $J_0 \subset J_1 \subset J_2 \subset \cdots$ because each time we add a monomial to a particular set $\mathrm{LM}(G_k)$ to form the set $\mathrm{LM}(G_{k+1})$, the monomial we choose is irreducible with respect to $\mathrm{LM}(G_k)$, and hence does not belong to the ideal J_k. However the Ascending Chain Condition tells us that such a chain of ideals must eventually become constant, so there must be some $i \geqslant 0$ such that $J_i = J_{i+1} = \cdots$. It follows that the algorithm will terminate once the set G_i has been constructed, as all of the S-polynomials left in A will now reduce to zero (if not, some S-polynomial left in A will reduce to a non-zero polynomial s_1' whose lead monomial is irreducible with respect to $\mathrm{LM}(G_i)$, allowing us to construct an ideal $J_{i+1} = \langle \mathrm{LM}(G_i) \cup \{\mathrm{LM}(s_1')\} \rangle \supset \langle \mathrm{LM}(G_i) \rangle = J_i$, contradicting the fact that $J_{i+1} = J_i$.)

- Page 41: I have changed the wordings of all four definitions of logged bases to follow the pattern for the following definition of logged commutative Gröbner Basis:

  Let $G = \{g_1, \dots, g_p\}$ be a Gröbner Basis computed from an initial basis $F = \{f_1, \dots, f_m\}$. We say that G is a *Logged Gröbner Basis* if, for each $g_i \in G$, we have an explicit expression of the form

  $$g_i = \sum_{\alpha=1}^{\beta} t_\alpha f_{k_\alpha},$$

  where the t_α are terms and $f_{k_\alpha} \in F$ for all $1 \leqslant \alpha \leqslant \beta$.

3 Chapter 3

- Page 55, Proposition 3.4.1: Indicated where the proof has been modified from:

  Proof (cf. [37], Appendix A): To be able to describe an S-polynomial corresponding to an overlap (as placed in the monomial $\ell_1 \mathrm{LM}(f) r_1$) between $\mathrm{LM}(h)$ and either $\mathrm{LM}(f)$ or $\mathrm{LM}(g)$, we introduce the following notation. …

4 Chapter 4

- Page 68: I have modified this section to better introduce the role of U.

In Definition 1.2.9, we saw that a commutative monomial u_1 is divisible by another monomial u_2 if there exists a third monomial u_3 such that $u_1 = u_2u_3$; we also introduced the notation $u_2 \mid u_1$ to denote that u_2 is a divisor of u_1, a divisor we shall now refer to as a *conventional* divisor of u_1. For a particular choice of an involutive division I, we say that u_2 is an *involutive* divisor of u_1, written $u_2 \mid_I u_1$, if, given a partitioning (by I) of the variables in the polynomial ring into sets of *multiplicative* and *nonmultiplicative* variables for u_2, all variables in u_3 are in the set of multiplicative variables for u_2.

Example 4.1 Let $u_1 := xy^2z^2$; $u_1' := x^2yz$ and $u_2 := xz$ be three monomials over the polynomial ring $\mathcal{R} := \mathbb{Q}[x, y, z]$, and let an involutive division I partition the variables in $\mathcal{R}$ into the following two sets of variables for the monomial u_2: multiplicative $= \{y, z\}$; nonmultiplicative $= \{x\}$. It is true that u_2 conventionally divides both monomials u_1 and u_1', but u_2 only involutively divides monomial u_1 as, defining $u_3 := y^2z$ and $u_3' := xy$ (so that $u_1 = u_2u_3$ and $u_1' = u_2u_3'$), we observe that all variables in u_3 are in the set of multiplicative variables for u_2, but the variables in u_3' (in particular the variable x) are not all in the set of multiplicative variables for u_2.

More formally, an involutive division I works with a set of monomials U over a polynomial ring $R[x_1, \ldots, x_n]$ and assigns a set of multiplicative variables $\mathcal{M}_I(u, U) \subseteq \{x_1, \ldots, x_n\}$ to each element $u \in U$. It follows that, *with respect to U*, a monomial w is divisible by a monomial $u \in U$ if $w = uv$ for some monomial v and all the variables that appear in v also appear in the set $\mathcal{M}_I(u, U)$.

Definition 4.2 Let M denote the set of all monomials in the polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$, and let $U \subset M$. The *involutive cone* $\mathcal{C}_I(u, U)$ of any monomial $u \in U$ with respect to some involutive division I is defined as follows.

$$\mathcal{C}_I(u,U) = \{uv \text{ such that } v \in M \text{ and } u \mid_I uv\}.$$

Remark 4.3 We may think of an involutive cone of a particular monomial u as containing all monomials that are involutively divisible by u.

Up to now, we have not mentioned any restriction on how we may assign multiplicative variables to a particular set of monomials. Let us now specify the rules that ensure that a particular scheme of assigning multiplicative variables may be referred to as an involutive division.

Definition 4.4 Let M denote the set of all monomials in the polynomial ring $\mathcal{R} = R[x_1, \dots, x_n]$. An *involutive division* I on M is defined if, given any finite set of monomials $U \subset M$, we can assign a set of *multiplicative variables* $\mathcal{M}_I(u,U) \subseteq \{x_1, \dots, x_n\}$ to any monomial $u \in U$ such that the following two conditions are satisfied.

(a) If there exist two monomials $u_1, u_2 \in U$ such that $\mathcal{C}_I(u_1,U) \cap \mathcal{C}_I(u_2,U) \neq \emptyset$, then either $\mathcal{C}_I(u_1,U) \subset \mathcal{C}_I(u_2,U)$ or $\mathcal{C}_I(u_2,U) \subset \mathcal{C}_I(u_1,U)$.

(b) If $V \subset U$, then $\mathcal{M}_I(v,U) \subseteq \mathcal{M}_I(v,V)$ for all $v \in V$.

Remark 4.5 Informally, . . .

- Page 80: Reworded Definition 4.3.5:

Definition 4.6 (Constructivity) Let I be an involutive division, and let U be an arbitrary set of monomials over a polynomial ring $R[x_1, \dots, x_n]$. We say that I is *constructive* if, given any monomial $u \in U$ and any nonmultiplicative variable $x_i \notin \mathcal{M}_I(u,U)$ satisfying the following two conditions, no monomial $w \in \mathcal{C}_I(U)$ exists such that $ux_i \in \mathcal{C}_I(w, U \cup \{w\})$.

(a) $ux_i \notin \mathcal{C}_I(U)$.

(b) If there exists a monomial $v \in U$ and a nonmultiplicative variable $x_j \notin \mathcal{M}_I(v,U)$ such that $vx_j \mid ux_i$ but $vx_j \neq ux_i$, then $vx_j \in \mathcal{C}_I(U)$.

- Page 81, line -3: Change from

  > Else let δ (where $1 \leqslant \delta < \alpha$) be the second greatest integer such that $e_\mu^\delta > 0$.

  to

  > Else let δ (where $1 \leqslant \delta < \alpha$) be the second greatest integer such that $e_{w_1}^\delta > 0$.

- Page 85: Definition changed to

  > An involutive division I is *Noetherian* if, given any finite set of monomials U, there is a finite Involutive Basis $V \supseteq U$ with respect to I and some arbitrary admissible monomial ordering O.

- Page 86, Remark 4.4.7: Change to

  > Stability ensures that any set of distinct monomials is autoreduced. In particular, if a set U of monomials is autoreduced, and if we add a monomial $u \notin U$ to U, then the resultant set $U \cup \{u\}$ is also autoreduced.

- Page 99: The set $d(G')$ need not be an Involutive Basis, but it will always be a Gröbner Basis. To make this clear, I have added the following remark to the end of Section 4.5.2:

  > **Remark 4.7** Although the set G returned by the procedure outlined in Definition 4.5.1 may not always be an Involutive Basis for the ideal generated by F, because the set G' will always be an Involutive Basis (and hence also a Gröbner Basis), we can state that G will always be a Gröbner Basis for the ideal generated by F (cf. Definition 2.5.7).

  I have also altered an example earlier on (see page 94) where the same observation was made (I have removed the observation from the example):

  > Taking the homogeneous route, we can homogenise F (with respect to Lex) to obtain the set $F' := \{x_1^2y + x_2^3,\ x_1y^2 + x_3^3\}$ over the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, y]$. Computing an Involutive Basis for F' with respect to the Janet involutive division, we obtain the set $G' := \{x_2^3y^3 + x_3^6,\ x_1x_2^2y^3 + x_2^2x_3^3y,\ x_1x_2y^3 + x_2x_3^3y,\ x_1y^3 + x_3^3y, x_1y^2 + x_3^3, x_1x_3^3y - x_2^3y^2,\ x_1^2x_3^2y + x_2^3x_3^2,$

$x_1^2x_3y + x_2^3x_3,\ x_1^2y + x_2^3,\ x_1x_3^3 - x_2^3y\}$. Finally, if we dehomogenise G', we obtain the set $H := \{x_2^3 + x_3^6,\ x_1x_2^2 + x_2^2x_3^3,\ x_1x_2 + x_2x_3^3,\ x_1 + x_3^3,\ x_1x_3^3 - x_2^3,\ x_1^2x_3^2 + x_2^3x_3^2,\ x_1^2x_3 + x_2^3x_3,\ x_1^2 + x_2^3\}$; however this set is not a Janet Involutive Basis for F, as can be verified by checking that (with respect to H) the variable x_3 is nonmultiplicative for the polynomial $x_2^3 + x_3^6$, and the prolongation of the polynomial $x_2^3 + x_3^6$ by the variable x_3 is involutively irreducible with respect to H.

- Page 100 (Too many for's): Paragraph changed to

  > In the former case, we can express g_i in terms of members of F by substitution because
  >
  > $$g_i = hx_j - \sum_{\alpha=1}^{\beta} t_\alpha h_{k_\alpha}$$
  >
  > for a variable x_j; terms t_α and polynomials h and h_{k_α} which we already know how to express in terms of members of F. In the latter case,
  >
  > $$g_i = h - \sum_{\alpha=1}^{\beta} t_\alpha h_{k_\alpha}$$
  >
  > for terms t_α and polynomials h and h_{k_α} which we already know how to express in terms of members of F, so it follows that we can again express g_i in terms of members of F.

  Similar changes made to Chapter 5 (Proposition 5.8.2, Page 160).

5 Chapter 5

- Page 102, First paragraph: For clarity, any mention of the set U at this stage has been deleted as all we are concerned about here is deciding whether a conventional divisor is an involutive divisor - not how the multiplicative variables were obtained.

- Page 102, line 6: Sentence changed to

  > In a noncommutative polynomial ring, an involutive division will again induce a restricted form of division.

- Page 103: Definition 5.1.3 changed to Remark.

- Page 103, Definition 5.1.5: I have changed the noncommutative definitions to match the commutative ones and I have also deleted the definition of 'left cone' and 'right cone' ($\mathcal{C}_I^L(u,U)$ and $\mathcal{C}_I^R(y,U)$) as they are not used anywhere else in the thesis. New text:

  > Let us now formally define what is meant by a (noncommutative) involutive division.
  >
  > **Definition 5.1** Let M denote the set of all monomials in a noncommutative polynomial ring $\mathcal{R} = R\langle x_1,\dots,x_n\rangle$, and let $U \subset M$. The involutive cone $\mathcal{C}_I(u,U)$ of any monomial $u \in U$ with respect to some involutive division I is defined as follows.
  >
  > $$\mathcal{C}_I(u,U) = \{v_1uv_2 \text{ such that } v_1, v_2 \in M \text{ and } u \mid_I v_1uv_2\}.$$

- Page 104: Definition 5.1.6 changed to:

  > **Definition 5.2** Let M denote the set of all monomials in a noncommutative polynomial ring $\mathcal{R} = R\langle x_1,\dots,x_n\rangle$. A *strong* involutive division I is defined on M if, given any finite set of monomials $U \subset M$, we can assign a set of left multiplicative variables $\mathcal{M}_I^L(u,U) \subseteq \{x_1,\dots,x_n\}$ and a set of right multiplicative variables $\mathcal{M}_I^R(u,U) \subseteq \{x_1,\dots,x_n\}$ to any monomial $u \in U$ such that the following three conditions are satisfied. ...

- Page 108 (and Page 75): 6 occurrences of $\mathrm{Rem}_I(f)$ changed to $\mathrm{Rem}_I(f,P)$ (3 on page 76; 3 on pages 108/109).

- Page 109: Sentence changed to

  > In the noncommutative case, we cannot hope to produce a carbon copy of the above results because a finitely generated basis may have an infinite Gröbner Basis, leading to the conclusion that ...

- Page 112, line 5: change made: disjoint → distinct.

- Page 112, Proposition 5.4.3: change made: 'an arbitrary' → 'a given'. Similar change for Proposition 4.3.3 (page 78).

- Page 112, Proposition 5.4.3: For clarity, I have changed the x_1 to y_1, x_2 to y_2, and so on. Similar changes made to the proof of Proposition 4.3.3 (page 78).

- Page 115, Proposition 5.4.7: change made: 'an arbitrary' → 'a given'.

- Page 122, Definition 5.5.10: Change 'set' → 'assign' in 2 places (one on page 122, one on page 123).

- Page 123, Remark 5.5.11: Small change to the first sentence:

  > One possible algorithm for the left overlap division is presented in Algorithm 13, where the reason...

- Page 124, Algorithm 13: Removed 'distinct' from the phrase 'distinct monomials' in the Input, which then causes a change from $u_1 > u_2 > \cdots > u_m$ to $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$. This change also applies to Algorithms 14 (page 133) and 15 (page 134).

- Page 131, 'We therefore propose...': paragraph reworded for clarification:

  > **(4 and 5)** In Definition 3.1.2, we defined a prefix overlap to be an overlap where, given two monomials m_1 and m_2 such that $\deg(m_1) \geqslant \deg(m_2)$, a prefix of m_1 is equal to a suffix of m_2; suffix overlaps were defined similarly. If we drop the condition on the degrees of the monomials, it is clear that every suffix overlap can be treated as a prefix overlap (by swapping the roles of m_1 and m_2); this allows us to deal with the case of a prefix overlap only.

  The same modification is made in Appendix A (page 206) as the same wording is used there.

- Page 132, line -6: change made: 'weak;' → 'weak,'.

- Pages 132–138: Because the Strong Left Overlap Division uses thick divisors, I have changed some material in Section 5.5.3 to make it clear under which circumstances (using thin/thick divisors) certain propositions hold.

  - Page 135: An extra proposition has been added after Proposition 5.5.18 to state that $\mathcal{S}$ is a Gröbner involutive division (this is for completeness).

    > **Proposition 5.3** *The strong left overlap division is a Gröbner involutive division.*

Proof: We refer to the proof of Proposition 5.5.16, replacing $\mathcal{O}$ by $\mathcal{S}$. $\square$

Remark 5.4 Propositions 5.5.18 and 5.3 apply either when using thin divisors or when using thick divisors.

- Page 135, Proposition 5.5.19: Changed to

  Proposition 5.5 *With respect to thick divisors, the strong left overlap division is a strong involutive division.*

- Page 135: Extra Proposition added after Proposition 5.5.19 to show that, with respect to thin divisors, the strong left overlap division is not strong.

  Proposition 5.6 *With respect to thin divisors, the strong left overlap division is a weak involutive division.*

  Proof: Let $U := \{xy\}$ be a set of monomials over the polynomial ring $\mathbb{Q}\langle x, y\rangle$. Here are the multiplicative variables for U with respect to the strong left overlap division $\mathcal{S}$.

u	$\mathcal{M}^L_{\mathcal{S}}(u, U)$	$\mathcal{M}^R_{\mathcal{S}}(u, U)$
xy	$\{x, y\}$	$\{y\}$

  For $\mathcal{S}$ to be strong with respect to thin divisors, the monomial xy^2xy, which is conventionally divisible by xy in two ways, must only be involutively divisible by xy in one way (this is the Unique Divisor condition of Definition 5.1.6). However it is clear that xy^2xy is involutively divisible by xy in two ways with respect to thin divisors, so $\mathcal{S}$ must be a weak involutive division with respect to thin divisors. $\square$

- Page 137, Example 5.5.20: Changed to:

  When we apply Algorithm 12 to F with respect to the DegLex monomial ordering, thick divisors and the strong left overlap division, F (as in Example 5.5.12) is returned to us as the output Locally Involutive Basis.

- Page 137, Remark 5.5.21: Changed to:

  What this means is that the involutive cones of F (and in general any Locally Involutive Basis with respect to $\mathcal{S}$ and thick divisors) will be disjoint (because $\mathcal{S}$ is strong), ...

- Page 138, 2nd paragraph: Changed to:

  Let us now summarise (with respect to thin divisors) the properties of the involutive divisions we have encountered so far, where we note that any strong and continuous involutive division is by default a Gröbner involutive division.

Division	Continuous	Strong	Gröbner
Left	Yes	Yes	Yes
Right	Yes	Yes	Yes
Left Overlap	Yes	No	Yes
Right Overlap	Yes	No	Yes
Strong Left Overlap	Yes	No	Yes
Strong Right Overlap	Yes	No	Yes

- Page 133–135: To make it clear that the algorithm *does* form part of the definition, I have made the following changes:

  - Page 132, Definition 5.5.17 and the paragraph following it on page 133 have been changed to give:

    Definition 5.7 (The Strong Left Overlap Division $\mathcal{S}$) Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials. Assign multiplicative variables to U according to Algorithm 15, which (in words) performs the following two tasks.

    (a) Assign multiplicative variables to U according to the left overlap division.

    (b) Using the recipe provided in Algorithm 14, ensure that at least one variable in every monomial $u_j \in U$ is right nonmultiplicative for each monomial $u_i \in U$.

    Remark 5.8 As Algorithm 15 expects any input set to be ordered with respect to DegRevLex, we may sometimes have to reorder a set of monomials U to satisfy this condition before we can assign multiplicative variables to U according to the strong left overlap division.

  - Page 136, end of first paragraph changed as follows:

    It therefore remains to show that if x is assigned right nonmultiplicative for u_2 in the latter case (which will happen during the final step of assigning multiplicative variables to U according to $\mathcal{S}$), then x is also assigned right nonmultiplicative for u_1. But this is clear when we consider that Algorithm 14 is used to perform this final step, because for u_1 and u_2 in Algorithm 14, we will always analyse each monomial in U in the same order.

- Page 137, last paragraph's wording: changed to

  > It follows that when involutively reducing a polynomial with respect to F, the reduction path will be unique but the correct remainder may not always be obtained (in the sense that some of the terms in our 'remainder' may still be conventionally reducible by members of F)

- Page 138, line -9: Sentence starting 'For better advice...' deleted, and in the next paragraph '...is another matter for further discussion' changed to '...is a matter for further discussion'.

- Definition 5.5.23 and the following paragraph (which includes a footnote) changed as follows:

  > **Definition 5.9 (The Two-Sided Left Overlap Division $\mathcal{W}$)** Consider a set $U = \{u_1, \ldots, u_m\}$ of monomials, where all variables are assumed to be left and right multiplicative for all elements of U to begin with. Assign multiplicative variables to U according to Algorithm 16, which (in words) performs the following tasks.
  >
  > (a) For all possible ways that a monomial $u_j \in U$ is a subword of a (different) monomial $u_i \in U$, so that
  >
  > $$\text{Subword}(u_i, k, k + \deg(u_j) - 1) = u_j$$
  >
  > for some integer k, assign the variable $\text{Subword}(u_i, k-1, k-1)$ to be left nonmultiplicative for u_j if u_j is a suffix of u_i; and assign the variable $\text{Subword}(u_i, k + \deg(u_j), k + \deg(u_j))$ to be right nonmultiplicative for u_j if u_j is not a suffix of u_i.
  >
  > (b) For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that
  >
  > $$\text{Prefix}(u_i, k) = \text{Suffix}(u_j, k)$$

for some integer k and u_i is not a subword of u_j or vice-versa, use the recipe provided in the second half of Algorithm 16 to ensure that at least one of the following conditions are satisfied: (i) the variable $\text{Subword}(u_i, k+1, k+1)$ is right nonmultiplicative for u_j; (ii) the variable $\text{Subword}(u_j, \deg(u_j)-k, \deg(u_j)-k)$ is left nonmultiplicative for u_i.

Remark 5.10 For task (b) above, Algorithm 16 gives preference to monomials which are greater in the DegRevLex monomial ordering (given the choice, it always assigns a nonmultiplicative variable to whichever monomial out of u_i and u_j is the smallest); it also attempts to minimise the number of variables made nonmultiplicative by only assigning a variable to be nonmultiplicative if both the variables $\text{Subword}(u_i, k+1, k+1)$ and $\text{Subword}(u_j, \deg(u_j)-k, \deg(u_j)-k)$ are respectively right multiplicative and left multiplicative. These refinements will become crucial when proving the continuity of the division.

Also the phrase '(as defined in Algorithm 16).' removed from Example 5.5.24 on Page 140.

- Page 143: Change made: 'less' → 'fewer'.
- Page 143, Definition 5.5.29: changed 'set' to 'assign' in (a) and (b) (and also later on in Definition 5.5.30). Note: For Definitions 5.5.29 and 5.5.30, the order of U is not important, unlike the previous two divisions (strong left overlap and two-sided left overlap) where the order of U is important and I have tied in the algorithm to the definition to indicate this.
- Page 144: Change made: ';' → ','.
- Page 145: change made 'always' → 'potentially'.
- Page 159, Definition 5.8.1: see the earlier comment about page 41.

6 Chapter 6

- I have altered the final paragraph on page 162 as follows:

> The Gröbner Walk, introduced by Collart, Kalkbrener and Mall in [18], forms part of a family of basis conversion algorithms that can convert Gröbner Bases with respect to 'fast' monomial orderings to Gröbner Bases with respect to 'slow' monomial orderings (see Section 2.5.4 for a brief discussion of other basis conversion algorithms). This process is often quicker than computing a Gröbner Basis for the 'slow' monomial ordering directly, as can be demonstrated by stating that in our test implementation of the Gröbner Walk, it only takes half a second to compute a Lex Gröbner Basis for the basis F defined above.
>
> In this chapter, we will first recall the theory of the (commutative) Gröbner Walk, based on [18] and a paper [1] by Amrhein, Gloor and Küchlin; the reader is encouraged to read these papers in conjunction with this Chapter. We then describe two generalisations of the theory to give (i) a commutative Involutive Walk (due to Golubitsky [30]); and (ii) noncommutative Walks between harmonious monomial orderings.

- Page 165, line -9 of algorithm: change $[0, 1] \rightarrow (0, 1]$. Explanation: By allowing zero to be found as the next value of t, we potentially allow an infinite loop, as if $t = 0$ then we stay where we are on the walk. However, the value $t = 0$ will never be found, as it is impossible to find two monomials whose ω-values differ but whose $\omega(0)$-values are the same, as $\omega = \omega(0)$. Thus the interval $[0, 1]$ is technically correct. However, to eliminate any doubt I have changed the interval to $(0, 1]$ to make it perfectly clear that $t = 0$ can never be found. This change also applies in the following places: Page 166, 3rd bullet point; Page 168, final paragraph of Pass 1; Page 170, middle of page and final line; Page 172, line -9 of algorithm.

- Page 173, Section 6.2: Added a back reference to the earlier definition:

> In the commutative case, any monomial ordering can be represented by a matrix that provides a decomposition of the ordering in terms of the rows of the matrix. This decomposition is then utilised in the Gröbner Walk algorithm when (for example) we use the first row of the matrix to provide a set of initials for a particular basis G (cf. Definition 6.1.6).

- Page 176, Definition 6.2.13 changed as follows:

> **Definition 6.1** Two noncommutative monomial orderings O_1 and O_2 are said to be *harmonious* if (i) there exist functional decompositions $\Theta_1 =$

$\{\theta_{1_1}, \theta_{1_2}, \ldots\}$ and $\Theta_2 = \{\theta_{2_1}, \theta_{2_2}, \ldots\}$ defining O_1 and O_2 respectively; and (ii) the ordering functions θ_{1_1} and θ_{2_1} are identical and extendible.

- Page 178, line -2: Change to

  > Because H' is a Gröbner Basis for the ideal $\text{in}_\theta(J)$ with respect to O', there must be a polynomial $h' \in H'$ such that $h' \mid \text{in}_\theta(q)$. Let ...

- Page 179, line 3: Change to

  > However θ is compatible with O', so that
  >
  > $$\sum_{i=1}^{j} \ell_i g_i r_i \mid q.$$

- Page 184, line 5: Change to

  > Because H' is an Involutive Basis for the ideal $\text{in}_\theta(J)$ with respect to I and O', there must be a polynomial $h' \in H'$ such that $h' \mid_I \text{in}_\theta(q)$. Let ...

- Page 184, line 9: Change to

  > However θ is compatible with O' (in particular the multiplicative variables for H' and H with respect to I and O' will be identical), so that
  >
  > $$\sum_{i=1}^{j} \ell_i g_i r_i \mid_I q.$$

- Page 187: 2nd paragraph of Section 6.2.6 (and the following diagram) deleted.

7 Other

- Page 4: Sentence changed to

  > This effect is achieved through assigning a set of *multiplicative variables* to each polynomial in an Involutive Basis H, imposing a restriction on how polynomials may be divided by H by only allowing any polynomial $h \in H$ to be multiplied by its corresponding multiplicative variables.

- Bibliography: Reference [31] (Opal) changed to the following reference:

```
@PhDThesis{Keller97,
  author       = {Keller, Benjamin J.},
  title        = {Algorithms and Orders for Finding
                  Noncommutative Gr\"{o}bner Bases},
  school       = {Virginia Tech},
  year        = 1997,
}
```

- Index: Added a few extra items, and also added alternate entries (for example, the entry “leading monomial” now also appears under “monomial!leading”).