G3M21 Computational Geometry: Exam Paper, January 2002

Answer 3 questions out of 5 (Questions Done: 1, 4, 5)

(1) Prove that the relation on $\mathbf{R}^3\backslash\{\mathbf{0}\}$ defined by
$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow (x_1, y_1, z_1) = r(x_2, y_2, z_2)$ for some $r \neq 0$
is an *equivalence relation*.

Explain how this equivalence is used to define the projective plane $\mathbf{P}^2$ and how *affine transformations* of $\mathbf{R}^2$ may be represented by *linear transformations* of $\mathbf{P}^2$. **[10 marks]**

Obtain the matrices associated to the following affine transformations:
(i) *Rotation* through an angle θ about the point (5, 7);
(ii) *Reflection* in the line y = x + 3. **[10 marks]**

(2) Let $\mathbf{v} = (l, m, n)$ be a unit vector in $\mathbf{R}^3$. Show that the *Householder matrix*
$H_\mathbf{v} = I_3 - 2\mathbf{v}^t\mathbf{v}$
is *symmetric* and *orthogonal*. Show also that $H_\mathbf{v}$ is the reflection matrix when $\mathbf{R}^3$ is reflected in the plane Π through **0** with normal **v**. **[8 marks]**
Hence obtain the projection matrix for the *parallel projection* of $\mathbf{R}^3$ onto Π. **[4 marks]**
Calculate the product $H_\mathbf{u}H_\mathbf{v}$ when $\mathbf{u} = ({}^1/_3, {}^2/_3, -{}^2/_3)$ and $\mathbf{v} = (-{}^2/_3, {}^2/_3, {}^1/_3)$.
Comment on the results of your calculation. **[8 marks]**

(3) Describe the *viewing pipeline* by which a three-dimensional object may be **visualised** on a two-dimensional display device. **[5 marks]**

When $\mathbf{n}.\mathbf{X} = lX+mY+nZ+pW = 0$ is the equation of a *plane* Π in $\mathbf{P}^3$, and $\mathbf{V} = (X_0, Y_0, Z_0, W_0)$ is a *viewpoint* also in $\mathbf{P}^3$, the *perspective projection matrix* for the mapping of $\mathbf{P}^3$ to Π from **V** is $M = \mathbf{n}^t\mathbf{V} - (\mathbf{n}.\mathbf{V})I_4$. (You are *not* required to prove this result!)

Use this formula to find the matrix M corresponding to the *projection* from P ≡ (3, 3, 3) $\in \mathbf{R}^3$ onto the plane Π with equation x+y+z-1 = 0. Calculate the *images* A', B', C' under this projection of the vertices of the right-angled triangle ABC in $\mathbf{R}^3$, where A ≡ (2, 2, 2), B ≡ (1, 2, 2), C ≡ (2, 1, 2), and find the *angle* between A'B' and A'C'. **[8 marks]**

Suggest a suitable *transformation* from Π to a screen with 400 × 300 pixels which displays A'B'C' near the **centre** of the screen. **[7 marks]**

(4) (a) The *Bernstein polynomials* are defined by
$B_{i,n}(t) = (^n{}_i)(1-t)^{n-i}t^i$ for $0 \le i \le n$, $0 \le t \le 1$; and $B_{i,n}(t) = 0$ otherwise.

Prove the following:

(i) $\Sigma_{i=0}{}^n B_{i,n}(t) = 1$,

(ii) $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$. **[8 marks]**

(b) A *quartic* Bezier curve B = B(t) has control points
$\mathbf{b}_0 \equiv (0, 2)$, $\mathbf{b}_1 \equiv (4, 2)$, $\mathbf{b}_2 \equiv (6, 6)$, $\mathbf{b}_3 \equiv (8, 2)$, $\mathbf{b}_4 \equiv (12, 2)$.

Evaluate the point B(½) in two ways:

(i) Substituting t = ½ into the *defining equation* of the Bezier curve;

(ii) Applying the *de Casteljau* algorithm.

Make a **sketch** illustrating the points derived when applying the de Casteljau algorithm and including your *best attempt* at B(t).

If the curve B is **subdivided** at t = ½ into two quartic segments B_{left} and B_{right}, what are the *control points* for these two segments? **[12 marks]**

(5) (a) Let $T = [t_0, t_1, ..., t_m]$ be a knot vector with $t_0 < t_1 < .. < t_m$.
The B-spline *basis functions* $N_{i,d}$ of degree d determined by T are defined recursively by

$N_{i,0}(t) = 1$ when $t_i \le t < t_{i+1}$, and $N_{i,0}(t) = 0$ otherwise;
$N_{i,j}(t) = ((t-t_i)/(t_{i+j}-t_i))N_{i,j-1}(t) + ((t_{i+j+1}-t)/(t_{i+j+1}-t_{i+1}))N_{i+1,n-1}(t)$,
for all $0 \le j \le d$ and $0 \le i \le m-1-j$.

Prove, for j > 0, that $N_{i,j}(t) = 0$ for all $t \in (-\infty, t_i] \cup [t_{i+j+1}, \infty)$. **[10 marks]**

(b) Let d = 2 and T = [0, 1, 2, 3, 4, 5, 6, 7].
Evaluate the basis function $N_{0,2}(t)$ and verify that it is *differentiable* at t = 1 and at t = 2. Sketch the graph of $N_{0,2}(t)$. **[10 marks]**