G3M15 Logic: Exam Paper, May 2002

Answer 3 questions out of 5 (Questions Done: 1, 3, 5)

(1) The following *equivalences* may be used in the following sub questions:

$\neg(A\vee B) \equiv \neg A\wedge\neg B$
$\neg(A\wedge B) \equiv \neg A\vee\neg B$
$A\rightarrow B \equiv \neg A\vee B$
$(A\vee B)\wedge C \equiv (A\wedge C)\vee(B\wedge C)$
$A\leftrightarrow B \equiv (A\rightarrow B)\wedge(B\rightarrow A)$
$(A\wedge B)\vee C \equiv (A\vee C)\wedge(B\vee C)$

(a) By defining appropriate *interpretations*, prove whether the following formulae are **satisfiable**, **valid** or **contradictory**:

(i) $(A\rightarrow(B\vee C))$
(ii) $(B\leftrightarrow(A\vee B))\wedge\neg(A\rightarrow B)$ **[4 marks]**

(b) Prove the following *equivalences*, showing *all* steps.

(i) $\neg(\neg r\wedge\neg(q\vee t)) \equiv r\vee q\vee t$
(ii) $((p\rightarrow q)\wedge\neg(q\rightarrow r)\wedge\neg r\wedge p) \equiv (q\wedge\neg r\wedge p)$ **[8 marks]**

(c) Prove, showing *all* steps, that if $U = \{p\leftrightarrow q, q\rightarrow r, p, \neg r\vee s\}$, then $U\models s$.
[8 marks]

(2) (a) Use *Ordered Binary Decision Diagrams* to prove that the following formula is valid: $((p\vee q)\wedge r)\vee\neg((p\vee q)\wedge r)$. **All** stages of the reduction process should be shown.
[5 marks]

(b) Using *Ordered Binary Decision Diagrams*, prove that the following formula is satisfiable but not valid: $(p\wedge q)\vee r$. **All** stages of the reduction process should be shown. **[6 marks]**

(c) Let $U = \{A_1, ..., A_n\}$. Prove that if $U\models B$ then $\models A_1\wedge A_2\wedge...\wedge A_n \rightarrow B$. **[9 marks]**

(3) The following two tables represent the α and β formulae used in the construction of semantic tableaux and are provided as an *aid* to the following sub questions:

α	α_1	α_2
$\neg\neg A_1$	A_1	
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \rightarrow A_2)$	A_1	$\neg A_2$
$A_1 \leftrightarrow A_2$	$A_1 \rightarrow A_2$	$A_2 \rightarrow A_1$

β	β_1	β_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \vee B_2$	B_1	B_2
$B_1 \rightarrow B_2$	$\neg B_1$	B_2
$\neg(B_1 \leftrightarrow B_2)$	$\neg(B_1 \rightarrow B_2)$	$\neg(B_2 \rightarrow B_1)$

(a) Construct the semantic tableaux for the following formulae and for each branch, where appropriate, construct the corresponding *Hintikka set*.

(i) $p \wedge (\neg q \vee \neg p)$

(ii) $(p \wedge q) \vee (\neg p \wedge \neg q)$ **[6 marks]**

(b) Use a semantic tableau to discover a *model* for the following formula and *define* that model: $p \wedge (q \vee r) \wedge (q \rightarrow \neg p)$. **[5 marks]**

(c) Prove the following formula is *valid* by showing that the completed semantic tableau for the *negation* of the formula is closed: $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$.
[9 marks]

(4) (a) For each set of clauses S below, derive the *simplest* possible set S' such that S is satisfiable if and only if S' is satisfiable, i.e. S ≈ S'. Show *all* stages of the derivation.

(i) $S = \{qr, \bar{q}p\bar{s}, \bar{p}pqs\}$

(ii) $S = \{pqr, \bar{q}, p\bar{r}s, qs, \bar{p}\bar{s}\}$ **[7 marks]**

(b) Determine the *satisfiability* of the following sets of clauses by applying the resolution procedure. No **simplification** of the sets should be carried out before applying the procedure.

(i) $\{\bar{p}\bar{q}r, pr, qr, \bar{r}\}$

(ii) $\{pqr, q\bar{r}\bar{s}, sp, \bar{p}, \bar{q}\}$ **[7 marks]**

(c) Let clause C be the *resolvent* of clauses C_1 and C_2. Prove that if C_1 and C_2 are *mutually satisfiable*, then C is also *satisfiable*. **[6 marks]**

(5) (a) Let F(x, y) mean '*x is the father of y*', and let M(x, y) mean '*x is the mother of y*'. Using these predicate symbols, translate the following sentences into predicate logic formulae:

(i) Everybody has a mother,
(ii) Everybody has a father and a mother,
(iii) Whoever has a mother has a father. **[8 marks]**

(b) Consider the formula $\forall x \exists y\ (p(x, a) \wedge p(y, x))$, where a is a *constant symbol*, p is a *predicate symbol*, and x and y are *variables*. Determine if the formula is true in the following interpretations, explaining your reasons:

(i) I = (**N**, {≥}, {0}), where **N** is the set of *positive integers* including zero.
(ii) I = (**Z**, {=}, {5}), where **Z** is the set of *all integers*, positive or negative.
[6 marks]

(c) Prove the formula $\forall x\ p(x) \rightarrow \exists x\ p(x)$ is *valid*. **[6 marks]**

G3M50 Wavelets: Exam Paper, May 2002

Answer 3 questions out of 5 (Questions Done: 1, 3, 4)

(1) (a) Write down the definition of the usual L_2 inner product for the vector space of **square-integrable complex valued functions of a real variable** defined on an interval [a, b]. **[2 marks]**

(b) Show that the set of *complex exponential functions* $\{e^{inx} : n \in \mathbf{Z}\}$ is an orthogonal set on the interval $[-\pi, \pi]$. **[4 marks]**

(c) State the **Euler formulae** for the coefficients $\{c_n\}$ of the complex form of the Fourier series of a function f(x) defined on the interval $[-\pi, \pi]$. **[2 marks]**

(d) Show that the *complex Fourier coefficients* of a **real-valued** function satisfy the relation $c_{-n} = \bar{c}_n$. **[3 marks]**

(e) Show that the complex Fourier coefficients of a *real-valued* function are **real** for an *even* function and **purely imaginary** for an *odd* function. **[4 marks]**

(f) Find the *complex form* of the Fourier series for the function

$f(x) = 1$ for $-{}^{\pi}/_{2} < x < {}^{\pi}/_{2}$, and
$f(x) = 0$ for $-\pi < x < -{}^{\pi}/_{2}$ and ${}^{\pi}/_{2} < x < \pi$. **[5 marks]**

(2) Let $\hat{f}(\omega)$ denote the *Fourier integral transform* of f(x).

(a) Derive expressions for the Fourier transforms of (i) $f(x-x_0)$ and (ii) $f(ax)$, where x_0 and a are *real constants*. **[5 marks]**

(b) Calculate the Fourier transform of the *characteristic function* $\chi_{[0,1)}(x)$ (also known as the indicator function, Haar scaling function or first order spline) defined by $\chi_{[0,1)}(x) = 1$ for $0 \le x < 1$, and $\chi_{[0,1)}(x) = 0$ otherwise. **[4 marks]**

(c) Use the *shift theorem result* in (a)(i) to obtain the Fourier transform of $\chi_{[-½,½)}(x)$ and deduce that it is *purely real*. **[7 marks]**

(d) Sketch *three graphs* showing the Fourier transform of $\chi_{[-½,½)}(x)$, the *amplitude spectrum* and the *phase spectrum*. **[4 marks]**

(3) (a) Explain what is meant by the term *direct sum* of two or more vector spaces.
[2 marks]

(b) Define a *multiresolution analysis* (MRA) of $L_2(\mathbf{R})$ in terms of the *scaling function spaces* V_j and derive the *two-scale refinement equation* for the scaling function ϕ in terms of the *coefficients* $\{h_k : k \in \mathbf{Z}\}$. **[6 marks]**

(c) Also define the *wavelet spaces* W_j, define the *wavelet* ψ and derive the *two-scale wavelet equation* in terms of the *coefficients* $\{g_k : k \in \mathbf{Z}\}$. **[4 marks]**

(d) Use the *two-scale relations* and the *conditions* that $\int\phi = 1$ and $\int\psi = 0$ to show that the *two-scale coefficients* $\{h_k : k \in \mathbf{Z}\}$ and $\{g_k : k \in \mathbf{Z}\}$ satisfy $\Sigma_k h_k = \surd(2)$ and $\Sigma_k g_k = 0$, and that the *Fourier transforms* of the scaling function and wavelet satisfy the following two expressions: $\hat{\phi}(0) = 1$ and $\hat{\psi}(0) = 0$. **[8 marks]**

(4) (a) Define the *z-transform* of a sequence. **[2 marks]**

(b) Define the convolution of two sequences and show that convolution reduces to multiplication in the z-domain. **[5 marks]**

(c) If $F(z)$ is the *z-transform* of a discrete signal $f(n)$ which is then **upsampled**, show that the resulting signal has z-transform given by $F(z^2)$. Similarly, show that **downsampling** results in a signal with z-transform given by $\tfrac{1}{2}\{F(z^{1/2})+F(-z^{1/2})\}$.
[7 marks]

(d) Consider analysis by the **fast wavelet transform** where the scaling function coefficients and wavelet coefficients are related by $c_{j-1,k} = \Sigma_l\, h_{l-2k}c_{j,l}$ and $d_{j-1,k} = \Sigma_l\, g_{l-2k}c_{j,l}$. Interpret these operations in terms of **convolution** and **upsampling** or **downsampling** filter bank operators and illustrate your answer by a *filter bank diagram*. **[6 marks]**

(5) Consider the *Haar* system of scaling functions and wavelets which you may assume define an *orthogonal multiresolution analysis*.

(a) Sketch different **translates** of the scaling functions and wavelets at *scale 0* and at *scale 1*. **[3 marks]**

(b) Write down the general *two-scale equations* and obtain the two-scale *coefficients* $\{h_k\}$ and $\{g_k\}$ for the Haar system. **[4 marks]**

(c) Define carefully the functions $\phi_{j,k}(x)$ and $\psi_{j,k}(x)$ for this system and sketch their *graphs*. **[3 marks]**

(d) Use the *general two-scale equations* $\phi_{j-1,k} = \Sigma_p\ h_{p-2k}\phi_{j,p}$ and $\psi_{j-1,k} = \Sigma_p\ g_{p-2k}\phi_{j,p}$ to derive expressions for the **analysis** stage of the fast wavelet transform, giving the scaling function coefficients $c_{j-1,k}$ and wavelet coefficients $d_{j-1,k}$ at scale j-1 in terms of the scaling function coefficients $c_{j,k}$ at scale j. **[4 marks]**

(e) Illustrate the *fast wavelet analysis* for the following function f(x) which is **piecewise constant** and is **zero** outside the interval [0, 3]. Note that $f(x) \in V_2(\mathbf{R})$, where $V_2(\mathbf{R})$ is the *Haar scaling function space* at scale 2, a scaling function space which has *basis set* $\{\phi_{2,k} : k \in \mathbf{Z}\}$. First express the function as a *sum of basis functions* at scale 2 in the form $f(x) = \Sigma_k\ c_{2,k}\phi_{2,k}(x)$. Then use the *fast wavelet analysis* to obtain the scaling function and wavelet coefficients at scale 1 and then at scale 0. Hence *sketch* the projection of f(x) onto the **lower scale** scaling function and wavelet spaces V_1, W_1, V_0 and W_0. **[6 marks]**