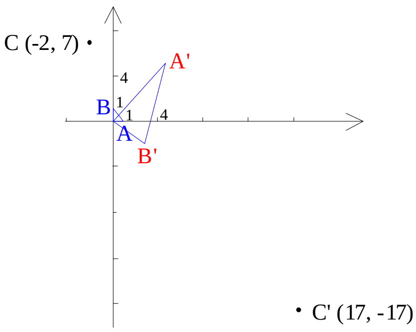
2nd October 2001

Transformations of the Plane

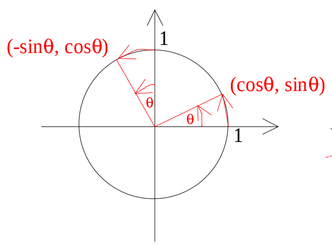
(1) *Linear/Homogeneous* which **fix** the origin. They are defined by images of any basis, e.g. T: $\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}2\\5\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}3\\-1\end{pmatrix}$. The associated matrix is $A = \begin{bmatrix}2 & 3\\5 & -1\end{bmatrix}$, and then $\begin{pmatrix}-2\\7\end{pmatrix} \to \begin{pmatrix}2 & 3\\5 & -1\end{pmatrix}\begin{pmatrix}-2\\7\end{pmatrix} = \begin{pmatrix}17\\-17\end{pmatrix}$. Check: $\begin{pmatrix}-2\\7\end{pmatrix} = -2\begin{pmatrix}1\\0\end{pmatrix} + 7\begin{pmatrix}0\\1\end{pmatrix} \to -2\begin{pmatrix}2\\5\end{pmatrix} + 7\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}-4\\-10\end{pmatrix} + \begin{pmatrix}21\\-7\end{pmatrix} = \begin{pmatrix}17\\-17\end{pmatrix}$. This uses the "*analytic convention*" of functions on the left: $(T_1 \bullet T_2)(\underline{x}) = T_1(T_2(\underline{x}))$, where $\underline{x}$ is a **column** vector.

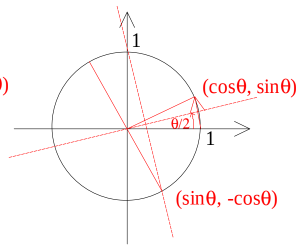


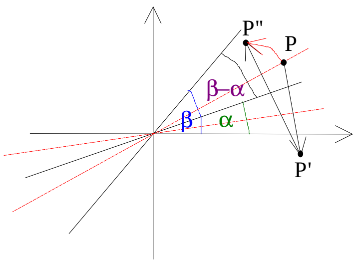
The "*algebraic convention*" has functions acting on the right: $\underline{x}(T_1 * T_2) = (\underline{x}T_1)T_2$, where $\underline{x}$ is a **row** vector. *Rewriting* our example, $(1, 0) \to (2, 5)$; $(0, 1) \to (3, -1)$; $A = \begin{bmatrix}2 & 5\\3 & -1\end{bmatrix}$; and $(-2\ 7) \to (-2\ 7)\begin{pmatrix}2 & 5\\3 & -1\end{pmatrix} = (17\ -17)$. Notice that $|A| = -17$ so that T involves a *reflection* and some *expansion*.

The 2×2 **non-singular** matrices form the general linear group $GL(2, \mathbf{R})$. The **subgroup** of matrices with $|A| = 1$ forms the special linear group $SL(2, \mathbf{R})$. The **orthogonal** matrices ($A^{-1} = A^t$) form the **orthogonal** group $O_2(\mathbf{R})$. The intersection $SL(2, \mathbf{R}) \cap O_2(\mathbf{R})$ forms the special orthogonal group $SO_2(R)$. *Note*: A orthogonal $\Rightarrow |A| = \pm 1$, and $|A| = 1$ — rotation; $|A| = -1$ — reflection.

Case $|A| = 1$: $A = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$. We have *anticlockwise rotation* about O with angle θ. Case $|A| = -1$: $A = \begin{bmatrix}\cos\theta & \sin\theta\\\sin\theta & -\cos\theta\end{bmatrix}$. We have *reflection* in the line $y = \tan(\theta/2).x$.







Product of reflections. $\begin{bmatrix}\cos\alpha & \sin\alpha\\\sin\alpha & -\cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & \sin\beta\\\sin\beta & -\cos\beta\end{bmatrix} = \begin{bmatrix}\cos(\beta-\alpha) & -\sin(\beta-\alpha)\\\sin(\beta-\alpha) & \cos(\beta-\alpha)\end{bmatrix}$. This is a *rotation* through $(\beta-\alpha)$, where **first** we reflect in $y = \tan(\alpha/2)x$, and **then** reflect in $y = \tan(\beta/2)x$.

Affine Transformations

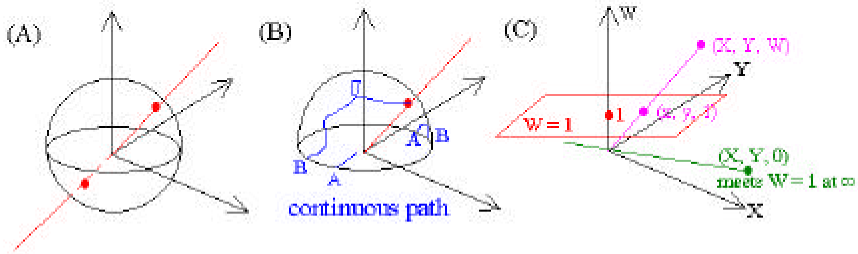
Consider the *vector space* $\mathbf{R}^3$. The <u>vector subspaces</u> are as follows: $\mathbf{R}^3$ itself, *planes* through $\underline{0}$, *lines* through $\underline{0}$, and $\{\underline{0}\}$. **Other** lines and planes are called *affine subspaces*. Affine transformations. Example: T: $(x, y) \to (a, b) + (x, y)A$. This involves a *change* of origin: $\underline{0} \to (a, b)$.

Homogeneous Co-ordinates (Projective Geometry)

<u>Definition</u>: There is an *equivalence relation* ~ on $\mathbf{R}^3\backslash\{0\}$ defined by $(x_1, y_1, z_1) \sim (x_2, y_2, z_2\} \Leftrightarrow (x_1, y_1, z_1) = r(x_2, y_2, z_2)$ for *some* $r \neq 0$. **<u>Proof</u>**. (i) *reflexive*: $(x_1, y_1, z_1) = 1(x_1, y_1, z_1)$ so that $(x_1, y_1, z_1) \sim (x_1, y_1, z_1)$. (ii) *symmetry*: $(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Rightarrow (x_1, y_1, z_1) = r(x_2, y_2, z_2)$, $r \neq 0 \Rightarrow (x_2, y_2, z_2) = {}^1/_r(x_1, y_1, z_1)$, $r \neq 0 \Rightarrow (x_2, y_2, z_2) \sim (x_1, y_1, z_1)$. (iii) *transitivity*: $\underline{v}_1 \sim \underline{v}_2$ and $\underline{v}_2 \sim \underline{v}_3 \Rightarrow \underline{v}_1 = r\underline{v}_2$ and $\underline{v}_2 = s\underline{v}_3$, with $r, s \neq 0$; $\Rightarrow \underline{v}_1 = rs\underline{v}_3 \Rightarrow \underline{v}_1 \sim \underline{v}_3$.

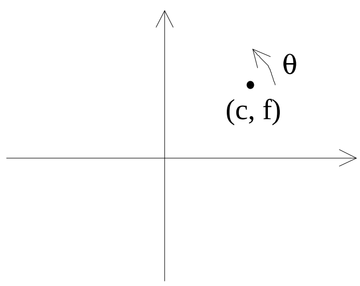
Definition: The projective plane $\mathbf{P}^2$ is the set of *equivalence* classes. $(X, Y, W) \in \mathbf{P}^2$. In $\mathbf{P}^2$, $(5, 6, ½) = (10, 12, 1) = (2½, 3, ¼) = \ldots$ When $W \neq 0$, (X, Y, W) has a *unique* representation $(x, y, 1)$, *where* $x = {}^{X}/_{W}$, and $y = {}^{Y}/_{W}$.

How to *visualise* $\mathbf{P}^2$? (1) The set of *lines* in $\mathbf{R}^3$ through <u>0</u>. (2) *Antipodal pairs* of points on a unit **sphere** (A) or points in the *upper hemisphere* with antipodal points on the equator identified (B). (3) The plane W = 1 with points at *infinity* (C).



Consider the *transformation* $(x', y', 1) = (x, y, 1)[{}^{a}b_{c}\ {}^{d}e_{f}\ {}^{0}0_{1}] = (ax+by+c, dx+ey+f, 1)$, i.e. $(x', y') = (x, y)[{}^{a}{}_{b}\ {}^{d}{}_{e}] + (c, f)$. If $(X, Y, W) = W(x, y, 1)$, then $(X', Y', W') = W(x, y, 1)A = (X, Y, W)A$, i.e. A transforms the *whole line* (X, Y, W) into the *new* line (X', Y', W').

<u>Special types of affine transformations</u>. **Translation**: $[{}^{1}0_{c}\ {}^{0}1_{f}\ {}^{0}0_{1}]$ i.e. $(x, y, 1) \to (x+c, y+f, 1)$. **Rotations/Reflections**: $A = [{}^{a}b_{0}\ {}^{d}e_{0}\ {}^{0}0_{1}]$ $A^{-1} = A^{t}$. **Scaling** about <u>0</u>: $A = [{}^{S_x}0_{0}\ {}^{0}S_{y0}\ {}^{0}0_{1}]$. **Rotation** about an **arbitrary** point (c, f) (see the *diagram*): *Translate* (c, f) to <u>0</u>; *Rotate*; and *translate* back. $[{}^{1}0_{-c}\ {}^{0}1_{-f}\ {}^{0}0_{1}][{}^{\cos\theta}{-\sin\theta}_{0}\ {}^{\sin\theta}\cos\theta_{0}\ {}^{0}0_{1}][{}^{1}0_{c}\ {}^{0}1_{f}\ {}^{0}0_{1}] = [{}^{\cos\theta}{-\sin\theta}_{(-c\cos\theta+f\sin\theta+c)}\ {}^{\sin\theta}\cos\theta_{(-c\sin\theta-f\cos\theta+f)}\ {}^{0}0_{1}]$. Q: Find a *reflection matrix* in the line $lx+my+n = 0$ $(n \neq 0)$.



A: (From the Book) (1) The line *intersects* the y-axis in the point $(0, -{}^{n}/_{m})$. (2) Make a **translation** mapping $(0, -{}^{n}/_{m})$ to the origin, and thus mapping the line to a line through the origin with *gradient* identical to the original line. (3) The **gradient** of the line is $\tan\theta = -{}^{l}/_{m}$, where θ is the angle the line makes with the x-axis. **Rotate** the line about the origin through an *angle* -θ. The line is thus **mapped** to the x-axis. (4) Apply a *reflection* in the x-axis. (5) Apply the <u>inverse</u> of the rotation above, followed by the *inverse* of the translation of step 2.

The **concatenation** of the above transformations is $({}^{1}0_{0}\ {}^{0}1_{n/m}\ {}^{0}0_{1})({}^{\cos\theta}\sin\theta_{0}\ {}^{-\sin\theta}\cos\theta_{0}\ {}^{0}0_{1})({}^{1}0_{0}\ {}^{0}{-1}_{0}\ {}^{0}0_{1})({}^{\cos\theta}{-\sin\theta}_{0}\ {}^{\sin\theta}\cos\theta_{0}\ {}^{0}0_{1})({}^{1}0_{0}\ {}^{0}1_{-n/m}\ {}^{0}0_{1}) = \ldots = ({}^{\cos^2\theta-\sin^2\theta}\ 2\cos\theta\sin\theta_{\ 2(n/m)\sin\theta\cos\theta}\ {}^{2\cos\theta\sin\theta}\ \sin^2\theta-\cos^2\theta_{\ (n/m)(\sin^2\theta-\cos^2\theta-1)}\ {}^{0}0_{1})$. *Since* $\tan\theta = {}^{\sin\theta}/_{\cos\theta} = -{}^{l}/_{m}$, it follows *that* $\sin\theta = l/(l^2+m^2)^{½}$, and that $\cos\theta = -m/(l^2+m^2)^{½}$. Hence $\cos^2\theta = {}^{m^2}/_{(l^2+m^2)}$, $\sin^2\theta = {}^{l^2}/_{(l^2+m^2)}$, $\cos\theta\sin\theta = {}^{-lm}/_{(l^2+m^2)}$, and $\cos^2\theta-\sin^2\theta = {}^{(m^2-l^2)}/_{(l^2+m^2)}$.

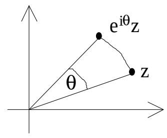
Finally, **substitution** for the *trigonometric functions* in the above matrix yields $({}^{m^2-l^2/l^2+m^2}{-(2lm)/l^2+m^2}_{\ -(2ln)/l^2+m^2}\ {}^{-2lm/l^2+m^2}{-(m^2-l^2)/l^2+m^2}_{\ -2mn/l^2+m^2}\ {}^{0}0_{\ 1}]$. Since in *homogeneous* co-ordinates multiplication by a **factor** does not affect the result, the above matrix can be *multiplied* by a factor of (l^2+m^2) to give the **general reflection matrix** $R_{(l,m,n)} = ({}^{m^2-l^2}{-2lm}_{\ -2ln}\ {}^{-2lm}{-m^2+l^2}_{\ -2mn}\ {}^{0}0_{\ l^2+m^2})$.

4th October 2001

Question: Construct a *rotation matrix* for the reflection in the line $y = \tan({}^{\theta}/_{2})x + f$. **Answer**: Let $A = [{}^{1}0_{0}\ {}^{0}1_{f}\ {}^{0}0_{1}]$, $B = [{}^{1}0_{0}\ {}^{0}1_{-f}\ {}^{0}0_{1}]$, and $Q = [{}^{\cos\theta}\sin\theta_{0}\ {}^{\sin\theta}{-\cos\theta}_{0}\ {}^{0}0_{1}]$. It follows that the matrix we *want* is given by $BQA = [{}^{\cos\theta}\sin\theta_{-f\sin\theta}\ {}^{\sin\theta}{-\cos\theta}_{f\cos\theta+f}\ {}^{0}0_{1}]$.

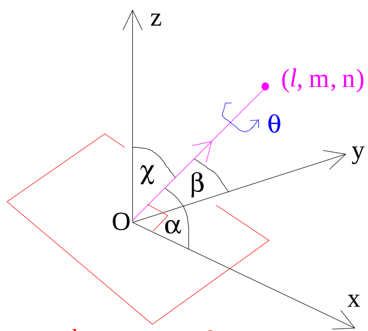
9th October 2001

Rotations in $\mathbf{R}^2$

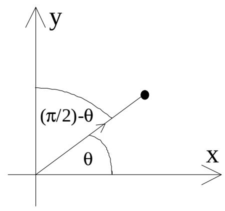


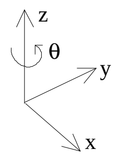
[Rotations in **C**: $i^2 = -1$; $e^{i\theta} = \cos\theta + i\sin\theta$. The map $z \to e^{i\theta}z$ is a *rotation*, as shown on the right]. In $\mathbf{R}^2$, let $J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, a *skew-symmetric matrix*. $J^2 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = -I$ $\Rightarrow J^3 = -J$ and $J^4 = I$. Now let us *calculate* $e^{J\theta}$. $e^{J\theta}$ = (by *definition*) = $\sum_{n=0}^{\infty} (J\theta)^n/n! = I + J\theta - I\theta^2/2 - J\theta^3/3! + I(\theta^4/4!) + \ldots = I\{1-\theta^2/2+(\theta^4/4!)-\ldots\} + J\{\theta-\theta^3/3!+(\theta^5/5!)-\ldots\} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$. So the map $(x\ y) \to (x\ y)e^{J\theta}$ gives a *rotation* in $\mathbf{R}^2$.

Direction Cosines

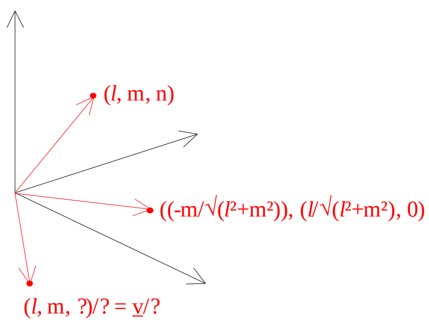


Consider the *unit* vector $(l, m, n) \in \mathbf{R}^3$. Of course, $l^2+m^2+n^2 = 1$. Now $(l, m, n).(1, 0, 0) = l$, etc. So $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\chi$, where α, β and χ are the *angles* of the coordinate axes. (Note: this *generalises* $\cos^2\theta+\sin^2\theta = 1 = \cos^2\theta+\cos^2(\pi/2-\theta)$ as shown on the *right*). Note that the vector (l, m, n) is **perpendicular** to the plane $lx+my+nz = 0$ as shown in red in the diagram. We want a *matrix* which rotates by θ about (l, m, n) as shown in blue.





Method (which is applied *numerically* with $l, m, n = (1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{6})$ in a worksheet): (1) Change **coordinates**, moving (l, m, n) to $(0, 0, 1)$ by *rotation*. (2) Rotate about Oz (as shown on the *right*). (3) Move $(0, 0, 1)$ back to (l, m, n).



Stages 1 and 3: Consider the *diagram* shown. Now $\underline{v}.(l, m, n) = l^2+m^2+n? = 0 \Rightarrow\ ? = {}^{-l^2-m^2}/_{n} = {}^{-1+n^2}/_{n}$. And $\|(l, m, {}^{-1+n^2}/_{n})\|^2 = l^2+m^2+(n^4-2n^2+1/n^2) = (l^2+m^2+n^2)-2+1/n^2 = -1+1/n^2$. Therefore, $\underline{v}/? = {}^{\pm n}/_{\sqrt{(1-n^2)}}(l, m, {}^{-(1-n^2)}/_{n})$. So *take* $A = \begin{bmatrix} ln/\sqrt{(1-n^2)} & mn/\sqrt{(1-n^2)} & -\sqrt{(1-n^2)} \\ -m/\sqrt{(l^2+m^2)} & l/\sqrt{(l^2+m^2)} & 0 \\ l & m & n \end{bmatrix}$: an **orthogonal** matrix, and the matrix for *step* (3). Now **calculate** $A^{-1}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}A$. [Aside: it is equal to ... = (also *using* $l^2+m^2 = 1-n^2$) = (by *column*) $[[({}^1/_{1-n^2}(ln(l n\cos\theta+m\sin\theta) - m(l\sin\theta-m\cos\theta)) + l^2), ({}^1/_{1-n^2}(ln(mn\cos\theta-l\sin\theta) - m(mn\sin\theta+l\cos\theta)) + lm), (-ln\cos\theta+m\sin\theta+ln)], [({}^1/_{1-n^2}(mn(ln\cos\theta+m\sin\theta) + l(l\sin\theta-m\cos\theta)) + lm), ({}^1/_{1-n^2}(mn(mn\cos\theta-l\sin\theta) + l(mn\sin\theta+l\cos\theta)) + m^2), (-mn\cos\theta-l\sin\theta+mn)], [(-ln\cos\theta-m\sin\theta+ln), (-mn\cos\theta+l\sin\theta+mn), ((1-n^2)\cos\theta+n^2)]]$]. Note: there is *more detail* on the above on page 5!

Now *consider* $J = \begin{bmatrix}0 & n & -m\\ -n & 0 & l\\ m & -l & 0\end{bmatrix}$, a *skew-symmetric* matrix. $J^2 = \begin{bmatrix}-n^2-m^2 & lm & ln\\ lm & -n^2-l^2 & mn\\ ln & mn & -m^2-l^2\end{bmatrix} = \begin{bmatrix}-1+l^2 & lm & ln\\ lm & -1+m^2 & mn\\ ln & mn & -1+n^2\end{bmatrix}$, i.e. $J^2 = -I+K$, *where* $K = \begin{bmatrix}l^2 & lm & ln\\ lm & m^2 & mn\\ ln & mn & n^2\end{bmatrix}$. *Further*, $J^3 = -J$, $J^4 = -J^2 = +I-K$, *etc.* Now $e^{J\theta} = I + J\theta + (-I+K)\theta^2/2 - J\theta^3/3! + (+I-K)(\theta^4/4!) + J(\theta^5/5!) - \ldots = I\{1-\theta^2/2+(\theta^4/4!)-\ldots\} + J\{\theta-\theta^3/3!+(\theta^5/5!)-\ldots\} + K\{\theta^2/2!-(\theta^4/4!)+\ldots\} = I\cos\theta+J\sin\theta+K(1-\cos\theta) = \begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)+n\sin\theta & ln(1-\cos\theta)-m\sin\theta \\ lm(1-\cos\theta)-n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)+l\sin\theta \\ ln(1-\cos\theta)+m\sin\theta & mn(1-\cos\theta)-l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix}$.

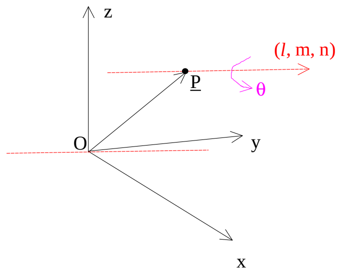
9th October 2001

Homogeneous Coordinates

Derive an *equivalence relation* on $\mathbf{R}^4 \backslash \{\underline{0}\}$ by $(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2)$ iff $x_1 = kx_2$, $y_1 = ky_2$, $z_1 = kz_2$ and $w_1 = kw_2$ for some $k \in \mathbf{R}^{\neq 0}$. Each *equivalence class* (X, Y, Z, W) has a *unique* representation (x, y, z, 1).

<u>The Projective 3-sphere $\mathbf{P}^3$</u>. This is the *set of equivalence classes*. We visualise $\mathbf{P}^3$ as the plane W = 1 in $\mathbf{R}^4$ together with *points* at ∞ (one in each *direction:* for $x = {}^X/_W$, $y = {}^Y/_W$ and $z = {}^Z/_W$, W = 0 gives points at ∞). Consider the *transformation* $(x, y, z, 1)\begin{bmatrix} a & e & i & 0 \\ b & f & j & 0 \\ c & g & k & 0 \\ d & h & l & 1 \end{bmatrix}$ = (ax+by+cz+d, ex+fy+gz+h, ix+jy+kz+l, 1). Matrices of this type map *one unique representation* to *another*. (0, 0, 0, 1) → (d, h, l, 1). **Closed** under multiplication: $\begin{bmatrix} A & \mathbf{0}^t \\ \boldsymbol{u} & 1 \end{bmatrix}\begin{bmatrix} B & \mathbf{0}^t \\ \boldsymbol{v} & 1 \end{bmatrix} = \begin{bmatrix} AB & \mathbf{0}^t \\ \boldsymbol{u}B+\boldsymbol{v} & 1 \end{bmatrix}$, where A and B are *3×3 matrices*.

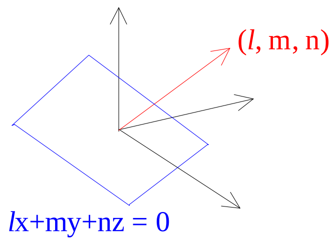
Affine Rotations



<u>Method</u>: (1) Change the *origin*, P → $\underline{0}$. (2) **Rotate** about the line through $\underline{0}$, with direction (l, m, n). (3) Shift the origin **back** again. Note that (3) uses T as shown on the *left*; that (1) uses T^{-1}, and that (2) uses *known* rotation.

$$T = \left[\begin{array}{ccc|c} 1 & & & \\ & 1 & & 0^t \\ & & 1 & \\ \hline & \underline{P} & & 1 \end{array}\right]$$

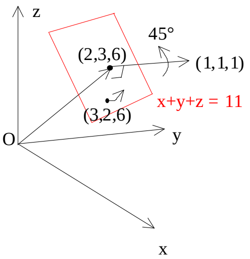
Reflections



The *reflection matrix* in the plane $lx+my+nz = 0$ (with $l^2+m^2+n^2 = 1$) is **given** by $V = \begin{bmatrix} 1-2l^2 & -2lm & -2ln \\ -2lm & 1-2m^2 & -2mn \\ -2ln & -2mn & 1-2n^2 \end{bmatrix} = I-2K$. [**Calculation**: $J^2 = -I+K \Rightarrow K = J^2+I$; $V^2 = (I-2K)^2 = (-I-2J^2)^2 = I^2+4J^2+4J^4 = I+4J^2-4J^2 = I$ (*using* $J^3 = -J$)].

(<u>From the Maple Worksheet</u>). Rotations about the *three coordinate axes* produces *familiar* matrices: x: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$; y: $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$; z: $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The direction (1, 1, 1) makes *equal angles* $\cos^{-1}({}^1/_{\sqrt{3}})$ with the three axes. A *rotation through 120°* **permutes** the three axes (R120 := $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$). Applying the transformation to an *arbitrary* vector permutes the *coordinates*.

As for *two dimensions*, a 3×3 matrix can be **extended** to a 4×4 matrix which can then be used to *transform* homogeneous coordinates. The method is to insert a **fourth row** and a **fourth column**, with *zero* entries except for the '*corner*' entry which is 1. To perform a **rotation** through an angle θ about a line with direction $\underline{u} = (l,m,n)$ through a point $\underline{p}$ = (a, b, c), the method is as follows: (1) move $\underline{p}$ to the *origin*; (2) apply the *rotation*; (3) move the origin *back* to $\underline{p}$.



For example, a rotation through $\pi/4$ about the line with *direction* (1, 1, 1) through the *point* (2, 3, 6) has **graphical** representation as shown, and $(1/\sqrt{2}+2+1/\sqrt{6}) + (3-1/\sqrt{2}+1/\sqrt{6}) + (6-2/\sqrt{6}) = 11$.

Now construct a *rotation matrix* in $\mathbf{R}^3$ about the line with direction cosines $(l, m, n) = (1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{6})$ by (1) rotating (l, m, n) in the z-axis, (2) applying a rotation through *theta* in the xy-plane, and (3) applying the **inverse** rotation to step (1). Here is the *rotation* matrix for step 2: $(^{\cos\theta}{-\sin\theta}_0\ {}^{\sin\theta}\cos\theta_0\ {}^{0}0_1)$.

We now set up **numeric values** for (l, m, n) where, of course, $l^2+m^2+n^2 = 1$; and choose **two** more directions forming an orthonormal basis for $\mathbf{R}^3$, namely *normalised* versions of $(-m, l, 0)$ and $(l, m, {}^{(-1+n^2)}/_n)$. So let $l = 1/\sqrt{2}$, $m = 1/\sqrt{3}$, $n = 1/\sqrt{6}$, $r = 1/\sqrt{(l^2+m^2)}$, $s = 1/\sqrt{((1/n^2)-1)}$, and A = a 3×3 matrix with *entries* (row by row) $(ls, ms, {}^{s(n^2-1)}/_n, -mr, lr, 0, l, m, n)$. We then *compute* A^{-1}, check that it is *orthogonal*, and compute the **transform** $A^{-1}RA$.

Now construct a *reflection matrix* in $\mathbf{R}^3$ through the plane $lx+my+nz = 0$ which is *perpendicular* to the line with direction cosines (l, m, n). The method is as follows: (1) *rotate* (l, m, n) into the z-axis; (2) apply a *reflection* in the xy-plane; and (3) apply the *inverse* rotation to step 1. For step 2, the **reflection** matrix is $V = [{}^{1}0_0\ {}^{0}1_0\ {}^{0}0_{-1}]$. Given *direction cosines* (l, m, n) where, of course, $l^2+m^2+n^2 = 1$, choose **two** more directions forming an orthonormal basis for $\mathbf{R}^3$ — namely *normalised versions* of $(-m, l, 0)$ and $(l, m, {}^{-1+n^2}/_n)$. The matrix (A) is as *shown*. Now calculate A^{-1}, *simplifying* using $l^2+m^2 = 1-n^2$, and then calculate the transform $C = A^{-1}VA$, simplifying *as before*. C is as shown on the *right*.

$$A = \begin{bmatrix} \frac{l}{\sqrt{-\frac{n^2-1}{n^2}}} & \frac{m}{\sqrt{-\frac{n^2-1}{n^2}}} & \frac{n^2-1}{\sqrt{-\frac{n^2-1}{n^2}}\,n} \\ -\frac{m}{\sqrt{l^2+m^2}} & \frac{l}{\sqrt{l^2+m^2}} & 0 \\ l & m & n \end{bmatrix}$$

$$C = \begin{bmatrix} -2l^2+1 & -2ml & -2nl \\ -2ml & -2m^2+1 & -2nm \\ -2nl & -2nm & 1-2n^2 \end{bmatrix}$$

11th October 2001

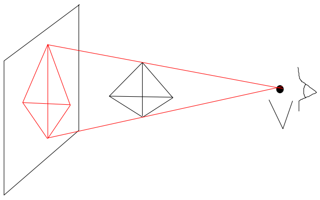
Tutorial

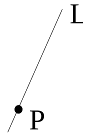
Q: Determine the 4×4 matrix of a *reflection* in the plane $lx+my+nz = k$. To do this, you must first choose a **point** in the plane. The obvious point to choose is on the line with *direction cosines* (l', m', n') (dashes = *normalised* versions). A: For the plane $lx+my+nz = k$, the point $({}^{k}/_{3l}, {}^{k}/_{3m}, {}^{k}/_{3n})$ is always a point in the plane. So we translate this point to the origin, do reflection V as defined in the *previous* lecture, and then translate back. So the **matrix** we want is given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k/3l & -k/3m & -k/3n & 1 \end{pmatrix} \begin{pmatrix} 1-2l'^2 & -2l'm' & -2l'n' & 0 \\ -2l'm' & 1-2m'^2 & -2m'n' & 0 \\ -2l'n' & -2m'n' & 1-2n'^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k/3l & k/3m & k/3n & 1 \end{pmatrix}$$

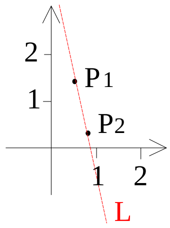
Q: Projection onto the plane $lx+my+nz = 0$. There are **two** possible types of projection: by *parallel rays*, or from a *single point*. We will only consider the *parallel* case now. When a point is **reflected** in a plane, think of the point as travelling *from* its initial position *along* a line perpendicular to the plane, *through* the plane, and onto its *final position*, the **reflected image.** The point in the plane is the *mid-point* between the **initial** and **final** positions! So the matrix of the projection is precisely *one half* of the sum of the *reflection* matrix and the *identity* matrix. Obtain the 4×4 version of this matrix. A: The 4×4 matrix we *want* is given by H = ½(I+G).

Projection

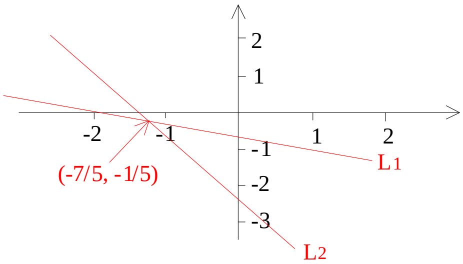




Points and lines in homogeneous coordinates. L has *equation* $lx+my+n = 0$, or $lX+mY+nW = 0$ (in *homogeneous* coordinates). It is defined by $\underline{l} = (l, m, n)$, the **homogeneous line coordinate vector**. P is the point (x, y) or (X, Y, W) = $\underline{P}$. P lies on L if $\underline{l}\bullet\underline{P} = lX+mY+nW = 0$.

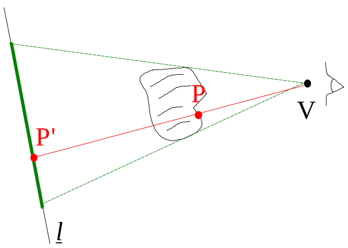


Line through P_1 and P_2. If P_1 and P_2 both lie on L, then $\underline{P_1}\bullet\underline{l} = 0 = \underline{P_2}\bullet\underline{l}$, i.e. is perpendicular to $\underline{P_1}$ and $\underline{P_2}$, i.e. $\underline{l} = \underline{P_1}\times\underline{P_2}$ (or some *multiple*). Example 1: If P_1 = (1, 3, 2) (≡ (½, $^3/_2$, 1)), and if P_2 = (2, 1, 3) (≡ $^2/_3$, $^1/_3$, 1), then $\underline{P_1}\times\underline{P_2} = |^{i}1_2\ ^{j}3_1\ ^{k}2_3|$ = (7 1, -5) = $\underline{l}$, and L has *equation* $7x+y-5 = 0$.

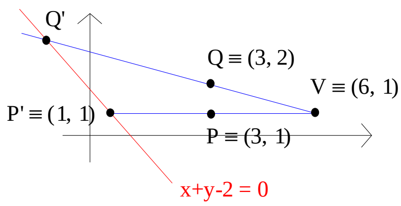


A point through L_1 and L_2. P lies on both L_1 and L_2 ⇒ $\underline{P}\bullet\underline{l_1} = 0 = \underline{P}\bullet\underline{l_2}$ ⇒ $\underline{P}$ is *perpendicular* to both $\underline{l_1}$ and $\underline{l_2}$ ⇒ $\underline{P} = \underline{l_1}\times\underline{l_2}$ (or some *multiple*). Example 2: L_1: $x+3y+2 = 0$; L_2: $2x+y+3 = 0$: $\underline{l_1}$ = (1, 3, 2); $\underline{l_2}$ = (2, 1, 3); and $\underline{l_1}\times\underline{l_2}$ = (7, 1, -5) (≡ (-$^7/_5$, -$^1/_5$, 1)) = $\underline{P}$. Conclusion: L_1 and L_2 *intersect* at (-$^7/_5$ -$^1/_5$).

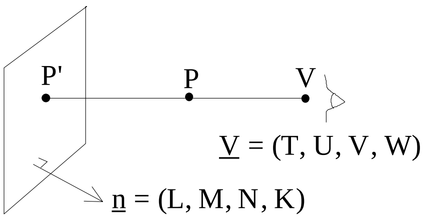
Projection onto a Line in R²



Theorem 4.1: The matrix of the *projection* is M = $\underline{l}^t\underline{V}-(\underline{l}\bullet\underline{V})I_3$ = $(^l m_n)(t\ u\ v) - (l\ m\ n)(^t u_v)(^1 0_0\ ^0 1_0\ ^0 0_1)$ = $(^{lt} mt_{nt}\ ^{lu} mu_{nu}\ ^{lv} mv_{nv})$ - $(^{lt+mu+nv} 0_0\ ^0 lt{+}mu{+}nv_0\ ^0 0_{lt+mu+nv})$ = $(^{-mu-nv} mt_{nt}\ ^{lu} {-lt-nv}_{nu}\ ^{lv} mv_{-lt-mu})$. **Proof**. Suppose the *projection* maps P to P' on L. The line *through* V and P has line *coordinates* $\underline{V}\times\underline{P}$. This line *intersects* L at the point $\underline{l}\times(\underline{V}\times\underline{P})$. Using the *identity* $\underline{A}\times(\underline{B}\times\underline{C})$ = $(\underline{C}\bullet\underline{A})\underline{B}-(\underline{A}\bullet\underline{B})\underline{C}$, $\underline{P'}$ = $(\underline{P}\bullet\underline{l})\underline{V}-(\underline{l}\bullet\underline{V})\underline{P}$ = $\underline{P}\underline{l}^t\underline{V}-\underline{P}(\underline{l}\bullet\underline{V})I_3$ = $\underline{P}\{\underline{l}^t\underline{V}-(\underline{l}\bullet\underline{V})I_3\}$.



Example: $\underline{l}$ = (1, 1, -2), $\underline{V}$ = (6, 1, 1): $\underline{l}\bullet\underline{V}$ = 5; M = $[^6 6_{-12}\ ^1 1_{-2}\ ^1 1_{-2}]$ - $[^5 0_0\ ^0 5_0\ ^0 0_5]$ = $[^1 6_{-12}\ ^1 {-4}_{-2}\ ^1 1_{-7}]$; and $\underline{P}M$ = [3, 1, 1]$[^1 6_{-12}\ ^1 {-4}_{-2}\ ^1 1_{-7}]$ = $[^{-3} 3_{-3}]$ ~ $[^1 1_1]$. Now $\underline{Q}$ = (3, 2, 1) ⇒ $\underline{Q}M$ = $[^3 {-7}_{-2}]$ ≡ $[^{-1½} 3½_1]$. *Repeat* with $\underline{V}$ ≡ (6, 1, 0): M = $(^1 1_{-2})$(6 1 0) - 7$(^1 0_0\ ^0 1_0\ ^0 0_1)$ = $(^6 6_{-12}\ ^1 1_{-2}\ ^0 0_0)$ - $(^7 0_0\ ^0 7_0\ ^0 0_7)$ = $(^{-1} 6_{-12}\ ^1 {-6}_{-2}\ ^0 0_{-7})$, so that $\underline{P}M$ = (3 1 1)$(^{-1} 6_{-12}\ ^1 {-6}_{-2}\ ^0 0_{-7})$ = $(^{-9} {-5}_{-7})$ ~ $(^{9/7}\ 5/7\ _1)$, and $\underline{Q}M$ = (3 2 1)$(^{-1} 6_{-12}\ ^1 {-6}_{-2}\ ^0 0_{-7})$ = $(^{-3} {-11}_{-7})$ ~ $(^{3/7}\ 11/7\ _1)$.



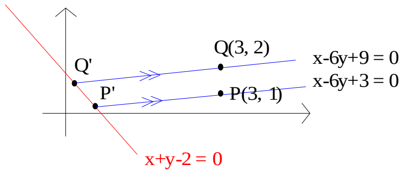
For the *diagram* shown, where V is **not** in the plane, we have the *projection matrix* M =

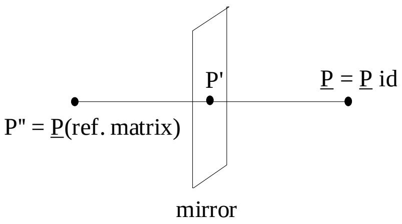
$$\begin{bmatrix} -MU-NV-KW & LU & LV & LW \\ MT & -LT-NV-KW & MV & MW \\ NT & NU & -LT-MU-KW & NW \\ KT & KU & KV & -LT-MU-NV \end{bmatrix}$$

$\underline{n}^t\underline{V}$ - $(\underline{n}\bullet\underline{V})I_4$ = the matrix shown *above*. **Proof**. Assume that $\underline{n}\bullet\underline{P}$ = 0, i.e. that P lies *in* the plane. We need to check that $\underline{P}M \equiv \underline{P}$. Now $\underline{P}M$ = $\underline{P}\underline{n}^t\underline{V}$ - $\underline{P}(\underline{n}\bullet\underline{V})I_4$. As $\underline{P}\underline{n}^t = \underline{P}\bullet\underline{n}$ = 0 by *assumption*, we have $\underline{P}M$ = $-(\underline{n}\bullet\underline{V})\underline{P}$, a *multiple* of $\underline{P}$ (*non-zero*).

Now **suppose** that $\underline{n}\bullet\underline{P} \neq 0$. So $\underline{P}' = \alpha\underline{P}+\beta\underline{V}$ for some $\alpha, \beta \in \mathbf{R}$. $\underline{P}'$ in the plane $\Rightarrow \underline{n}\bullet(\alpha\underline{P}+\beta\underline{V}) = 0 \Rightarrow \alpha(\underline{n}\bullet\underline{P})+\beta(\underline{n}\bullet\underline{V}) = 0 \Rightarrow \alpha = -\beta[(\underline{n}\bullet\underline{V})/(\underline{n}\bullet\underline{P})]$. So $\underline{P}' = -\beta[(\underline{n}\bullet\underline{V})/(\underline{n}\bullet\underline{P})]\underline{P} + \beta\underline{V} = \beta/(\underline{n}\bullet\underline{P})\{-(\underline{n}\bullet\underline{V})\underline{P}+(\underline{n}\bullet\underline{P})\underline{V}\} = -(\underline{n}\bullet\underline{V})\underline{P} + (\underline{n}\bullet\underline{P})\underline{V}$ = (*because* $(\underline{n}\bullet\underline{P}) = \underline{P}\underline{n}^t$) $= \underline{P}\{\underline{n}^t\underline{V} - (\underline{n}\bullet\underline{V})I_4\}$. The case W = 0 gives *parallel* projection = projection *from* ∞ in the (T, U, V)-*direction*.

Let us go back to the *previous example*. Recall that when we considered V = (6, 1, 0), we had $M = [{}^{-1}6_{-12}\ {}^{1}{-6}_{-2}\ {}^{0}0_{-7}]$, $\underline{P}M = (1\tfrac{2}{7}, \tfrac{5}{7}, 1)$, and $\underline{Q}M = (\tfrac{3}{7}, 1\tfrac{4}{7}, 1)$. The blue lines in the diagram are *parallel* lines with gradient $\tfrac{1}{6}$. Finally, we can think of *parallel projection* as the **average** ½(identity+reflection). The **mid-point** of x and y is ½(x+y) so that $\underline{P}' = \underline{P}\{$½(I+the reflection matrix)$\}$. In *non-homogeneous coordinates*, we have $\tfrac{1}{2}\{({}^{1}0_{0}\ {}^{0}1_{0}\ {}^{0}0_{1}) + ({}^{1-2l^2}{-2lm}_{-2ln}\ {}^{-2lm}1{-2m^2}_{-2mn}\ {}^{-2ln}{-2mn}_{1-2n^2})\} = ({}^{1-l^2}{-lm}_{-ln}\ {}^{-lm}1{-m^2}_{-mn}\ {}^{-ln}{-mn}_{1-n^2})$. Since $l^2+m^2+n^2 = 1$, then we *have* $({}^{-m^2-n^2}{-lm}_{-ln}\ {}^{-lm}{-l^2-n^2}_{-mn}\ {}^{-ln}{-mn}_{-l^2-m^2})$ = the *previous* matrix with $\underline{V} \equiv \underline{l}$.

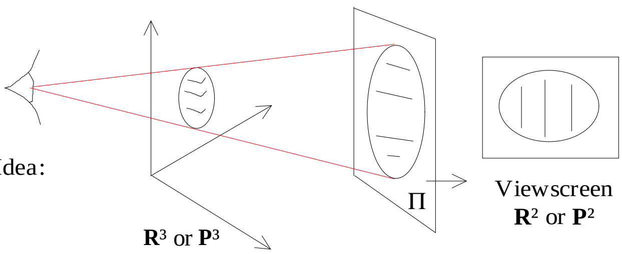




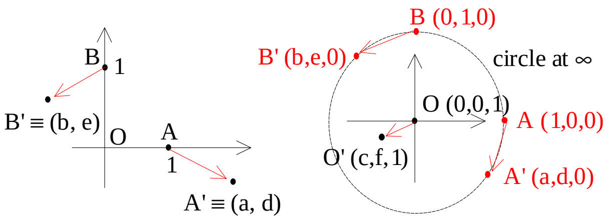
23rd October 2001

Rows in Transformation Matrices

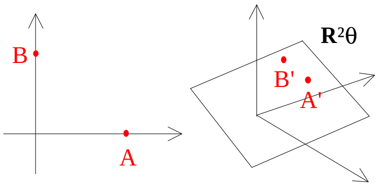
Idea:



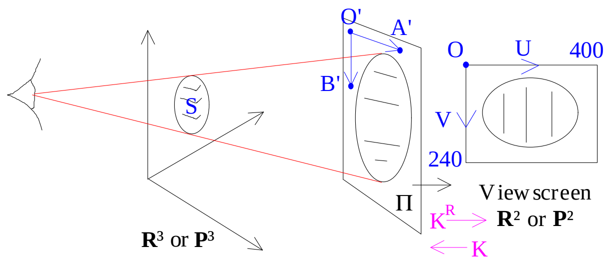
Linear Transformation: θ: $\mathbf{R}^2 \to \mathbf{R}^2$ with matrix M (w.r.t. *standard* bases). $\underline{x}M = (x\ y)({}^{a}{}_{b}\ {}^{d}{}_{e})$, with (1 0)M = (a d) and (0 1)M = (b e). The **rows** of M are the images of the *basis* vectors. If now θ: $\mathbf{P}^2 \to \mathbf{P}^2$, then $M = ({}^{a}b_{c}\ {}^{d}e_{f}\ {}^{0}0_{1})$. The three *rows* of M are: (1 0 0)θ, (0 1 0)θ, and (0 0 1)θ ((a d 0), (b e 0), and (c f 1)).



θ: $\mathbf{R}^3 \to \mathbf{R}^3$ is *determined* by (1 0 0)θ, (0 1 0)θ and (0 0 1)θ (a *linear* transformation). θ: $\mathbf{P}^3 \to \mathbf{P}^3$ is determined by (1 0 0 0)θ, (0 1 0 0)θ, (0 0 1 0)θ (three points at *infinity*) and (0 0 0 1)θ, the new *origin*. If we want θ: $\mathbf{R}^2 \to \mathbf{R}^3$, then $M = ({}^{a}{}_{b}\ {}^{d}{}_{e}\ {}^{g}{}_{h})$, with (1 0)θ = (a d g) and (0 1)θ = (b e h). For θ: $\mathbf{P}^2 \to \mathbf{P}^3$, $M = ({}^{a}b_{c}\ {}^{d}e_{f}\ {}^{g}h_{k}\ {}^{0}0_{1})$ = K (as in the *book*), with (1 0 0)θ = (a d g 0), (0 1 0)θ = (b e h 0), and (0 0 1)θ = (c f k 1) = O'.

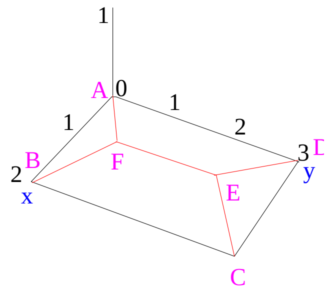


We seek a *right inverse* K^R for K (K is not *square*, so we cannot have an **inverse**). K^R is a 4×3 matrix so that $KK^R = I_3$ — *but* $K^RK \neq I_4$. Now KK^T (K^T = the *transpose* of K) is a 3×3 matrix so that $(KK^T)^{-1}$ **exists** (if $|KK^T| \neq 0$), so that $KK^T(KK^T)^{-1} = I_3$. **Define** $K^R = K^T(KK^T)^{-1}$, where K^R is a matrix associated to an *affine transformation* $\mathbf{P}^3 \to \mathbf{P}^2$.



In the diagram, the surface S is *projected* onto the plane Π. In Π, choose the **origin** O' and the **unit** vectors $\overrightarrow{O'A'}$ and $\overrightarrow{O'B'}$ (probably *orthogonal* to each other).

Construct K as a *matrix* for θ: $\mathbf{P}^2 \to \mathbf{P}^3$ mapping (a) O (on the *viewscreen*) to O', (b) A point at *infinity* in the direction (1, 0) to a point at *infinity* in the direction O'A', and (c) A point at *infinity* in the direction (0, 1) to a point at *infinity* in the direction O'B'. So if O'A' ≡ $(r_1, r_2, r_3, 1)$, with $\sqrt{(r_1^2+r_2^2+r_3^2)} = 1$; if O'B' ≡ $(s_1, s_2, s_3, 1)$, with $\sqrt{(s_1^2+s_2^2+s_3^2)} = 1$; and if O' ≡ $(q_1, q_2, q_3, 1)$, then $K = ({}^{r_1}s_{1q_1}\ {}^{r_2}s_{2q_2}\ {}^{r_3}s_{3q_3}\ {}^{0}0_1)$. Finally, *construct* $K^R = K^T(KK^T)^{-1}$.

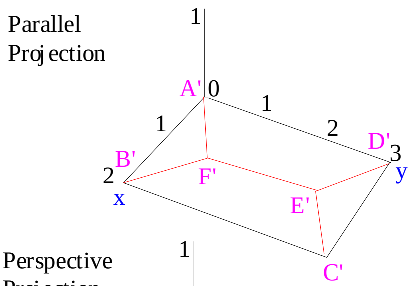


Consider a *parallel projection* of the prism shown in the diagram onto the plane z = 0 in a direction **parallel** to the z-axis. The *viewpoint* is **V**(0, 0, 1, 0), the point at infinity in the direction of the *z-axis*; and the viewplane has equation 0x+0y+1z+0 = 0, so that **n** = (0, 0, 1, 0). Thus M

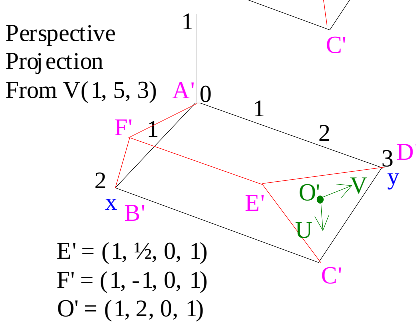
$$= \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}(0\ 0\ 1\ 0) - 1\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} -1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&-1 \end{pmatrix}.$$

The prism has vertices **A**(0, 0, 0), **B**(2, 0, 0), **C**(2, 3, 0), **D**(0, 3, 0), **E**(1, 2, 1) and **F**(1, 1, 1). Applying the *projection* to the vertices of the prism gives

$$\begin{pmatrix} A'\\B'\\C'\\D'\\E'\\F' \end{pmatrix} = \begin{pmatrix} 0&0&0&1\\2&0&0&1\\2&3&0&1\\0&3&0&1\\1&2&1&1\\1&1&1&1 \end{pmatrix}\begin{pmatrix} -1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&-1 \end{pmatrix} = \begin{pmatrix} 0&0&0&-1\\-2&0&0&-1\\-2&-3&0&-1\\0&-3&0&-1\\-1&-2&0&-1\\-1&-1&0&-1 \end{pmatrix}.$$



Following the *usual procedure* of dividing each point by its **fourth** coordinate yields the *Cartesian* coordinates **A'**(0, 0, 0), **B'**(2, 0, 0), **C'**(2, 3, 0), **D'**(0, 3, 0), **E'**(1, 2, 0) and **F'**(1, 1, 0). The image of the prism is shown in the *first* diagram on the right.



Consider a perspective projection onto the plane z = 0 from the viewpoint (1, 5, 3). The viewpoint has *homogeneous coordinates* **V**(1, 5, 3, 1), and the viewplane vector is **n** = (0, 0, 1, 0). Thus M =

$$\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}(1\ 5\ 3\ 1) - 3\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} -3&0&0&0\\0&-3&0&0\\1&5&0&1\\0&0&0&-3 \end{pmatrix}.$$

Applying the *projection* to the vertices of the prism yields

$$\begin{pmatrix} A'\\B'\\C'\\D'\\E'\\F' \end{pmatrix} = \begin{pmatrix} 0&0&0&1\\2&0&0&1\\2&3&0&1\\0&3&0&1\\1&2&1&1\\1&1&1&1 \end{pmatrix}\begin{pmatrix} -3&0&0&0\\0&-3&0&0\\1&5&0&1\\0&0&0&-3 \end{pmatrix} = \begin{pmatrix} 0&0&0&-3\\-6&0&0&-3\\-6&-9&0&-3\\0&-9&0&-3\\-2&-1&0&-2\\-2&2&0&-2 \end{pmatrix}.$$

The images have *Cartesian coordinates* **A'**(0, 0, 0), **B'**(2, 0, 0), **C'**(2, 3, 0), **D'**(0, 3, 0), **E'**(1, 0.5, 0), and **F'**(1, -1, 0). The image of the prism is as shown in the *second* diagram above.

Now set up a *viewscreen* as in Exercise 4.5 (in the book), with origin (1, 2, 0, 1), U-direction (3, 4, 0), and V-direction (-4, 3, 0) (these directions are *normalised* to produce unit vectors, so that we obtain **basis vectors** $({}^3/_5, {}^4/_5, 0, 1)$ and $(-{}^4/_5, {}^3/_5, 0, 1)$).

So $K = \begin{pmatrix} 3/5 & 4/5 & 0 & 0 \\ -4/5 & 3/5 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ and $K^R = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 0 \\ -11/5 & -2/5 & 1 \end{pmatrix}$, where $K^R = K^T(KK^T)^{-1}$. **Check**: $KK^R = I_3$, $K^RK = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, so that K^RK is **not** a left inverse. Now

$$\begin{pmatrix} 0 & 0 & 0 & -3 \\ -6 & 0 & 0 & -3 \\ -6 & -9 & 0 & -3 \\ 0 & -9 & 0 & -3 \\ -2 & -1 & 0 & -2 \\ -2 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 0 \\ -11/5 & -2/5 & 1 \end{pmatrix} = \begin{pmatrix} 33/5 & 6/5 & -3 \\ 3 & 6 & -3 \\ -21/5 & 3/5 & -3 \\ -3/5 & -21/5 & -3 \\ 12/5 & 9/5 & -2 \\ 24/5 & 18/5 & -2 \end{pmatrix}$$

, *so that* A'' = $(-^{11}/_5, -^2/_5, 1)$, B'' = (-1, -2, 1), C'' = $(^7/_5, -^1/_5, 1)$, D'' = $(^7/_5, -^1/_5, 1)$, E'' = $(-^6/_5, -^9/_{10}, 1)$, and F'' = $(-^{12}/_5, -^9/_5, 1)$; and

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ -2 & 0 & 0 & -1 \\ -2 & -3 & 0 & -1 \\ 0 & -3 & 0 & -1 \\ -1 & -2 & 0 & -1 \\ -1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 0 \\ -11/5 & -2/5 & 1 \end{pmatrix} = \begin{pmatrix} 11/5 & 2/5 & -1 \\ 1 & 2 & -1 \\ -7/5 & 1/5 & -1 \\ -1/5 & -7/5 & -1 \\ 0 & 0 & -1 \\ 4/5 & 3/5 & -1 \end{pmatrix}$$

, *so that* A'' = $(-^{11}/_5, -^2/_5, 1)$, B'' = (-1, -2, 1), C'' = $(^7/_5, -^1/_5, 1)$, D'' = $(^7/_5, -^1/_5, 1)$, E'' = (0, 0, 1), and F'' = $(-^4/_5, -^3/_5, 1)$.

Exercises. Q: Determine the *projection matrix* for a parallel projection in the direction (-1, 4) and viewline 2x-y+8 = 0. Apply to the **triangle** ABC, where A = (2, 2), B = (4, 3), and C = (3, 5). A: The *viewline* has equation 2x-y+8 = 0, so that *l* = (2, -1, 8). The *viewpoint* is V = (-1, 4, 0), so that $l \bullet \underline{V} = -6$, and so $M = l^T\underline{V} - (l \bullet \underline{V})I_3 = ({}^{2}{-1}_{8})(-1\ 4\ 0) + 6({}^{1}0_{0}\ {}^{0}1_{0}\ {}^{0}0_{1}) = ({}^{-2}1_{-8}\ {}^{8}{-4}_{32}\ {}^{0}0_{0}) + ({}^{6}0_{0}\ {}^{0}6_{0}\ {}^{0}0_{6}) = ({}^{4}1_{-8}\ {}^{8}2_{32}\ {}^{0}0_{6})$. The **new** vertices are given by $({}^{A'}B'_{C'}) = ({}^{A}B_{C})M = ({}^{2}4_{3}\ {}^{2}3_{5}\ {}^{1}1_{1})({}^{4}1_{-8}\ {}^{8}2_{32}\ {}^{0}0_{6}) = ({}^{1/3}11/6_{3/2}\ {}^{26/3}35/3_{11}\ {}^{1}1_{1})$, *so that* A' = $(^1/_3, ^{26}/_3)$, B' = $(^{11}/_6, ^{35}/_3)$, and C' = $(^3/_2, 11)$.

Q: Determine the *projection matrix* for a perspective projection from V(2, -1, 1) onto -x+3y+2z-4 = 0. A: The *viewpoint* has **homogeneous** coordinates V(2, -1, 1, 1), and the *viewplane vector* is n = (-1, 3, 2, -4). **Thus**

$$M = \begin{pmatrix} -1 \\ 3 \\ 2 \\ -4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & 1 \end{pmatrix} + 7 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 & -1 \\ 6 & -2 & 3 & 3 \\ 4 & -2 & 2 & 2 \\ -8 & 4 & -4 & -4 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 1 & -1 & -1 \\ 6 & 4 & 3 & 3 \\ 4 & -2 & 9 & 2 \\ -8 & 4 & -4 & 3 \end{pmatrix}.$$

Q: Is $(K^TK)^{-1}(K^TK) = I_4$? A: K is a 3×4 matrix; K^T is a 4×3 matrix; and K^TK is a 4×4 matrix, so that $(K^TK)^{-1}$ *exists* & $(K^TK)^{-1}K^TK = I_4$ *as long as* $|K^TK| \neq 0$. Q: Is $K^L = (K^TK)^{-1}K^T$ a *left inverse*? A: Using the *previous* answer, $K^L = (K^TK)^{-1}K^T$ is a *left* inverse for K, i.e. $K^LK = I_4$.

25th October 2001

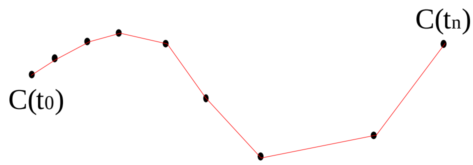
Tutorial

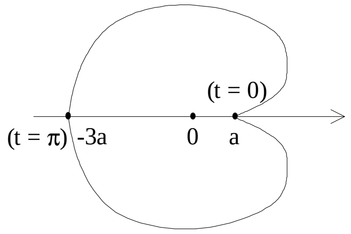
In *Maple*, we plotted **triangles** in the plane using a predefined procedure (the triangles were defined by *points* or by *lines*). We then **rotated** a particular triangle in the plane (with vertices not on the coordinate axes) by an angle *theta* around the origin; **projected** the triangle onto a line *l*x+my = 0; generated a **matrix** representing the *affine rotation* of $\mathbf{P}^2$ about a point Q through an angle *theta* (using *homogeneous* coordinates); and generated a matrix representing the *affine projection* of $\mathbf{P}^2$ onto a line *l*x+my+nz = 0 (using *homogeneous* coordinates).

30th October 2001

Curves in R^2

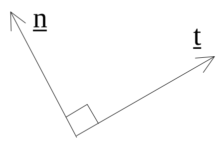
Define *parametrically*, e.g. $x = a\cos t$, $y = b\sin t$ — **easier** than $x^2/a^2 + y^2/b^2 = 1$. Consider the curve C or C(t) *defined* on $t \in I = [a, b]$ by the set of *points* $(x(t), y(t))$. C can be **rendered** by plotting n+1 points $(x(t_i), y(t_i))$, where $t_i = a + (i/n)(b-a)$, $0 \le i \le n$; and then *joining* these points by line sequences as shown. The speed of C is given by $v(t) = \sqrt{(\dot{x}^2+\dot{y}^2)}$, where $\dot{x} = dx/dt$. C is **regular** if $v(t) \neq 0$ for all $t \in I$.





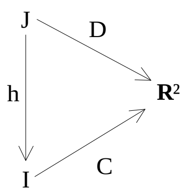
Example: the *cardioid* $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t$. *Here*, $v^2(t) = 8a^2(1-\cos t)$. **Recall**: $1-\cos t = 2\sin^2(t/2)$, and $1+\cos t = 2\cos^2(t/2)$. Therefore, $v(t) = 4a|\sin t/2|$, which is 0 at t = 0, and 4a at $t = \pi$. Note: $v(t)$ is **not** the radius of curvature, which is $\varrho = 1/\kappa$, where $\kappa = \dot{x}\ddot{y} - \ddot{x}\dot{y} / v(t)^{3/2}$.

Tangents and Normals (perhaps not important?): $\underline{t}(t) = (\dot{x}(t), \dot{y}(t))$ is the *tangent vector*. Further, $(\dot{x}(t)/v(t), \dot{y}(t)/v(t))$ is the *unit tangent vector*, the *tangent line* is given by $y-y(t) = (\dot{y}(t)/\dot{x}(t))(x-x(t))$ (or $\dot{x}(t)(y-y(t)) = \dot{y}(t)(x-x(t))$), the unit *normal* vector is given by $(-\dot{y}(t)/v(t), \dot{x}(t)/v(t))$, and the *normal* line is given by the following expression: $\dot{x}(t)(x-x(t)) + \dot{y}(t)(y-y(t)) = 0$.

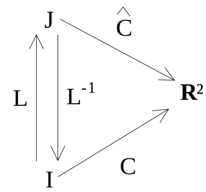


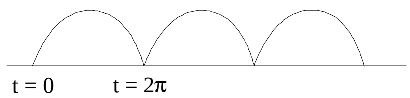
Arc Length: $s = \int_{x=a}^{b} \sqrt{(1+(dy/dx)^2)}dx$, which is *obtained* from $\Sigma\sqrt{(\delta x_i^2+\delta y_i^2)}$. *Parametrically*, $s = \int_{t=a}^{b} \sqrt{(\dot{x}^2+\dot{y}^2)}dt = \int_a^b v(t)dt$. For the *cardioid*, $s = \int_0^{2\pi} 4a|\sin t/2|dt = 4a[-2\cos t/2]_0^{2\pi} = 4a\{2+2\} = 16a$.

Reparametrisation

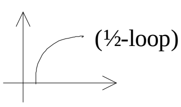


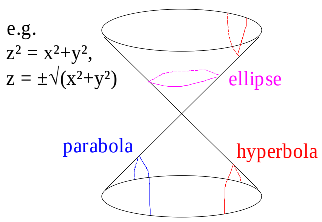
Idea: we want *constant* speed. Let I and J be *real intervals*, and let $h: J \to I$ be a *differentiable* function with $\dot{h}(t) \neq 0$ for all $t \in J$. We define $D(t) = C(h(t))$ for all $t \in J$ — a *reparametrisation* of C. **Theorem (5.1)**: If $C(t) = (x(t), y(t))$ has *length function* $L: I \to J$ ($t \mapsto$ arc *length* from t = a to t = t), i.e. J = [0, *length* of C], then $\hat{C}(s) = (x(L^{-1}(s)), y(L^{-1}(s)))$ is a *unit speed curve*. Note: L is **monotonic increasing**.





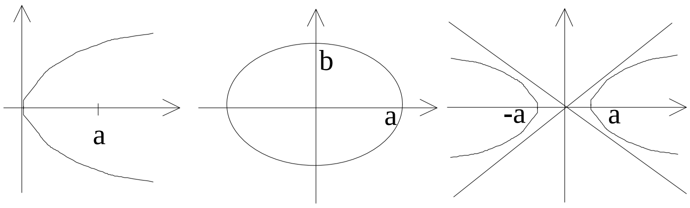
Example: *Cycloid*: $x = a(t+\sin t)$, $y = a(1-\cos t)$. Here, $v(t) = 2a|\cos t/2|$, and $L(t) = \int_0^t v(u)du = \int_0^t 2a\cos u/2\, du = [4a\sin u/2]_0^t = 4a\sin t/2 = s$ (for $0 \le t \le \pi$). Further, $L^{-1}(s) = t = 2\arcsin(s/4a)$, so that $x = a(2\arcsin(s/4a) + \sin(2\arcsin(s/4a)))$, and $y = a(1-\cos(2\arcsin(s/4a)))$ (for $0 \le s \le 4a$).





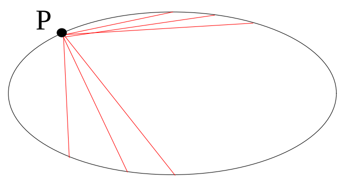
Review of Conics. Conics can be thought of as the *intersection* of a cone with a plane. Parabola: $y^2 = 4ax$, or $x = at^2$, $y = 2at$. Ellipse: $x^2/a^2 + y^2/b^2 = 1$, or $x = a\cos t$, $y = b\sin t$. Hyperbola: $x^2/a^2 - y^2/b^2 = 1$, or $x = a\sec t$, $y = b\tan t$.

General Conic: $ax^2+2hxy+by^2+2gx+2fy+c = 0$, or $aX^2+2hXY+bY^2+2gXW+2fYW+cW^2 = 0$ (---(*)). Associated *matrix*: $M = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$. It follows that (*) is given by $[X\ Y\ W]M\begin{bmatrix} X \\ Y \\ W \end{bmatrix} = 0$.



Parametrisation of a Conic

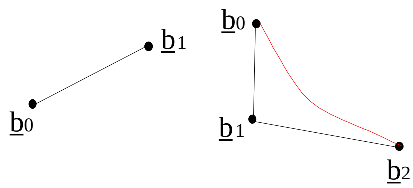
(1) Choose a *point* $P(p_1, p_2)$ on C. (2) Consider the *family of lines* L_t through P. Each line meets C in a *second* point Q_t, so that L_t has equation $y-p_2 = t(x-p_1)$. (3) **Eliminate** y from C and L_t to get a quadratic equation in x with *one solution* $x = p_1$ — the other solution is the **x-coordinate** of Q_t. Substitute in L_t to get the *y-coordinate* of Q_t.



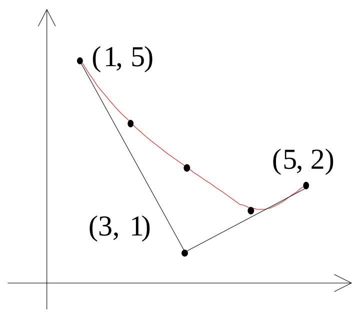
Example: $x^2+2y^2+5xy = 3x+4y+1$. Take $P \equiv (1, 1)$ so that L_t is given by $y = 1+t(x-1)$. It follows that $x^2 + 2\{1+2t(x-1)+t^2(x-1)^2\} + 5x\{1+t(x-1)\} - 3x - 4\{1+t(x-1)\} - 1 = 0$; $x^2(1+2t^2+5t) + x(4t-4t^2+5-5t-3-4t) + (2-4t+2t^2-4+4t-1) = 0$; $x^2(2t^2+5t+1) + x(-4t^2-5t+2) + (2t^2-3) = 0$; $(x-1)\{x(2t^2+5t+1)-(2t^2-3)\} = 0$ (by long division) $\Rightarrow$ at Q_t, $x_t = {}^{2t^2-3}/_{2t^2+5t+1}$. Note: $x_t = 1$ when $5t+1 = -3$, i.e. $t = -{}^4/_5$ = the *gradient* at P ($Q_t \equiv P$ on *tangent*). Now $y_t = 1+t(x_t-1) = 1 + t({}^{2t^2-3}/_{2t^2+5t+1} - 1) = {}^{2t^2+5t+1+2t^3-3t-2t^3-5t^2-t}/_{2t^2+5t+1} = {}^{-3t^2+t+1}/_{2t^2+5t+1}$.

Bezier Curves

Linear: 2 *control points*, $\underline{b}_0$ and $\underline{b}_1$. $B(t) = (1-t)\underline{b}_0+t\underline{b}_1$ ($t \in [0, 1]$) = the *line segment* from $\underline{b}_0$ to $\underline{b}_1$. $\dot{B}(t) = -\underline{b}_0+\underline{b}_1$. **Quadratic**: 3 *control points*, $\underline{b}_0$, $\underline{b}_1$ and $\underline{b}_2$. $B(t) = (1-t)^2\underline{b}_0+2t(1-t)\underline{b}_1+t^2\underline{b}_2$ ($t \in [0, 1]$). $B(0) = \underline{b}_0$, $B(1) = \underline{b}_2$, $B(½) = ¼(\underline{b}_0+2\underline{b}_1+\underline{b}_2)$. $\dot{B}(t) = -2(1-t)\underline{b}_0 + 2(1-t)\underline{b}_1 - 2t\underline{b}_1 + 2t\underline{b}_2 = 2\{(1-t)(\underline{b}_1-\underline{b}_0) + t(\underline{b}_2-\underline{b}_1)\}$. $\dot{B}(0) = 2(\underline{b}_1-\underline{b}_0)$, and $\dot{B}(1) = 2(\underline{b}_2-\underline{b}_1)$.



Example: $\underline{b}_0 = (1, 5)$, $\underline{b}_1 = (3, 1)$, and $\underline{b}_2 = (5, 2)$. We can *tabulate* as shown, where we use $B(t) = (1+4t, 5t^2-8t+5)$ (which comes from *applying* $B(t) = (1-t)^2\underline{b}_0+2t(1-t)\underline{b}_1+t^2\underline{b}_2$).



t	0	¼	½	¾	1
B(t)	(1, 5)	(2, ${}^{53}/_{16}$)	(3, 2¼)	(4, $1^{13}/_{16}$)	(5, 2)

6th November 2001

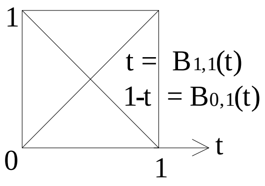
Bernstein Polynomials

$B_{i,n} = \binom{n}{i}t^i(1-t)^{n-i}$ for $0 \le t \le 1$, and $B_{i,n} = 0$ otherwise. Properties: (a) $\Sigma_{i=0}^{n} B_{i,n}(t) = 1$. (b) $B_{i,n}(t) = tB_{i-1,n-1}(t)+(1-t)B_{i,n-1}(t)$ (*recurrence relation*). "**Bernstein's Triangle**" is as shown on the right. (c) $B_{i,n}(t) = B_{n-i,n}(1-t)$. (d) $0 \le B_{i,n}(t) \le 1$ (*indeed* $B_{i-1,n-1}(t) \le B_{i,n}(t) \le B_{i,n-1}(t)$).

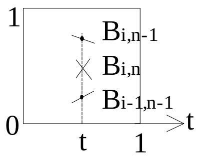
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & (1-t) & & t & & \\ & (1-t)^2 & & 2t(1-t) & & t^2 & \\ (1-t)^3 & & 3t(1-t)^2 & & 3t^2(1-t) & & t^3 \\ & & & \text{etc.} & & & \end{array}$$

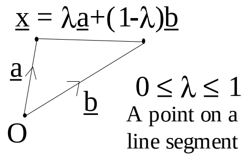
Proofs. (a) Uses the *Binomial Theorem*: $(a+b)^n = \Sigma_{i=0}^n \binom{n}{i} a^i b^{n-i}$. Recall that for *Pascal's triangle*, we have $\Sigma_{i=0}^n \binom{n}{i} = (1+1)^n = 2^n$. *Similarly*, $\Sigma_{i=0}^n (-1)^i \binom{n}{i} = (-1+1)^n = 0$. Therefore, $\Sigma_{i=0}^n B_{i,n}(t) = (1-t+t)^n = 1$.

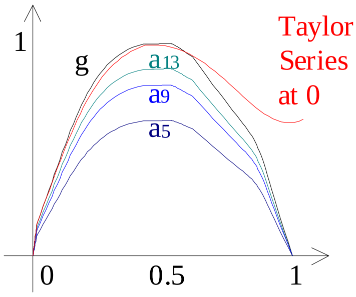
(b) Uses the *binomial recurrence* $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$. Proof. *Algebraically*, RHS = $^{(n-1)!}/_{(i-1)!(n-i)!} + {}^{(n-1)!}/_{i!(n-1-i)!} = {}^{(n-1)!}/_{i!(n-i)!}\{i+(n-i)\} = {}^{n!}/_{i!(n-i)!}$ = LHS. *Combinatorially*, $\binom{n}{i}$ = the number of ways of *choosing* an unordered i-element subset from {1, 2, ..., n} = (the number of *such subsets* which **include** the last element n) + (the number of *such subsets* which do **not** include n) = $\binom{n-1}{i-1} + \binom{n-1}{i}$.



Proof. RHS = $t\binom{n-1}{i-1}t^{i-1}(1-t)^{(n-1)-(i-1)} + (1-t)\binom{n-1}{i}t^i(1-t)^{(n-1)-i} = \{\binom{n-1}{i-1}+\binom{n-1}{i}\}t^i(1-t)^{n-i} = B_{i,n}(t)$. (c) *Trivial*. (d) $0 \le B_{i,n}(t) \le 1$ for $t \in [0, 1]$. Proof by *induction*. Case n = 1: see the diagram on the **left**. *General* case: $B_{i,n}(t) = tB_{i-1,n-1}(t)+(1-t)B_{i,n-1}(t)$ *says* that $B_{i-1,n-1}(t) \le B_{i,n}(t) \le B_{i,n-1}(t)$, *or* that $B_{i-1,n-1}(t) \ge B_{i,n}(t) \ge B_{i,n-1}(t)$ (see the diagrams on the **right**).

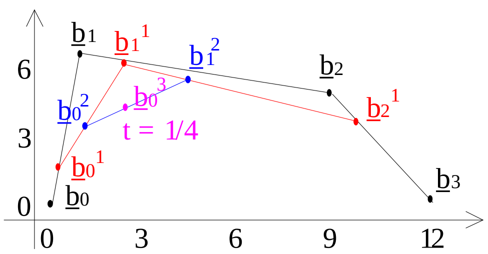






Exercise: f is a *continuous* function with domain [0, 1]. f has *Bernstein approximation* $B_n(t)$ of degree n, $B_n(t) = \Sigma_{i=0}^n f(t_i)B_{i,n}(t)$, where $t_i = {}^i/_n$. *Example*: $B_3(t) = f(0)B_{0,3} + f(^1/_3)B_{1,3} + f(^2/_3)B_{2,3} + f(1)B_{3,3}$. Now consider the function g: $x \to \sin(\pi x)$ in the region [0, 1]. Consider the *Bernstein approximations* for n = 5, 9 and 13 as shown in the *diagram* on the left.

de Casteljau Algorithm



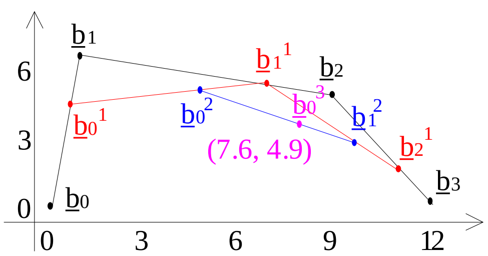
This algorithm finds a *point* on a Beizer curve corresponding to a parameter t. **Recursive** formula for $\underline{b}_i^j$ given *control points* $\underline{b}_0, ..., \underline{b}_n$: $\underline{b}_i^0 = \underline{b}_i$, $\underline{b}_i^j = (1-t)\underline{b}_i^{j-1} + t\underline{b}_{i+1}^{j-1}$.

b_0^0 b_1^0 b_2^0 b_3^0
b_0^1 b_1^1 b_2^1
b_0^2 b_1^2
b_0^3

(1, 1) (2, 7) (8, 6) (12, 2)
(5/4, 10/4) (14/4, 27/4) (36/4, 20/4)
(29/16, 57/16) (78/16, 101/16)
t = 1/4 (165/64, 272/64) ≈ (2.58, 4.25)

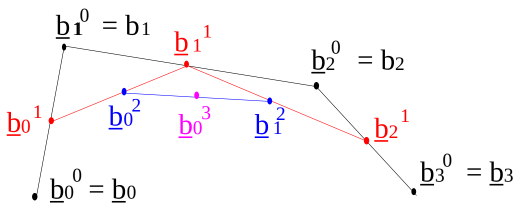
(1, 1) (2, 7) (8, 6) (12, 2)
(5/3, 15/3) (18/3, 19/3) (32/2, 10/3)
(41/9, 53/9) (82/9, 39/9)
t = 2/3 (205/27, 181/27) ≈ (7.6, 4.9)

(1, 1) (2, 7) (8, 6) (12, 2)
(1, 1) (2, 7) (8, 6)
(1, 1) (2, 7)
t = 0 (1, 1) (gives the start point)



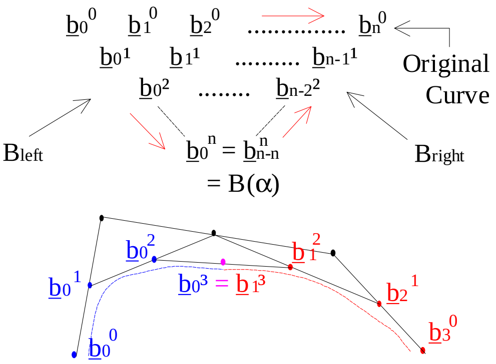
Example: for n = 3, consider $\underline{b}_0 = (1, 1)$, $\underline{b}_1 = (2, 7)$, $\underline{b}_2 = (8, 6)$, and $\underline{b}_3 = (12, 2)$. We have a *triangular table* of $\underline{b}_i^j$ for $0 \le j \le 3$ and $0 \le i \le n-j$ as *shown*. We consider three examples above on the right: t = ¼, t = $^2/_3$, and t = 0. The *diagrams* for t = ¼ and t = $^2/_3$ are shown above on the **left**.

Exercise 6.28: $\underline{b}_k^j = \Sigma_{i=0}^j B_{i,j}(t)\underline{b}_{i+k}$. *Recall* the diagram shown. $\underline{b}_0^3 = B(t)$. *Recurrence*: $\underline{b}_i^0 = \underline{b}_i$, $\underline{b}_i^j = (1-t)\underline{b}_i^{j-1}+t\underline{b}_{i+1}^{j-1}$. Proof (by *induction* on j): Case j = 0: $\underline{b}^0_k = B_{00}(t)\underline{b}_{0+k} = \underline{b}_k$ (since $B_{i,j}(t) = 0$ *when* $i > j$ or when $i < 0$).



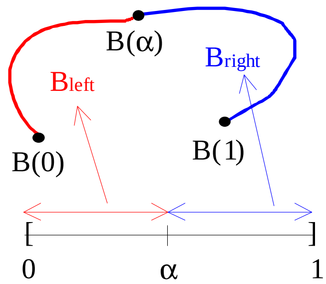
In **general**, $\underline{b}_k^j = (1-t)\underline{b}_k^{j-1}+t\underline{b}_{k+1}^{j-1} = (1-t)\Sigma_{i=0}^{j-1}B_{i,j-1}(t)\underline{b}_{i+k} + t\Sigma_{i=0}^{j-1}B_{i,j-1}(t)\underline{b}_{i+k+1}$. In the *first* part, replace i by h; and in the *second* part, replace i+1 by h (*or* i by h-1) — we obtain $(1-t)\Sigma_{h=0}^{j-1}B_{h,j-1}(t)\underline{b}_{h+k} + t\Sigma_{h=1}^{j}B_{h-1,j-1}(t)\underline{b}_{h+k}$. In the **first** summation, change to $\Sigma_{h=0}^j$ (*because* $B_{j,j-1} = 0$); and in the **second** summation, change to $\Sigma_{h=0}^j$ (*because* $B_{-1,j-1} = 0$). It follows that $\Sigma_{h=0}^j\{(1-t)B_{h,j-1}(t)+tB_{h-1,j-1}(t)\}\underline{b}_{h+k} = \Sigma_{h=0}^j B_{h,j}(t)\underline{b}_{h+k}$.

Exercise 6.29: Prove that $B_{i,n}(\alpha t) = \Sigma_{j=0}^n B_{i,j}(\alpha)B_{j,n}(t)$. Proof. $B_{i,n}(\alpha t) = \binom{n}{i}(1-\alpha t)^{n-i}(\alpha t)^i = \binom{n}{i}((1-t)+t(1-\alpha))^{n-i}(\alpha t)^i = \Sigma_{k=0}^{n-i} \binom{n}{i}\binom{n-i}{k}(1-t)^{n-i-k}t^k(1-\alpha)^k(\alpha t)^i = \Sigma_{k=0}^{n-i} {}^{n!}/_{i!(n-i)!}{}^{(n-i)!}/_{k!(n-i-k)!}(1-t)^{n-i-k}t^{i+k}(1-\alpha)^k\alpha^i$. *Substitution*: j = i+k or k = j-i: we **get** $\Sigma_{j=i}^n {}^{n!}/_{i!}{}^{j!}/_{j!}{}^{1}/_{(j-i)!(n-j)!}(1-t)^{n-j}t^j(1-\alpha)^{j-i}\alpha^i = \Sigma_{j=i}^n {}^{n!}/_{j!(n-j)!}(1-t)^{n-j}t^j({}^{j!}/_{i!(j-i)!})(1-\alpha)^{j-1}\alpha^i = \Sigma_{j=i}^n B_{j,n}(t)B_{i,j}(\alpha)$. This is the *same* as $\Sigma_{j=0}^n$ because $B_{i,j} = 0$ when $j < i$. Note: the *case* i = n is $B_{nn}(\alpha t) = B_{nn}(\alpha)B_{nn}(t) = (\alpha t)^n = \alpha^n t^n$, with only *1 non-zero* term.



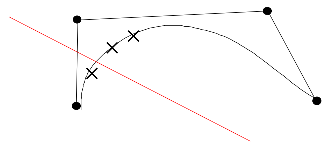
6.4 The Subdivision Theorem.

This theorem gives us *control points* for B_{left} and B_{right} in the diagram. The **required** control points are the sets of diagonal points as shown in the *diagram* on the left. Note that this is **well defined** in $\mathbf{R}^3$.

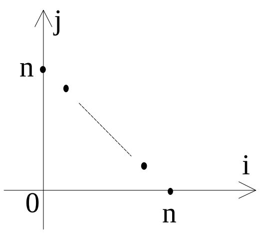


6.10: Applications.

(1) Rendering a *curve* by *linear approximations*. Algorithm: subdivide into "*almost linear*" pieces and **plot** the linear approximations. (2) Intersection of a *curve* with a *line*. (3) Intersection of *2 Bezier curves*. If C_1 is Bezier of degree n, and if C_2 is Bezier of degree m, then there may be up to *mn intersections*. Subdivide **both**.

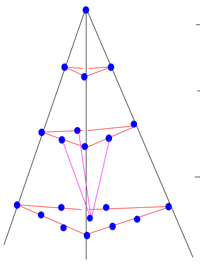


Symmetric Form for the Binomial Coefficients



$\binom{n}{i}$ is the number of ways of *choosing an i-element subset* from {1, ..., n} = the number of ways of *partitioning* {1, ..., n} into an **i-element** subset and a **j-element** subset, where j = n-i, i.e. an ordered pair of subsets, $= {}^{n!}/_{i!j!}$ = (by *definition*) $= \binom{n}{i,j}$. *Binomial Expansion Formula:* $(a+b)^n = \Sigma_{0\le i,j\le n,\ i+j=n} \binom{n}{i,j}a^ib^j$. The *recurrence relation* is given by $\binom{n}{i,j} = \binom{n-1}{i-1,j} + \binom{n-1}{i,j-1}$.

Trinomial Coefficients. $\binom{n}{i,j,k} = {}^{n!}/_{i!j!k!}$, where $0 \le i,j,k \le n$, and i+j+k = n. Now $(a+b+c)^n = \Sigma_{i,j,k} \binom{n}{i,j,k}a^ib^jc^k$, and $\binom{n}{i,j,k} = \binom{n-1}{i-1,j,k} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}$ = the number of ways of *partitioning* {1,...,n} into [subset 1, subset 2, subset 3], where |subset 1| = i, |subset 2| = j, *and* |subset 3| = k.



1

1 1
1

1 2 1
2 2
1

1 3 3 1
3 ? 3
3 3
1

Pascal 3-Simplex

Each *face* in the diagram is a *Pascal triangle*. ? is the coefficient of abc in $(a+b+c)^3 = \binom{3}{1,1,1} = \binom{2}{0,1,1} + \binom{2}{1,0,1} + \binom{2}{1,1,0} = 2+2+2 = 6$. The *next* levels are shown on the **right**.

1 4 6 4 1
4 12 12 4
6 12 6
4 4
1

1 5 10 10 5 1
5 20 30 20 5
10 30 30 10
10 20 10
5 5
1

Σentries = 243 = 3^5

1 6 15 20 15 6 1
6 30 60 60 30 6
15 60 90 60 15
20 60 60 20
15 30 15
6 6
1

$90 = \binom{6}{2,2,2}$

Section 7.3: Conversions Between Representations

Given a *polynomial curve* in $\mathbf{R}^2$, $A = \underline{a}_0+\underline{a}_1t+\ldots+\underline{a}_nt^n$, where $\underline{a}_i \in \mathbf{R}^2$, this is *equivalent* to $[p_0+p_1t+\ldots+p_nt^n,\ q_0+q_1t+\ldots+q_nt^n]$, where $\underline{a}_i = [p_i, q_i]$. Suppose that this has *Bezier representation* B $= \underline{b}_0(1-t)^n + \underline{b}_1n(1-t)^{n-1}t + \ldots + \underline{b}_nt^n$ (---(*)). Comparing the *coefficients* of the t^i, we get $[\underline{a}_0\ \underline{a}_1\ \ldots\ \underline{a}_n]^T = M_n[\underline{b}_0\ \underline{b}_1\ \ldots\ \underline{b}_n]^T$. *Case* n = 3: $\underline{a}_0+\underline{a}_1t+\underline{a}_2t^2+\underline{a}_3t^3 = \underline{b}_0(1-3t+3t^2-t^3) + 3\underline{b}_1(t-2t^2+t^3) + 3\underline{b}_2(t^2-t^3) + \underline{b}_3(t^3)$. So we have what is *shown* on the *right*, where the 4×4 matrix is M_3.

$$\begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -3 & 3 & & \\ 3 & -6 & 3 & \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 1 & \frac{1}{3} & & \\ 1 & \frac{2}{3} & \frac{1}{3} & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

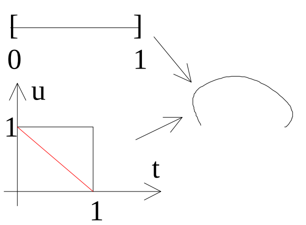
The *coefficient* of t^i in (*) is given by $\underline{b}_0(-1)^i\binom{n}{i,n-i} + \underline{b}_1(-1)^{i-1}\binom{n-1}{i-1,n-i}\binom{n}{1,n-1} + \underline{b}_2(-1)^{i-2}\binom{n}{2,n-2}\binom{n-2}{i-2,n-i} + \ldots + \underline{b}_i(-1)^0\binom{n}{i,n-i}\binom{n-i}{0,n-i}$. The *coefficient* of $\underline{b}_j$ in this is *given* by $(-1)^{i-j}\binom{n}{j,n-j}\binom{n-j}{i-j,n-i} = (-1)^{i-j}\left(\frac{n!}{j!(n-j)!}\right)\frac{(n-j)!}{(i-j)!(n-i)!} = (-1)^{i-j}\binom{n}{j,i-j,n-i}$, *for* $0 \le j \le i \le n$. Now M_3^{-1} is the matrix shown on the *left*. **Exercise**: write down M_4 and *calculate* M_4^{-1}.

Answer: Case n = 4: $\underline{a}_0+\underline{a}_1t+\underline{a}_2t^2+\underline{a}_3t^3+\underline{a}_4t^4 = \underline{b}_0(1-4t+6t^2-4t^3+t^4) + \underline{b}_1(4t-12t^2+12t^3-4t^4) + \underline{b}_2(6t^2-12t^3+6t^4) + \underline{b}_3(4t^3-4t^4) + \underline{b}_4(t^4)$. So we have what is shown on the *right*, where the **middle** matrix is M_4. Calculating the *inverse* in the usual way (i.e. *manipulating* $[M_4 \mid I]$ to get $[I \mid M_4^{-1}]$, we get M_4^{-1} as shown on the *left*.

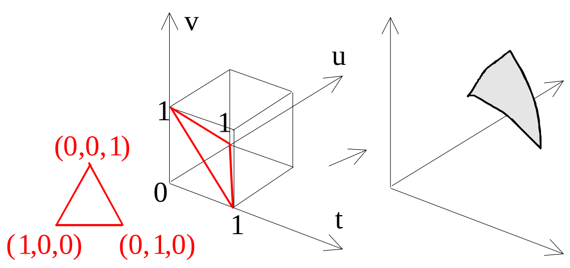
$$\begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \\ \boldsymbol{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -4 & 4 & & & \\ 6 & -12 & 6 & & \\ -4 & 12 & -12 & 4 & \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_0 \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \\ \boldsymbol{b}_4 \end{bmatrix}$$

$$M_4^{-1} = \begin{bmatrix} 1 & & & & \\ 1 & \frac{1}{4} & & & \\ 1 & \frac{1}{2} & \frac{1}{6} & & \\ 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

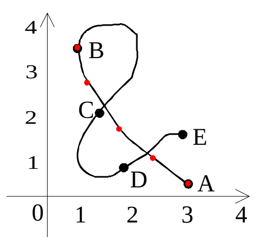
Symmetrising B(t)



$B_n(t) = \Sigma_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^i\underline{b}_i$, for $t \in [0, 1]$. This *becomes* $B_n(t,u) = \Sigma_{i,j}\binom{n}{i,j}t^iu^j\underline{b}_i$, *where* $0 \le i,j \le n$, $i+j = n$, $t+u = 1$, and $0 \le t,u \le 1$. With *3 parametres*, we have the diagram on the **right**, and $B_n(t,u,v) = \Sigma_{i,j,k}\binom{n}{i,j,k}t^iu^jv^k\underline{b}_i$ for $0 \le i,j,k \le n$, $i+j+k = n$, $0 \le t,u,v \le 1$, *and* $t+u+v = 1$.

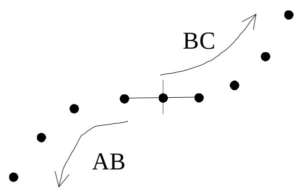


15th November 2001



Tutorial. We can *join* together Bezier curves to draw a **chosen** curve. Consider an attempt to draw an **ampersand**. First, chop the sketch up into *pieces*, e.g. AB, BC, CD and DE. For each piece, choose *control points*, e.g. AB = [AB0, AB1, AB2, AB3, AB4] = [[3.2, 0.3], [2.4, 0.5], [1.6, 1.5], [1.4, 2.5], [1.3, 3.2]].

We must apply a *continuity condition*, e.g. AB4 = BC0 and AB4-AB3 = BC1-BC0, so that BC = [AB4, 2*AB4-AB3, BC2, BC3, BC4] etc. Now **plot** these curves using *procedures* defined in Maple.



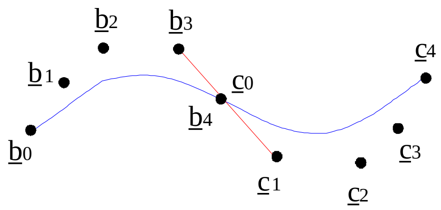
20th November 2001

Derivatives

$^d/_{dt}B_{i,n}(t) = {}^{(i-nt)}/_{t(1-t)}B_{i,n}(t) = n(B_{i-1,n-1}(t)-B_{i,n-1}(t))$. Example: $^d/_{dt}B_{2,4}(t) = {}^d/_{dt}6(1-t)^2t^2 = 6\times2(-1)(1-t)t^2 + 6\times2(1-t)^2t = 4\{3(1-t)^2t-3(1-t)t^2\} = 4\{B_{1,3}(t)-B_{2,3}(t)\} = 4t-11t^2+7t^3$. Corollary 7.2: If $B(t) = \Sigma_{i=0}^{n} \underline{b}_iB_{i,n}(t)$, then $^d/_{dt}B(t) = \Sigma_{i=0}^{n-1}n(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-1}(t)$, $^{d^2}/_{dt^2}B(t) = \Sigma_{i=0}^{n-2}n(n-1)(\underline{b}_{i+2}-2\underline{b}_{i+1}+\underline{b}_i)B_{i,n-2}(t)$, ..., $(d^r/dt^r)B(t) = \Sigma_{i=0}^{n-r}\,{}^{n!}/_{(n-r)!}\{\Sigma_{j=0}^{r}(-1)^j\binom{r}{j}\underline{b}_{i+r-j}\}B_{i,n-r}(t)$.

Piecewise Bezier Curves

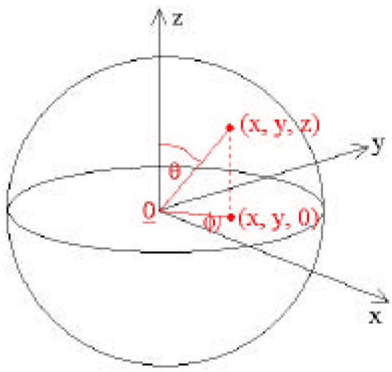
The diagram on the *right* shows two quartic curves joined **together**. If $\underline{c}_0 = \underline{b}_4$, and if $\underline{c}_1-\underline{c}_0 = \mu(\underline{b}_4-\underline{b}_3)$, then the join is "*visually tangent continuous*". If $\underline{c}_0 = \underline{b}_4$, and if $\underline{c}_1-\underline{c}_0 = \underline{b}_4-\underline{b}_3$, then the join is "*tangent continuous*" or "*C^1-continuous*". The formula for $^d/_{dt}B(t)$ when n = 4 is: at t = 1, $4(\underline{b}_4-\underline{b}_3)$ on the *first* curve; and, at t = 0, $4(\underline{c}_1-\underline{c}_0)$ on the *second* curve. The **second** derivatives are: at t = 1, $12(\underline{b}_4-2\underline{b}_3+\underline{b}_2)$ on the *first* curve; and, at t = 0, $12(\underline{c}_2-2\underline{c}_1+\underline{c}_0)$ on the *second* curve.



So we make the *join* C^1-continuous by setting $\underline{c}_0 = \underline{b}_4$ and $\underline{c}_1 = \underline{c}_0+\underline{b}_4-\underline{b}_3 = 2\underline{b}_4-\underline{b}_3$ — and C^2-continuous by *setting* $\underline{c}_2 = 2\underline{c}_1-\underline{c}_0+\underline{b}_4-2\underline{b}_3+\underline{b}_2 = 2(2\underline{b}_4-\underline{b}_3)-\underline{b}_4+\underline{b}_4-2\underline{b}_3+\underline{b}_2 = 4\underline{b}_4-4\underline{b}_3+\underline{b}_2$.

Surfaces

$S = \{(x, y, z) \in \mathbf{R}^3 \mid F(x, y, z) = 0\}$ for some *function* F is an **implicit** surface. Example: $x^2+y^2+z^2 = a^2$, a *sphere* with *centre* $\underline{0}$ and *radius* a. S is **algebraic** if F is **polynomial**. A point on S is *singular* if $^{\partial F}/_{\partial x} = {}^{\partial F}/_{\partial y} = {}^{\partial F}/_{\partial z} = 0$ at P, and *regular* otherwise. For a **parametric** surface, we have $S: U \subset \mathbf{R}^2 \to \mathbf{R}^3$, $(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$, e.g. $(\theta, \phi) \mapsto (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$. In the *diagram* shown, we have $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$.

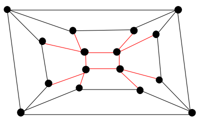


Tangent vectors at P(s, t). $S_s(s, t) = ({}^{\partial x}/_{\partial s}, {}^{\partial y}/_{\partial s}, {}^{\partial z}/_{\partial s})$, and $S_t(s, t) = ({}^{\partial x}/_{\partial t}, {}^{\partial y}/_{\partial t}, {}^{\partial z}/_{\partial t})$. These define the *tangent plane* if **not** linearly dependent. In the *example*, $S_\theta(\theta, \phi) = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta)$, and $S_\phi(\theta, \phi) = (-a\sin\theta\sin\phi, a\sin\theta\cos\phi, 0)$. The *normal* at P is the **vector product** $S_s(s, t)\times S_t(s,t) = N(s, t)$. In the *example*, $N(\theta, \phi) = a^2\sin\theta(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = a\sin\theta(x, y, z)$!

Maple Note: to *specify* $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, we use A := matrix(2, 2, [[1, 2], [3, 4]]); or A := matrix(2, 2, [1, 2, 3, 4]). We have to use the *former* version for $A = \begin{bmatrix}[1] & [2,3]\\ [6,7] & [8]\end{bmatrix}$.

Bernstein Polynomials in 3-D

$B_{i,j,n}(s, t)$ = (by *definition*) = $B_{i,n}(s)B_{j,n}(t) = \binom{n}{i}\binom{n}{j}(1-s)^{n-i}s^i(1-t)^{n-j}t^j$, where $s, t \in [0, 1]$. A Bezier *surface* of degree n has $(n+1)^2$ control points $\underline{b}_{i,j}$ $(0 \le i, j \le n)$. Bezier surface. $B(s, t) = \Sigma_{i=0}^{n}\Sigma_{j=0}^{n} B_{i,j,n}(s,t)\underline{b}_{i,j}$. 4 *special* values: $B(0,0) = \underline{b}_{0,0}$, $B(1,0) = \underline{b}_{n,0}$, $B(0,1) = \underline{b}_{0,n}$, and $B(1,1) = \underline{b}_{n,n}$ — because $(1-s)^{n-i}s^i(1-t)^{n-j}t^j = 0$ *unless* $i, j \in \{0, n\}$.

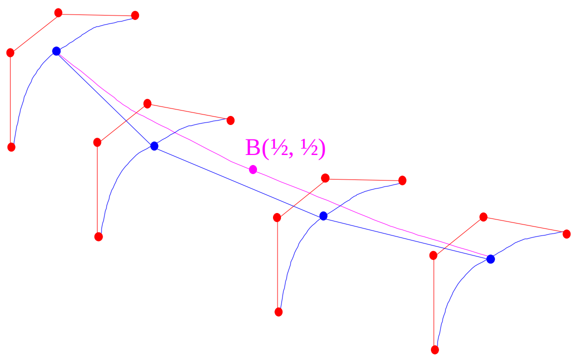


de Casteljau Algorithm. Consider the *table* and *diagram* shown and calculate B(½, ½). **Method**: Apply the 2-D algorithm to each *row* with s = ½. Row 1: As shown *below*.

i\j	0	1	2	3
0	$\underline{b}_{00}$ = [0,0,6]	$\underline{b}_{01}$ = [1,0,8]	$\underline{b}_{02}$ = [2,0,8]	$\underline{b}_{03}$ = [3,0,6]
1	[0,1,8]	[1,1,9]	[2,1,9]	[3,1,8]
2	[0,2,8]	[1,2,9]	[2,2,9]	[3,2,8]
3	[0,3,6]	[1,3,8]	[2,3,8]	[3,3,6]

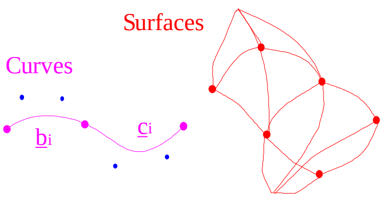
[0,0,6] [1,0,8] [2,0,8] [3,0,6]
[0.5,0,7] [1.5,0,8] [2.5,0,7]
[1,0,7.5] [2,0,7.5]
[1.5,0,7.5]

Row 2: Similarly, [0.5,1,8.5], [1.5,1,9], [2.5,1,8.5] → [1,1,8.75], [2,1,8.5] → [1.5,1,8.75]. Row 3: [0.5,2,8.5], [1.5,2,9], [2.5,2,8.5] → [1,2,8.75], [2,2,8.75] → [1.5,2,8.75]. Row 4: [0.5,3,7], [1.5,3,8], [2.5,3,7] → [1,3,7.5], [2,3,7.5] → [1.5,3,7.5].

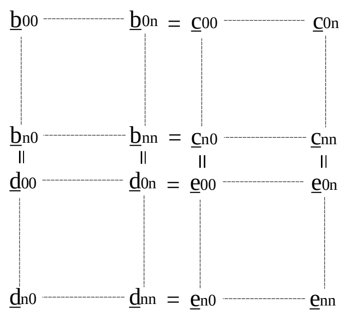


Now apply the 2-D algorithm to this *column* of answers: [1.5,0,7.5], [1.5,1,8.75], [1.5,2,8.75], [1.5,3,7.5] → [1.5,0.5,8.125], [1.5,1.5,8.75], [1.5,2.5,8.125] → [1.5,1.8,4.375], [1.5,2,8.4375] → [1.5,1.5,8.4375]. So B(½, ½) = [1.5, 1.5, 8.4375].

Piecewise Bezier Surfaces



We can connect *surfaces* much the same way that we connected *curves*. To be **tangent** C^1-continuous, we *require* $\underline{c}_{i0} = \underline{b}_{in}$, $\underline{d}_{0i} = \underline{b}_{ni}$, $\underline{e}_{0i} = \underline{c}_{ni}$, $\underline{e}_{i0} = \underline{b}_{in}$, $\underline{c}_{i1} = 2\underline{b}_{in}-\underline{b}_{i,n-1}$, $\underline{d}_{i1} = 2\underline{b}_{ni}-\underline{b}_{n-1,i}$, $\underline{e}_{1i} = 2\underline{c}_{ni}-\underline{c}_{n-1,i}$, *and* $\underline{e}_{i1} = 2\underline{b}_{in}-\underline{b}_{i,n-1}$.



27th November 2001

Integral B-spline curves

Data: An *interval* [a, b]; a *degree* d; a *knot vector* $t_0, t_1, ..., t_d, t_{d+1}, ..., t_{m-d-1}, t_{m-d}, ..., t_m$, where $t_d = a$, $t_{m-d} = b$; and we have the *end* from t_0 to t_d, the *interior* from t_{d+1} to t_{m-d-1}, and another *end* from t_{m-d} to t_m. **Note** that $t_i \le t_{i+1}$, and that we also have *control points* $\underline{b}_0, \underline{b}_1, ..., \underline{b}_n$.

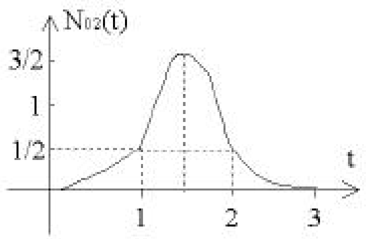
Basis Functions, $N_{i,d}(t)$. They are determined by the *knot vector* and the recursive *formula* $N_{i,0}(t) = 1$ if $t \in [t_i, t_{i+1})$, $N_{i,0}(t) = 0$ otherwise, and $N_{i,d}(t) = ((t-t_i)/(t_{i+d}-t_i))N_{i,d-1}(t) + ((t_{i+d+1}-t)/(t_{i+d+1}-t_{i+1}))N_{i+1,d-1}(t)$. Notes: $^0/_0$ is *assumed* to be 0, (*notation*) the formulae **should** be $N_{i,j}(t) = ...$ for $1 \le j \le d$; and the *degree* of $N_{i,j}(t)$ is j. Definition: The B-spline curve of *degree* d is defined on $[a, b] = [t_d, t_{m-d}]$ by $B(t) = \Sigma_{i=0}^{n} \underline{b}_i N_{i,d}(t)$.

<u>Example</u>: d = 3, with $0 = t_0 = t_1 = t_2 = t_3$ ($t_3 = a$), and $1 = t_4 = t_5 = t_6 = t_7$ ($t_4 = b$). *Now* $N_{00} = 1$ if $t \in [0, 0)$ ($= \phi$), and $N_{00} = 0$ *otherwise*. So $N_{00} = N_{10} = N_{20} \equiv 0$. $N_{30} = 1$ if $t \in [0, 1)$, and $N_{30} = 0$ otherwise. *Again* $N_{40} = N_{50} = N_{60} \equiv 0$. $N_{01} = ({}^{t-0}/_{0-0})N_{00} + ({}^{0-t}/_{0-0})N_{10} = t({}^{0}/_{0})-t({}^{0}/_{0}) = 0$ on $[0, 0)\cup[0, 0)$, and $N_{01} = 0$ *elsewhere*. So $N_{01} = N_{11} \equiv 0$. $N_{21} = {}^{(t-0)}/_{(0-0)}N_{20} + {}^{(1-t)}/_{(1-0)}N_{30} = 0 + (1-t)$ (on [0, 1]), so that $N_{21} = 1-t$ for $t \in [0, 1)$, and $N_{21} = 0$ *elsewhere*. $N_{31} = {}^{(t-0)}/_{(1-0)}N_{30} + {}^{(1-t)}/_{(1-1)}N_{40} = t$ on [0, 1), and $N_{31} = 0$ *elsewhere*. $N_{41} = N_{51} = 0$.

<u>Level 2</u>. $N_{02} \equiv 0$. $N_{12} = {}^{(t-0)}/_{(0-0)}N_{11} + {}^{(1-t)}/_{(1-0)}N_{21} = (1-t)^2$ on [0, 1). $N_{22} = {}^{(t-0)}/_{(1-0)}N_{21} + {}^{(1-t)}/_{(1-0)}N_{31} = t(1-t)+(1-t)t = 2t(1-t)$ *on* [0, 1). $N_{32} = t^2$, and $N_{42} \equiv 0$. <u>Level 3</u>. $N_{03} = (1-t)^3$, $N_{13} = 3t(1-t)^2$, $N_{23} = 3t^2(1-t)$, and $N_{33} = t^3$. So we have the diagram as shown on the *right* — cubic Bernstein polynomials.

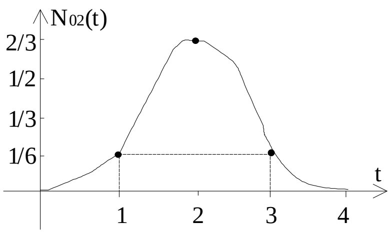
0 0 0 1 0 0 0
0 0 1-t t 0 0
0 (1-t)² 2t(1-t) t² 0
(1-t)³ 3t(1-t)² 3t²(1-t) t³

<u>Exercise</u>: *Repeat* the above using the knot vector $t_0 = 0$, $t_1 = 1$, ..., $t_7 = 7$ (a *uniform* B-spline). $N_{00} = 1$ for $t \in [0, 1)$, and $N_{00} = 0$ *otherwise*. $N_{10} = 1$ for $t \in [1, 2)$; $N_{20} = 1$ for $t \in [2, 3)$; ...; $N_{60} = 1$ for $t \in [6, 7)$. $N_{01} = ({}^{t-0}/_{1-0})(1) + ({}^{2-t}/_{2-1})(1) = t+(2-t) = t$ for $t \in [0, 1)$, $2-t$ for $t \in [1, 2)$, and 0 *otherwise*. $N_{11} = ({}^{t-1}/_{2-1})(1) + ({}^{3-t}/_{3-2})(1) = (t-1)+(3-t) = t-1$ for $t \in [1, 2)$, $3-t$ for $t \in [2, 3)$, and 0 *otherwise*. $N_{21} = ({}^{t-2}/_{3-2})(1) + ({}^{4-t}/_{4-3})(1) = (t-2)+(4-t) = t-2$, $t \in [2, 3]$; $4-t$, $t \in [3, 4]$; and 0 *otherwise*; and so on. $N_{02} = ({}^{t-0}/_{2-0})N_{01} + ({}^{3-t}/_{3-1})N_{11} = {}^{t}/_{2}N_{01}+\frac{1}{2}(3-t)N_{11} = {}^{t^2}/_{2} = \frac{1}{2}(t-0)^2$, $t \in [0, 1)$; ${}^{t}/_{2}(2-t)+\frac{1}{2}(3-t)(t-1) = -t^2+3t-{}^{3}/_{2} = \frac{3}{4}-(t-{}^{3}/_{2})^2$, $t \in [1, 2)$; $\frac{1}{2}(3-t)^2 = \frac{1}{2}(3-t)^2$, $t \in [2, 3)$; and 0 *otherwise*, etc.



From the graph above, N_{02} is continuous. Is it *differentiable*? ${}^{d}/_{dt}N_{02}(t) = t$ for $0 < t < 1$, $-2(t-{}^{3}/_{2})$ for $1 < t < 2$, and $-(3-t)$ for $2 < t < 3$. *Note* that it is 1 at 1 and -1 at 2. Expect that it is **not** twice differentiable: ${}^{d^2}/_{dt^2}N_{02}(t) = 1$ for $0 < t < 1$, -2 for $1 < t < 2$, and 1 for $2 < t < 3$.

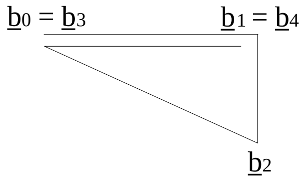
Now $N_{12}(t) = \frac{1}{2}(t-1)^2$ for [1, 2), $\frac{3}{4}-(t-{}^{5}/_{2})^2$ for [2, 3), and $\frac{1}{2}(4-t)^2$ for [3, 4), etc. Note: *reflect* $t \to 5-t$ to get the **third** expression from the **first**. $N_{03}(t) = ({}^{t-0}/_{3-0})N_{02} + ({}^{4-t}/_{4-1})N_{12} = {}^{1}/_{3}tN_{02} + {}^{1}/_{3}(4-t)N_{12}$. For [0, 1), *we have* ${}^{1}/_{3}t(\frac{1}{2}t^2) = {}^{1}/_{6}(t-0)^3$ ($= {}^{1}/_{6}$ at t = 1). For [1, 2), *we have* ${}^{1}/_{3}t({}^{3}/_{4}-(t-{}^{3}/_{2})^2) + {}^{1}/_{3}(4-t)(\frac{1}{2}(t-1)^2) = {}^{1}/_{12}(3t-t(2t-3)^2+2(4-t)(t^2-2t+1)) = {}^{1}/_{12}(3t + (-4t^3+12t^2-9t) + (8t^2-16t+8) + (-2t^3+4t^2-2t)) = {}^{1}/_{12}(-6t^2+24t^2-24t+8) = {}^{1}/_{12}((-6t^2+36t^2-72t+48) + (-12t^2+48t-48) + (8)) = {}^{1}/_{12}(-6(t-2)^3 - 12(t-2)^2 + 8) = -\frac{1}{2}(t-2)^3 - (t-2)^2 + {}^{2}/_{3}$ ($= {}^{1}/_{6}$ at t = 1). It has a *stationary point* at t = 2.



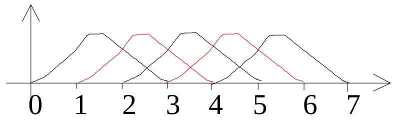
In *summary*, $N_{03}(t) = {}^{1}/_{6}(t-0)^3$ for [0, 1), ${}^{2}/_{3}-{}^{1}/_{2}(t-2)^3-(t-2)^2$ for [1, 2), ${}^{2}/_{3}-{}^{1}/_{2}(2-t)^3-(2-t)^2$ for [2, 3), and ${}^{1}/_{6}(4-t)^3$ for [3, 4). Note that we can get the *third* and *fourth* from the *first* and *second* by doing $t \to 4-t$. *Similarly*, $N_{13}(t) = {}^{1}/_{6}(t-1)^3$ for [1, 2), ${}^{2}/_{3}-{}^{1}/_{2}(t-3)^2-(t-3)^2$ for [2, 3), ${}^{2}/_{3}-{}^{1}/_{2}(3-t)^3-(3-t)^2$ for [3, 4), and ${}^{1}/_{6}(5-t)^3$ for [4, 5), etc.

<u>Exercise</u>: $N_{04}(t) = {}^{1}/_{4}tN_{03}+{}^{1}/_{4}(5-t)N_{13}$ on [0, 5]. For [0, 1), *we have* ${}^{1}/_{24}(t-0)^4$. For [1, 2), *we have* ${}^{1}/_{4}t({}^{2}/_{3}-{}^{1}/_{2}(t-2)^3-(t-2)^2) + {}^{1}/_{4}(5-t){}^{1}/_{6}(t-1)^3 = \ldots$ (*multiply* out, *gather* terms, *factorise* and *tidy up*). *Similarly* for [2, 3): get a polynomial in $(t-{}^{5}/_{2})^2$. Obtain the *expression* for [3, 4) **from** the one for [1, 2). [4, 5): ${}^{1}/_{24}(5-t)^4$.

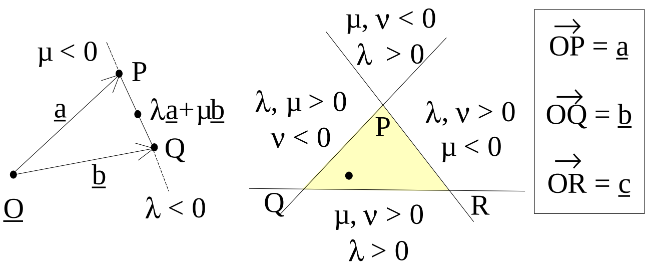
<u>Example</u>. $T = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $m = 7$, $d = 2$. *Control points*: $\underline{b}_0 = (8, 10)$, $\underline{b}_1 = (10, 10)$, $\underline{b}_2 = (10, 8)$, $\underline{b}_3 = (8, 10)$, and $\underline{b}_4 = (10, 10)$. $n = 4$ and $m = n+d+1$. *Basis functions*: $N_{00} = 1$ on $[0, 1)$ and 0 otherwise, ..., $N_{60} = 1$ on $[6, 7)$ and 0 otherwise. $N_{01} = t$ on $[0, 1)$ and $2-t$ on $[1, 2)$, ..., $N_{51} = (t-5)$ on $[5, 6)$ and $(7-t)$ on $[6, 7)$. $N_{02} = ½t^2$ on $[0, 1)$, $¾-(t-{}^3/_2)^2$ on $[1, 2)$ and $½(t-3)^2$ on $[2, 3)$, ..., $N_{42} = ½(t-4)^2$ on $[4, 5)$, $¾-(t-5½)^2$ on $[5, 6)$, and $½(t-7)^2$ on $[6, 7)$.



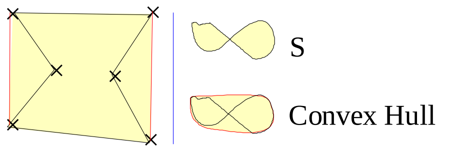
B-spline: $B(t) = \Sigma_{i=0}^{n} \underline{b}_i N_{i,d}(t) = \Sigma_{i=0}^{4} \underline{b}_i N_{i,2}(t)$. <u>Note</u>: here, there are only $d+1$ *non-zero* terms. Example: $B(3½) = \underline{b}_1 N_{12}(3½) + \underline{b}_2 N_{22}(3½) + \underline{b}_3 N_{32}(3½) = (10, 10)(½(3½-4)^2) + (10, 8)({}^3/_4-(3½-3½)^2) + (8, 10)(½(3½-3)^2) = {}^1/_8(10, 10) + {}^3/_4(10, 8) + {}^1/_8(8, 10)$ (<u>Check</u>: ${}^1/_8+{}^3/_4+{}^1/_8 = 1$, OK) $= ({}^5/_4, {}^5/_4) + (7½, 6) + (1, {}^5/_4) = (9{}^3/_4, 8½)$.



<u>Note</u>: $\Sigma_{i=0}^{n} N_{id}(t) = 1$ so that $B(t)$ is of the *form* $\lambda_0\underline{b}_0+\lambda_1\underline{b}_1+...+\lambda_n\underline{b}_n$, with $0 \le \lambda_i \le 1$, and $\Sigma\lambda_i = 1$, i.e. $B(t)$ is in the *convex hull* of the control points. In the **first** example on the right, $\lambda+\mu = 1$, $0 \le \lambda,\mu \le 1$, and the line segment PQ is the *convex hull* of P and Q. In the **second** example, $\lambda\underline{a}+\mu\underline{b}+\nu\underline{c}$ is a general point, with $0 \le \lambda, \mu, \nu \le 1$ in the *shaded* region, and $\lambda+\mu+\nu = 1$ always. The *convex hull* of {P, Q, R} is the **triangle**.



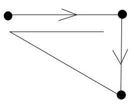
<u>Definition</u>: If $S = \{\underline{x}_1, ..., \underline{x}_n\} \subseteq \mathbf{R}^m$, then the *convex hull* of S is given by $\{\lambda_1\underline{x}_1+...+\lambda_n\underline{x}_n \mid 0 \le \lambda_i \le 1, \Sigma\lambda_i = 1\}$. Also, if $\underline{x}, \underline{y} \in S$, then the *line segment* $|\underline{x}-\underline{y}|$ is **in** the convex hull.



<u>Special Types of B-spline</u>

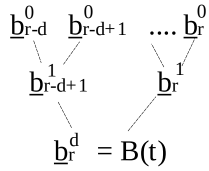
An **open** B-spline of degree d has $t_0 = t_1 = ... = t_d$ and $t_{m-d} = ... = t_{m-1} = t_m$. Then $B(t_d) = \underline{b}_0$, $B(t_{m-d}) = \underline{b}_n$, $B'(t_d) = [d/(t_{d+1}-t_1)](\underline{b}_1-\underline{b}_0)$, and $B'(t_{m-d}) = [d/(t_{m-1}-t_{m-d-1})](\underline{b}_n-\underline{b}_{n-1})$ (same *start*, *finish* and *tangency* behaviour as for Bezier curves). A **uniform** B-spline has $t_{i+1}-t_i$ = constant (as in our *example*, where $t_{i+1}-t_i = 1$).

A **periodic** B-spline of degree d with n+1 control points is obtained by *choosing* $t_0 \le t_1 \le ... \le t_n$ and then **setting** $t_{n+i} = t_{n+i-1} + (t_i-t_{i-1})$ *up to* t_m, e.g. $n = 3$, $d = 2$, $T = \{0, 1, 3, 6, ...\}$ (*differences* = 1, 2, 3, ...): $i = 1$: $t_4 = t_3 + (t_1-t_0) = 6+1 = 7$; $i = 2$: $t_5 = t_4 + (t_2-t_1) = 7+2 = 9$; $i = 3$: $t_6 = t_5 + (t_3-t_2) = 9+3 = 12$. So $T = \{0, 1, 3, 6, 7, 9, 12\}$, *with* $m = 6$. A **closed periodic** B-spline of degree d with n+1 control points is given by a *periodic knot* vector of length $(n+2d+2) = (n+d+2)+d$, and **repeating** control points: $\underline{b}_{n+1} = \underline{b}_0$, $\underline{b}_{n+2} = \underline{b}_1$, ..., $\underline{b}_{n+d} = \underline{b}_{d-1}$, e.g. as shown on the *left*.

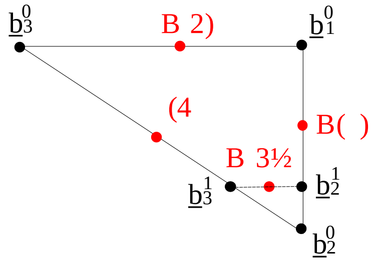


The de Boor Algorithm

Evaluate B(t) for $t \in [t_r, t_{r+1})$. Method: start with $\underline{b}_j^0 = \underline{b}_j$ for $r\text{-}d \le j \le r$, and then use the following recursive definition: $\underline{b}_i^j(t) = (1-\alpha_i^j(t))\underline{b}_{i-1}^{j-1}(t) + \alpha_i^j(t)\underline{b}_i^{j-1}(t)$ (on a *line segment*), where $\alpha_i^j(t) = (t-t_i)/(t_{i+d-j+1}-t_i)$ for $1 \le j \le d$ and $r\text{-}d+j \le i \le r$ — giving a *triangle* as shown on the right. Q: Attempt B(3½) as in the **previous** example. A: Now 3½ ∈ [3, 4) (= $[t_3, t_4)$), so that r = 3.



d = 2. $\underline{b}_0 = (8, 10) = \underline{b}_3$, $\underline{b}_1 = (10, 10) = \underline{b}_4$, $\underline{b}_2 = (10, 8)$. $\underline{b}_1^0 = (10, 10)$, $\underline{b}_2^0 = (10, 8)$, $\underline{b}_3^0 = (8, 10)$. Now $\alpha_2^1 = (t-t_2)/(t_4-t_2) = (^{t-2}/_2) = (^{(3/2)}/_2) = ¾$, and $\alpha_3^1 = (t-t_3)/(t_5-t_3) = {}^{t-3}/_2 = {}^{½}/_2 = ¼$. So $\underline{b}_2^1$ = ¼(10, 10) + ¾(10, 8) = (10, 8½), and $\underline{b}_3^1$ = ¾(10, 8) + ¼(8, 10) = (9½, 8½). Now $\alpha_3^2 = (t-t_3)/(t_4-t_3) = {}^{t-3}/_1 = {}^{½}/_1 = ½$. So $\underline{b}_3^2$ = ½(10, 8½) + ½(9½, 8½) = (9¾, 8½).



Another example: t = 3. $\underline{b}_1^0 = (10, 10)$, $\underline{b}_2^0 = (10, 8)$, $\underline{b}_3^0 = (8, 10)$, $\alpha_2^1 = ½$, $\alpha_3^1 = 0$, $\underline{b}_2^1 = (10, 9)$, $\underline{b}_3^1 = (10, 8)$, $\alpha_3^2 = 0$, and $\underline{b}_3^2 = (10, 9)$. *Example* 3: B(2): 2 ∈ [2, 3) so that r = 2. We also have d = 2. $\underline{b}_0^0 = (8, 10)$, $\underline{b}_1^0 = (10, 10)$, $\underline{b}_2^0 = (10, 8)$. Now $\alpha_1^1 = (t-t_1)/(t_3-t_1) = {}^{2-1}/_2 = ½$, and $\alpha_2^1 = (t-t_2)/(t_4-t_2) = {}^{2-2}/_2 = 0$. Therefore, $\underline{b}_1^1$ = ½(8, 10) + ½(10, 10) = (9, 10), and $\underline{b}_2^1$ = 1(10, 10) + 0 = (10, 10). Further, $\alpha_2^2 = (t-t_2)/(t_3-t_2) = {}^{2-2}/_1 = 0$, so that $\underline{b}_2^2$ = B(2) = 1(9, 10) = (9, 10).

Example 4: B(4): 4 ∈ [4, 5) so that r = 4. We again have d = 2. $\underline{b}_2^0 = (10, 8)$, $\underline{b}_3^0 = (8, 10)$, $\underline{b}_4^0 = (10, 10)$. $\alpha_3^1 = (t-t_3/t_5-t_3) = {}^{4-3}/_2 = ½$, and $\alpha_4^1 = (t-t_4)/(t_6-t_4) = 0$. So $\underline{b}_3^1$ = ½(10, 8) + ½(8, 10) = (9, 9), and $\underline{b}_4^1 = (8, 10)$. Now $\alpha_4^2 = (t-t_4/t_5-t_4) = {}^{4-4}/_1 = 0$. So $\underline{b}_4^2$ = B(4) = (9, 9).

Definition: The *multiplicity* (t) for t ∈ T is the number of times t occurs in T. **Breakpoints** = $\{u_1, ..., u_r\}$, where $u_1 < u_2 < ... < u_r$ is the set of distinct *values* of interior knots. Claim: If an interior knot t_i has *multiplicity* m_i, then $N_{i,k}(t)$ is *differentiable* $k\text{-}m_i$ times at $t = t_i$. With d = 2, expect B(t) to have a '*corner*' at t = 4, with T = [0, 1, ..., 4, 4, ..., 7].

Sample Paper

Q: Prove that the *relation* on $\mathbf{R}^4\backslash\{\underline{0}\}$ defined by $(x_1, y_1, z_1, w_1) \sim (x_2, y_2, z_2, w_2) \Leftrightarrow (x_1, y_1, z_1, w_1) = r(x_2, y_2, z_2, w_2)$ (for some r ≠ 0) is an *equivalence relation*. Explain how this equivalence is used to **define** the projective plane $\mathbf{P}^3$, and how *affine* transformations of $\mathbf{R}^3$ may be represented by *linear transformations* of $\mathbf{P}^3$. Obtain the *matrices* associated to the following affine transformations: (i) **Rotation** through an angle $^{\pi}/_4$ about the line x = 4, y = 5; (ii) **Reflection** in the plane x+y+z = 6.

A: First part — *book work* — generalise the *proof* on page 1 to the *definition* on page 4. $\mathbf{P}^3$ is the set of equivalence classes $(\mathbf{R}^4\backslash\{\underline{0}\})/\sim$. Except for points at *infinity* (when W = 0), elements (X, Y, Z, W) ∈ $\mathbf{P}^3$ have *standard representative* (x, y, z, 1), where $x = {}^X/_W$, etc. Aside: **Affine** transformations of $\mathbf{R}^2$ such as (x', y') = (x, y)$[^a{}_b\ ^d{}_e]$ + (c, f) can be represented as a **linear** transformation of $\mathbf{P}^2$ as (x', y', 1) = (x, y, 1)$[^a{}_{bc}\ ^d{}_{ef}\ ^0{}_{01}]$.

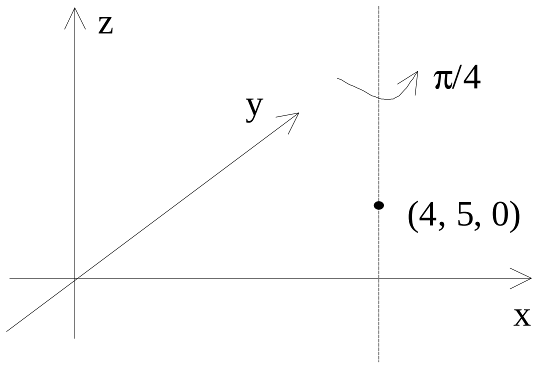
An *affine* transformation of $\mathbf{R}^3$ such as $x' = ax+by+cz+d$, $y' = ez+fy+gz+h$, and $z' = ix+jy+kz+l$, may be represented by the **linear** transformation of $\mathbf{P}^3$ given by $(x', y', z', 1) = (x, y, z, 1)A$, where A is as shown on the *right*.

$$A = \begin{bmatrix} a & e & i & 0 \\ b & f & j & 0 \\ c & g & k & 0 \\ d & h & l & 1 \end{bmatrix}$$

(i) The required matrix is given by $T^{-1}RT$, where T *translates* (0, 0, 0) to (4, 5, 0), and R *rotates* about the z-axis by $^{\pi}/_4$.

$$T^{-1}AT = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -4 & -5 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ 0 & 0 & 1 & \\ \frac{1}{\sqrt{2}} & -\frac{9}{\sqrt{2}} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 4 & 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4+\frac{1}{\sqrt{2}} & 5-\frac{9}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$



(ii) Here, if we consider the *reflection* in the plane $lx+my+nz = k$, we have $l = m = n = 1$, $l' = m' = n' = {}^1/_{\sqrt{3}}$, and $k = 6$. Looking at page 5, we see that the matrix we need is **given** by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k/3l & -k/3m & -k/3n & 1 \end{pmatrix}\begin{pmatrix} 1-2l'^2 & -2l'm' & -2l'n' & 0 \\ -2l'm' & 1-2m'^2 & -2m'n' & 0 \\ -2l'n' & -2m'n' & 1-2n'^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k/3l & k/3m & k/3n & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -2 & -2 & -2 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 & 0 \\ -2 & 1 & -2 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 & 0 \\ -2 & 1 & -2 & 0 \\ -2 & -2 & 1 & 0 \\ 6 & 6 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 & 0 \\ -2 & 1 & -2 & 0 \\ -2 & -2 & 1 & 0 \\ 8 & 8 & 8 & 1 \end{pmatrix}.$$

Q: Let (l, m, n) be the *direction cosines* of a line L in $\mathbf{R}^3$ and let A be as shown. Prove that A is an *orthogonal* matrix, and that $(l, m, n)A = (0, 0, 1)$. Explain how this matrix A may be used to **calculate** the rotation matrix $R_{L,\theta}$ for the rotation of $\mathbf{R}^3$ about L through an angle θ. Calculate the *rotation matrix* for the rotation about the line with direction cosines $({}^1/_{\sqrt{3}}, {}^1/_{\sqrt{3}}, {}^1/_{\sqrt{3}})$ through an angle $^{2\pi}/_3$. *Comment* on the results of your calculation.

$$A = \begin{pmatrix} \frac{ln}{\sqrt{1-n^2}} & \frac{-m}{\sqrt{1-n^2}} & l \\ \frac{mn}{\sqrt{1-n^2}} & \frac{l}{\sqrt{1-n^2}} & m \\ -\sqrt{1-n^2} & 0 & n \end{pmatrix}$$

A: To prove that A is *orthogonal*, we proceed as follows: Let R_1, R_2 and R_3 be the three rows of A^T. Then $R_3.R_3 = l^2+m^2+n^2 = 1$, $R_2.R_2 = (l^2+m^2)/(1-n^2) = (1-n^2)/(1-n^2) = 1$, $R_1.R_1 = (l^2+m^2)n^2/(1-n^2) + (1-n^2) = n^2+(1-n^2) = 1$, $R_1.R_2 = (-lmn+lmn)/(1-n^2) + 0 = 0$, $R_1.R_3 = ... = 0$, $R_2.R_3 = ... = 0$. The three rows are **orthonormal**, so A is **orthogonal**.

Now we must show that $(l, m, n)A = (0, 0, 1)$. The **columns** of A are the **rows** of A^T so we have shown that $(l, m, n)A = (0, 0, 1)$.

The matrix A **rotates** the line L into the z-axis. We then rotate a point about the z-axis by an angle θ using the standard 3×3 matrix (say R) for this operation, and we then rotate the line **back** to its original position. In other words, $R_{L,\theta} = ARA^{-1}$ as shown above. For the **rest** of the question, we substitute in the *appropriate values* for *l*, m, n and θ and work out the **required** 3×3 matrix, which is given by

$$R_{L,\theta} = \begin{pmatrix} \frac{ln}{\sqrt{1-n^2}} & \frac{-m}{\sqrt{1-n^2}} & l \\ \frac{mn}{\sqrt{1-n^2}} & \frac{l}{\sqrt{1-n^2}} & m \\ -\sqrt{1-n^2} & 0 & n \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{ln}{\sqrt{1-n^2}} & \frac{mn}{\sqrt{1-n^2}} & -\sqrt{1-n^2} \\ \frac{-m}{\sqrt{1-n^2}} & \frac{l}{\sqrt{1-n^2}} & 0 \\ l & m & n \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt6} & -\frac{\sqrt3}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{\sqrt3}{\sqrt6} & \frac{1}{\sqrt3} \\ -\frac{\sqrt6}{3} & 0 & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} -\frac12 & \frac{\sqrt3}{2} & 0 \\ \frac{\sqrt3}{2} & -\frac12 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{\sqrt6}{3} \\ -\frac{\sqrt3}{\sqrt6} & \frac{\sqrt3}{\sqrt6} & 0 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt6} & \frac{\sqrt3}{\sqrt6} & \frac{1}{\sqrt3} \\ -\frac{2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \\ \frac{\sqrt6}{6} & -\frac{\sqrt6\ \sqrt3}{6} & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{\sqrt6}{3} \\ -\frac{\sqrt3}{\sqrt6} & \frac{\sqrt3}{\sqrt6} & 0 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

<u>Comment</u>: The calculated matrix is the matrix for a *rotation* through 120° (applying the transformation to an *arbitrary* vector **permutes** the coordinates, i.e. (1, 0, 0) → (0, 1, 0) → (0, 0, 1) → (1, 0, 0)) This is just what we expect since L makes *equal angles* with the three axes.

Q: Let V = (X_0, Y_0, Z_0, W_0) be a viewpoint in $\mathbf{P}^3$, and let Π be the plane in $\mathbf{P}^3$ with equation <u>n</u>•<u>X</u> = *l*X+mY+nZ+pW = 0. Prove that the *perspective projection matrix* for the mapping of $\mathbf{P}^3$ to Π from <u>V</u> is given by M = <u>n</u>^{t}<u>V</u> - (<u>n</u>•<u>V</u>)I_4. Find the matrix M *corresponding* to the projection from P ≡ (3, 4, 5) ∈ $\mathbf{R}^3$ onto the *plane* Π with equation x+y-z+2 = 0.

Calculate the *images* A', B' and C' under this projection of the **vertices** of the equilateral triangle ABC in $\mathbf{R}^3$, where A ≡ (1, 0, 0), B ≡ (0, 1, 0), and C ≡ (0, 0, 1) — and then find the *angle* between B'A' and B'C'. Suggest a suitable **transformation** from Π onto a *display device* with 1024×640 pixels which displays A'B'C' near the **centre** of the display.

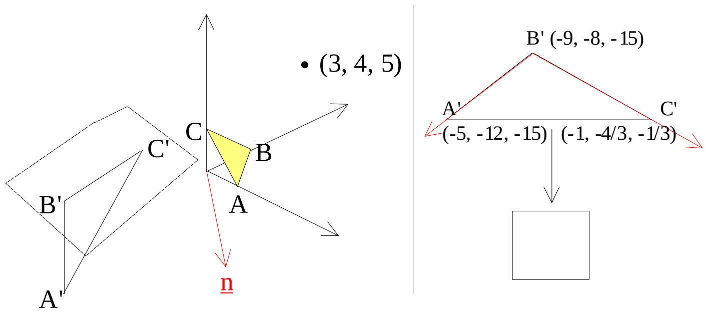
A: **Proof**. Let <u>P</u> (≠ <u>V</u>) be a point to be *projected*. If <u>P</u> lies in Π then <u>n</u>.<u>P</u> = 0 and we need to *verify* that <u>P</u>M ~ <u>P</u>: <u>P</u>M = <u>P</u>(n^t<u>V</u>) - <u>P</u>(<u>n</u>.<u>V</u>)I_4 = (<u>P</u>.<u>n</u>)<u>V</u> - (<u>n</u>.<u>V</u>)<u>P</u>I_4 (since <u>n</u>.<u>V</u> is *scalar*) = 0 - (<u>n</u>.<u>V</u>)<u>P</u> ~ <u>P</u> (since <u>n</u>.<u>V</u> ≠ 0 because <u>V</u> is *not* in Π). Now *suppose* that <u>P</u> does not lie in Π (so that <u>n</u>.<u>P</u> ≠ 0). It follows that **every** point on the line through <u>P</u> and <u>V</u> has *homogeneous coordinates* α<u>P</u>+β<u>V</u>.

The line *intersects* Π when <u>n</u>.(α<u>P</u>+β<u>V</u>) = 0. In *this* case, α(<u>n</u>.<u>P</u>) + β(<u>n</u>.<u>V</u>) = 0 and α = -((<u>n</u>.<u>V</u>)/(<u>n</u>.<u>P</u>))β (with β ≠ 0). Thus the line *meets* Π at <u>P</u>', where <u>P</u>' = -((<u>n</u>.<u>V</u>)/(<u>n</u>.<u>P</u>))β<u>P</u> + β<u>V</u> ~ -(<u>n</u>.<u>V</u>)<u>P</u> + (<u>n</u>.<u>P</u>)<u>V</u> (*multiply* by <u>n</u>.<u>P</u>/β), i.e. <u>P</u>' ~ -<u>P</u>(<u>n</u>.<u>V</u>)I_4 + (<u>P</u>n^t)<u>V</u> = <u>P</u>(<u>n</u>^{t}<u>V</u> - (<u>n</u>.<u>V</u>)I_4). *Hence* M = <u>n</u>^{t}<u>V</u> - (<u>n</u>.<u>V</u>)I_4.

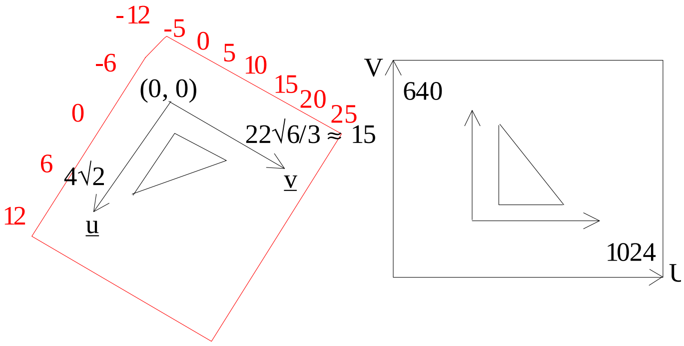
Now P = (3, 4, 5) ∈ $\mathbf{R}^3$ ⇒ <u>V</u> = (3, 4, 5, 1) ∈ $\mathbf{P}^3$. Π has equation x+y-z+2 = 0 or X+Y-Z+2W = 0 (in $\mathbf{P}^3$). So <u>n</u> = (1, 1, -1, 2). *Therefore*, M = $(1\ 1\ -1\ 2)^T(3\ 4\ 5\ 1) - 4I_4$ = the matrix shown on the *right*. Now consider the **triangle** specified in $\mathbf{R}^3$. In *homogeneous* coordinates, A ≡ (1, 0, 0, 1) → (5, 12, 15, -1) (in Π) = (-5, 12, 15, 1). So A' ≡ (-5, -12, -15). *Similarly*, B' ≡ (-9, -8, -15), and C' ≡ (-1, -$\frac{4}{3}$, -$\frac{1}{3}$).

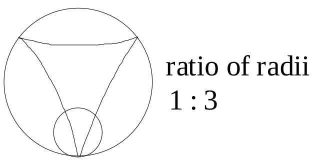
$$\begin{bmatrix} -1 & 4 & 5 & 1 \\ 3 & 0 & 5 & 1 \\ -3 & -4 & -9 & -1 \\ 6 & 8 & 10 & -2 \end{bmatrix}$$

Looking at the *sketch* shown (not *needed* in an exam), $\overrightarrow{B'A'}$ = $\underline{B}$'-$\underline{A}$' = (-4, 4, 0), and $\overrightarrow{B'C'}$ = $\underline{B}$'-$\underline{C}$' = (-8, $-6^2/_3$, $-14^2/_3$) = $-^1/_3$(24, 20, 44). Now *using* $\cos\theta = \underline{u}\bullet\underline{v}/\sqrt{(\underline{u}\bullet\underline{u})}\sqrt{(\underline{v}\bullet\underline{v})}$, we *get* $\cos\theta = (^{16}/_3)/4\sqrt{2}\bullet{}^4/_3\sqrt{282} \approx {}^1/_{2\sqrt{141}}$, so that θ is *approximately* $^\pi/_2$ (86.8°). Now for the **transformation**, we need to choose an *origin* and some *axes*. Let us choose B' as the **origin** and B'A' as one of the **axes**, so that B'A' = (-4, 4, 0), and $\underline{n}$ = (1, 1, -1). *2nd* axis: B'A'×$\underline{n}$ = (-4, -4, -8) ~ (1, 1, 2) (or -(1, 1, 2)).



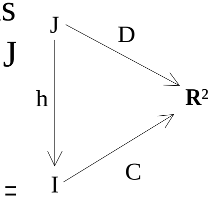
One *suitable transformation* Π → display device is given by (-9, -8, -15) + $^u/_{\sqrt{2}}$(-1, 1, 0) + $^v/_{\sqrt{6}}$(1, 1, 2) ↦ (U, V) ≡ (500+8u, 160+8v). (Aside: The *unit vectors* are given by $^1/_{\sqrt{2}}$(-1, 1, 0) and $-^1/_{\sqrt{6}}$(1, 1, 2). Further, B'C' = $-^2/_3$(12, 10, 22) ≈ $-^{20}/_3$(1, 1, 2) = $^{20\sqrt{6}}/_3\times(-^1/_{\sqrt{6}})$(1, 1, 2) ≈ $15\times(-^1/_{\sqrt{6}})$(1, 1, 2). To get the *transformation*, note that u = -12 → U = 0, that u = 20 → U = 1024, and that $^{1024}/_{32}$ = 32. Also, v = -5 → V = 0, and v = 15 → V = 640 (a bit *tight*!) So we need U = 30(u+12) and V = 30(v+5), *say*).





Q: Let C = C(t) be the segment of a *curve* in $\mathbf{R}^2$ with points {x(t), y(t) | t ∈ [a, b]}. How can C be *rendered* on a display device? Define the terms **speed** and **regular** in this context. Explain the notion of *reparametrisation*. The deltoid is a *three-cusped plane curve* given by x = 2c(cos(t))+c(cos(2t)), y = 2c(sin(t))-c(sin(2t)), t ∈ [0, 2π]. Prove that the *arc length* from t = 0 to t = τ is given by $s(\tau) = {}^8/_3c(1-\cos^3/_2\tau)$. Hence obtain a *reparametrisation* of the deltoid as a **unit speed curve**.

A: **First** part — *book work* — see page 10 (e.g. *Reparametrisation* is obtained using a **differentiable function** h: J→I = [a, b], with $\dot{h}$ (t) ≠ 0 for all t ∈ J and we define D(t) = C(h(t)) for all t ∈ J, with the diagram shown on the right).



Second part: $\dot{x}$ = -2csint-2csin2t, $\dot{x}^2$ = 4c²sin²t+8c²sintsin2t+4c²sin²2t, $\dot{y}$ = 2c(cost)-2c(cos2t), $\dot{y}^2$ = 4c²cos²t-8c²costcos2t+4c²cos²2t. Now $\dot{x}^2+\dot{y}^2$ = 4c²(2+2sintsin2t-2costcos2t) = 8c²(1+sintsin2t-costcos2t) = 8c²(1-(costcos2t-sintsin2t)) = (*using* cos(A ± B) = cosAcosB ∓ sinAsinB) = 8c²(1-cos3t) = (*using* cost = 1-2sin²($^t/_2$)) = 16c²sin²($^{3t}/_2$). So ν(t) = 4c|sin$^{3t}/_2$|. Arc length: $s(\tau) = \int_0^\tau 4c|\sin{}^{3t}/_2|dt = 4c[-{}^2/_3\cos{}^{3t}/_2]_0^\tau = -{}^{8c}/_3(\cos{}^{3\tau}/_2-1) = {}^{8c}/_3(1-\cos{}^{3\tau}/_2)$. QED.

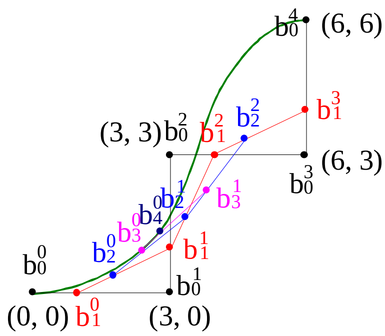
Reparametrisation: We have $s = {}^8/_3c(1-\cos{}^{3\tau}/_2)$, $^{3s}/_{8c} = 1-\cos{}^{3\tau}/_2$, $1-{}^{3s}/_{8c} = \cos{}^{3\tau}/_2$, $^{3\tau}/_2 = \arccos(1-{}^{3s}/_{8c})$, $\tau = {}^2/_3\arccos(1-{}^{3s}/_{8c})$. So the *reparametrisation required* is given by $x = 2c(\cos({}^2/_3\arccos(1-{}^{3s}/_{8c}))) + c(\cos({}^4/_3\arccos(1-{}^{3s}/_{8c})))$, and $y = 2c\sin({}^2/_3\arccos(1-{}^{3s}/_{8c})) - c\sin({}^4/_3\arccos(1-{}^{3s}/_{8c}))$.

Q: (a) The **Bernstein polynomials** are defined by $B_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i$ for $0 \le i \le n$ and $0 \le t \le 1$, and $B_{i,n}(t) = 0$ otherwise. Prove the *following*: (i) $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$, (ii) $0 \le B_{i,n}(t) \le 1$ and $B_{i,n}(t)$ lies between $B_{i-1,n-1}(t)$ and $B_{i,n-1}(t)$. (b) A *quartic Bezier curve* $B = B(t)$ has **control points** $\mathbf{b}_0 \equiv (0, 0)$, $\mathbf{b}_1 \equiv (3, 0)$, $\mathbf{b}_2 \equiv (3, 3)$, $\mathbf{b}_3 \equiv (6, 3)$, and $\mathbf{b}_4 \equiv (6, 6)$.

Evaluate the point $B(^1/_3)$ in **two** ways: (a) By substituting $t = {}^1/_3$ into the *defining equation* of the Bezier curve, and (b) By applying the *de Casteljau algorithm*. Make a **sketch** illustrating the points derived when applying the de Casteljau algorithm. Use *symmetry* to deduce the point $B(^2/_3)$, and include in your sketch your **best attempt** at $B(t)$.

A: (i) *Book work* — see page 12. (ii) When $t \notin [0, 1]$, all three terms are *zero*, so equal. We prove by **induction** that $0 \le B_{i,n}(t) \le 1$. This is clearly true *when* $n = 1$ — recall the **diagram** on page 12. The recurrence relation in (i) above with $0 \le t \le 1$ states that $B_{i,n}(t)$ is in the *interval* between $B_{i,n-1}(t)$ and $B_{i-1,n-1}(t)$ (though we do not know **which** of these two is the *greater*). Since these two *bounding values* lie in [0, 1], it follows that $B_{i,n}(t)$ also lies in [0, 1].

(0, 0)   (3, 0)   (3, 3)   (6, 3)   (6, 6)
(1, 0)   (3, 1)   (4, 3)   (6, 4)
(5/3, 1/3)   (10/3, 5/3)   (14/3, 10/3)
(20/9, 7/9)   (34/9, 20/9)
(74/27, 34/27)



(a) We get the defining equation ($B(t) = (1-t)^4\mathbf{b}_0 + 4t(1-t)^3\mathbf{b}_1 + 6t^2(1-t)^2\mathbf{b}_2 + 4t^3(1-t)\mathbf{b}_3 + t^4\mathbf{b}_4$) by using *"Bernstein's triangle"* — and then we *substitute in* $t = {}^1/_3$ to get the point $(^{74}/_{27}, {}^{34}/_{27})$. (b) We apply the algorithm as shown *above*. The **sketch** is as shown on the right We then use *symmetry* (about the point (3, 3)) to get $B(^2/_3) = (^{88}/_{27}, {}^{128}/_{27})$. The attempt at B(t) is as shown in green.

Q: Prove that, *when* $B(t) = \Sigma_{i=0}^{n} \underline{b}_i B_{i,n}(t)$, (i) $^d/_{dt}B(t) = \Sigma_{i=0}^{n-1} n(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-1}(t)$, and (ii) $^{d^2}/_{dt^2}B(t) = \Sigma_{i=0}^{n-2} n(n-1)(\underline{b}_{i+2}-2\underline{b}_{i+1}+\underline{b}_i)B_{i,n-2}(t)$. Deduce that the *second derivative* of a Bezier curve of degree n is given by $B''(t) = \Sigma_{i=0}^{n-2} \underline{b}_i^{(2)}B_{i,n-2}(t)$, where $\underline{b}_i^{(2)} = n(n-1)(\underline{b}_{i+2}-2\underline{b}_{i+1}+\underline{b}_i)$. Determine *control points* for the **first** and **second** derivatives of the cubic Bezier curve with *control points* $\underline{b}_0 \equiv (6, 3)$, $\underline{b}_1 \equiv (4, 3)$, $\underline{b}_2 \equiv (1, 2)$, and $\underline{b}_3 \equiv (-1, 2)$.

A: We first need to *prove* that $B'_{i,n} = n(B_{i-1,n-1} - B_{i,n-1})$. Now $B_{i,n} = \binom{n}{i}t^i(1-t)^{n-i} \Rightarrow B'_{i,n} = \binom{n}{i}it^{i-1}(1-t)^{n-i} - \binom{n}{i}(n-i)t^i(1-t)^{n-i-1} = {}^{n!}/_{i!(n-i)!}it^{i-1}(1-t)^{n-i} - {}^{n!}/_{i!(n-i)!}(n-i)t^i(1-t)^{n-i-1} = n(^{(n-1)!}/_{(i-1)!(n-1)!})t^{i-1}(1-t)^{n-i} - n(^{(n-1)!}/_{i!(n-i-1)!})t^i(1-t)^{n-i-1} = n\binom{n-1}{i-1}t^{i-1}(1-t)^{n-i} - n\binom{n-1}{i}t^i(1-t)^{n-i-1} = n(B_{i-1,n-1} - B_{i,n-1})$. QED. Applying the *formula* and using the **fact** that $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$ gives $^d/_{dt}B(t) = \Sigma_{i=0}^{n} \underline{b}_iB'_{i,n}(t) = \Sigma_{i=0}^{n} \underline{b}_i n(B_{i-1,n-1}(t)-B_{i,n-1}(t)) = \Sigma_{i=0}^{n} n\underline{b}_iB_{i-1,n-1}(t) - \Sigma_{i=0}^{n} n\underline{b}_iB_{i,n-1}(t) = \Sigma_{i=1}^{n} n\underline{b}_iB_{i-1,n-1}(t) - \Sigma_{i=0}^{n-1} n\underline{b}_iB_{i,n-1}(t)$. *Renumbering* the **first** summation gives $^d/_{dt}B(t) = \Sigma_{i=0}^{n-1} n\underline{b}_{i+1}B_{i,n-1}(t) - \Sigma_{i=0}^{n-1} n\underline{b}_iB_{i,n-1}(t) = \Sigma_{i=0}^{n-1} n(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-1}(t)$.

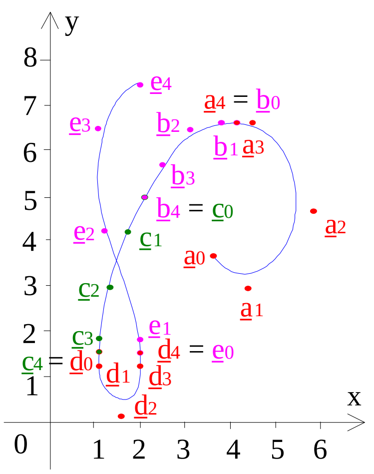
Now $^{d^2}/_{dt^2}B(t) = \Sigma_{i=0}^{n-1} n(\underline{b}_{i+1}-\underline{b}_i)B'_{i,n-1}(t) = \Sigma_{i=0}^{n-1} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)(B_{i-1,n-2}(t)-B_{i,n-2}(t)) = \Sigma_{i=0}^{n-1} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)B_{i-1,n-2}(t) - \Sigma_{i=0}^{n-1} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-2}(t) = \Sigma_{i=1}^{n-1} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)B_{i-1,n-2}(t) - \Sigma_{i=0}^{n-2} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-2}(t)$. Renumbering the *first summation* gives $^{d^2}/_{dt^2}B(t) = \Sigma_{i=0}^{n-2} n(n-1)(\underline{b}_{i+2}-\underline{b}_{i+1})B_{i,n-2}(t) - \Sigma_{i=0}^{n-2} n(n-1)(\underline{b}_{i+1}-\underline{b}_i)B_{i,n-2}(t) = \Sigma_{i=0}^{n-2} n(n-1)(\underline{b}_{i+2}-2\underline{b}_{i+1}+\underline{b}_i)B_{i,n-2}(t))$. QED. The deduction follows *immediately* by definition as $^{d^2}/_{dt^2}B(t) = B''(t)$.

The *differences* of the control points are (4, 3) - (6, 3) = (-2, 0), (1, 2) - (4, 3) = (-3, -1), and (-1, 2) - (1, 2) = (-2, 0). **Multiply** each difference by 3 to give the control points of the first derivative $\underline{b}_0^{(1)}$ = (-6, 0), $\underline{b}_1^{(1)}$ = (-9, -3), and $\underline{b}_2^{(1)}$ = (-6, 0). Aside: Therefore, the *derivative* of the cubic is the **quadratic** Bezier curve $(1-t)^2(-6, 0) + 2(1-t)t(-9, -3) + t^2(-6, 0)$. To determine the **second derivative**, take the differences of the control points of the *first derivative:* (-9, -3) - (-6, 0) = (-3, -3), and (-6, 0) - (-9, -3) = (3, 3). **Multiply** each difference by (n-1) = 2 to give $\underline{b}_0^{(2)}$ = (-6, -6), and $\underline{b}_1^{(2)}$ = (6, 6). Aside: Therefore, the *second derivative* of the cubic is the **linear** Bezier curve (1-t)(-6, -6) + t(6, 6).

Q: The *trinomial coefficients* are defined by $\binom{n}{i,j,k} = {}^{n!}/_{i!j!k!}$ when i, j, k ∈ {0, 1, ..., n} and i+j+k = n, and $\binom{n}{i,j,k} = 0$ *otherwise*. **Prove** the following: (i) $\binom{n}{i,j,k} = \binom{n-1}{i-1,j,k} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}$, (ii) $(a+b+c)^n = \Sigma_{i,j,k} \binom{n}{i,j,k} a^i b^j c^k$, (iii) $\Sigma_{i,j,k} \binom{n}{i,j,k} = 3^n$, and (iv) $\Sigma_{i,j,k}\, \omega^{j+2k} \binom{n}{i,j,k} = 0$ when ω is a *complex cube root* of unity. Define the **Bernstein surface polynomials** by $B_{i,j,k}(t, u, v) = \Sigma_{i,j,k} \binom{n}{i,j,k} t^i u^j v^k$, where $0 \le i,j,k \le n$, $i+j+k = n$, $0 \le t,u,v \le 1$, and $t+u+v = 1$. *Prove that* $B_{i,j,k}(t, u, v) = tB_{i-1,j,k}(t,u,v) + uB_{i,j-1,k}(t,u,v) + vB_{i,j,k-1}(t,u,v)$.

A: (i) RHS = ${}^{(n-1)!}/_{(i-1)!j!k!} + {}^{(n-1)!}/_{i!(j-1)!k!} + {}^{(n-1)!}/_{i!j!(k-1)!} = {}^{(n-1)!}/_{i!j!k!}[i+j+k] = {}^{n!}/_{i!j!k!}$ = LHS. QED. (ii) and (iv): *Pass*! (iii) Follows from (ii) by setting a = b = c = 1. In the **last part** of the question, just apply the *definition*: RHS = $tB_{i-1,j,k}(t,u,v) + uB_{i,j-1,k}(t,u,v) + vB_{i,j,k-1}(t,u,v) = t\Sigma_{i,j,k} \binom{n}{i-1,j,k} t^{i-1}u^j v^k + u\Sigma_{i,j,k} \binom{n}{i,j-1,k} t^i u^{j-1} v^k + v\Sigma_{i,j,k} \binom{n}{i,j,k-1} t^i u^j v^{k-1} = \Sigma_{i,j,k} [\binom{n}{i-1,j,k}+\binom{n}{i,j-1,k}+\binom{n}{i,j,k-1}] t^i u^j v^k$ = (by (i)) = $\Sigma_{i,j,k} \binom{n}{i,j,k} t^i u^j v^k$ = LHS. QED.

Q: Let C_1 and C_2 be two *quartic Bezier curves* with control points $\underline{b}_i$ and $\underline{c}_j$ respectively, **joined** together at $\underline{b}_4 = \underline{c}_0$. Explain what it means for the join to be *visually tangent continuous* and *tangent continuous*. Construct sets of control points for an explicit piecewise tangent continuous Bezier curve with at most **5 components**, in order to approximate the character  shown on the right in the box $0 \le x \le 6$ and $0 \le y \le 8$.



A: If $\underline{c}_0 = \underline{b}_4$, and if $\underline{c}_1 - \underline{c}_0 = \mu(\underline{b}_4 - \underline{b}_3)$, then the join is "*visually tangent continuous*". If $\underline{c}_0 = \underline{b}_4$, and if $\underline{c}_1 - \underline{c}_0 = \underline{b}_4 - \underline{b}_3$, then the join is "*tangent continuous*" or "*C^1-continuous*". To approximate the character, we first **draw** it and **guess** the control points as shown. They are *written down* as follows: $\underline{a}_0$ = (3.5, 3.8), $\underline{a}_1$ = (4.3, 2.9), $\underline{a}_2$ = (5.8, 4.3), $\underline{a}_3$ = (4.5, 6.5), $\underline{a}_4$ = (4.3, 6.5), $\underline{b}_0$ = (4.3, 6.5), $\underline{b}_1$ = (4.9, 6.5), $\underline{b}_2$ = (3.1, 6.3), $\underline{b}_3$ = (2.5, 5.7), $\underline{b}_4$ = (2, 5), $\underline{c}_0$ = (2, 5), $\underline{c}_1$ = (1.5, 4.3), $\underline{c}_2$ = (1.2, 3), $\underline{c}_3$ = (1, 1.9), $\underline{c}_4$ = (1, 1.6), $\underline{d}_0$ = (1, 1.6), $\underline{d}_1$ = (1, 1.3), $\underline{d}_2$ = (1.5, 0.1), $\underline{d}_3$ = (2, 1.3), $\underline{d}_4$ = (2, 1.6), $\underline{e}_0$ = (2, 1.6), $\underline{e}_1$ = (2, 1.9), $\underline{e}_2$ = (1, 4.2), $\underline{e}_3$ = (1, 6.3), $\underline{e}_4$ = (2, 7.5). *Notice* that "$\underline{b}_4 = \underline{c}_0$" and "$\underline{b}_4 - \underline{b}_3 = \underline{c}_1 - \underline{c}_0$" in *all* cases.

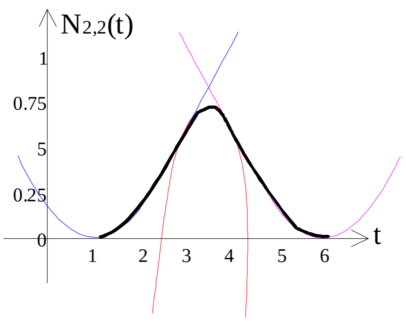
Q: (a) Let $T = [t_0, t_1, ..., t_m]$ be a *knot vector* with $t_0 < t_1 < ... < t_m$. The **B-spline basis functions** $N_{i,d}$ of degree d determined by T are defined *recursively* by $N_{i,0}(t) = 1$ when $t_i \leq t < t_{i+1}$, $N_{i,0}(t) = 0$ otherwise, and $N_{i,j}(t) = (t-t_i/t_{i+j}-t_i)N_{i,j-1}(t) + (t_{i+j+1}-t/t_{i+j+1}-t_{i+1})N_{i+1,j-1}(t)$ for all $0 \leq j \leq d$ and all $0 \leq i \leq m-1-j$. Prove that (for $j > 0$) we have $N_{i,j}(t) = 0$ for all $t \in (-\infty, t_i]\cup[t_{i+j+1}, \infty)$. (b) Let d = 2 and let T = [0, 0, 1, 3, 4, 6, 7, 7]. Evaluate the *basis function* $N_{2,2}(t)$ and verify that it is **differentiable** at t = 3 and at t = 4. Sketch the **graph** of $N_{2,2}(t)$.

A: (a) We prove by *induction on j*. The **initial** induction step j = 1 is as follows: when j = 1, $N_{i,1}(t) = ((t-t_i)/(t_{i+1}-t_i))N_{i,0}(t) + ((t_{i+2}-t)/(t_{i+2}-t_{i+1}))N_{i+1,0}(t)$. We need to *prove* that $N_{i,1}(t) = 0$ for all $t \in (-\infty, t_i]\cup[t_{i+2}, \infty)$. Now if $t \in (-\infty, t_i)$, then $N_{i,0}(t) = 0$ for all i by *definition*, and $N_{i+1,0}(t) = 0$ for all i by definition. **Therefore**, when $t \in (-\infty, t_i)$, $N_{i,1}(t) = 0+0 = 0$.

Similarly, if $t = t_i$, then $N_{i,1}(t) = 0 + ((t_{i+2}-t)/(t_{i+2}-t_{i+1}))N_{i+1,0}(t)$. But because $N_{i+1,0}(t) = 0$ for $t = t_i$ (by *definition*), then when $t = t_i$, we automatically have $N_{i,1}(t) = 0$. **Finally**, when $t \in [t_{i+2}, \infty)$, then $t \notin [t_i, t_{i+1})$ so that $N_{i,0}(t) = 0$, and $t \notin [t_{i+1}, t_{i+2})$ so that $N_{i+1,0}(t) = 0$. It follows that when $t \in [t_{i+2}, \infty)$, we have $N_{i,1}(t) = 0$.

The *induction hypothesis* is that all basis functions of degree j, $N_{i,0}(t), ..., N_{i,j}(t)$, satisfy the property that we are trying to *prove*. Now $N_{i,j+1}(t) = ((t-t_i)/(t_{i+j+1}-t_i))N_{i,j}(t) + ((t_{i+j+2}-t)/(t_{i+j+2}-t_{i+1}))N_{i+1,j}(t)$, where $N_{i,j}(t) = 0$ for $t \notin (t_i, t_{i+j+1})$, and $N_{i+1,j}(t) = 0$ for $t \notin (t_{i+1}, t_{i+j+2})$. We need to *prove* that $N_{i,j+1}(t) = 0$ for $t \in (-\infty, t_i]\cup[t_{i+j+2}, \infty)$, i.e. that $N_{i,j+1}(t) = 0$ for $t \notin (t_i, t_{i+j+2})$. Suppose that $t \notin (t_i, t_{i+j+2})$, so that $t \notin (t_i, t_{i+j+1})$ and $t \notin (t_{i+1}, t_{i+j+2})$. Thus $N_{i,j}(t) = 0$, $N_{i+1,j}(t) = 0$, and hence $N_{i,j+1}(t) = 0$ as *required*. QED.

(b) $N_{2,2} = {}^{t-1}/_{4-1}N_{21} + {}^{6-t}/_{6-3}N_{31} = {}^{1}/_{3}(t-1)N_{21} + {}^{1}/_{3}(6-t)N_{31}$. $N_{2,1} = {}^{t-1}/_{2}N_{20} + {}^{4-t}/_{4-3}N_{30} = ½(t-1)N_{20} + (4-t)N_{30}$. $N_{3,1} = {}^{t-3}/_{1}N_{30} + {}^{6-t}/_{6-4}N_{40} = (t-3)N_{30} + ½(6-t)N_{40}$. $N_{2,0} = 1$ if $t \in [1, 3)$, and $N_{2,0} = 0$ *elsewhere*. $N_{3,0} = 1$ if $t \in [3, 4)$. $N_{4,0} = 1$ if $t \in [4, 6)$. So $N_{2,2} = {}^{1}/_{3}(t-1)N_{21} + {}^{1}/_{3}(6-t)N_{31} = {}^{1}/_{3}(t-1)[½(t-1)N_{20} + (4-t)N_{30}] + {}^{1}/_{3}(6-t)[(t-3)N_{30} + ½(6-t)N_{40}] = {}^{1}/_{6}(t-1)^2N_{20} + {}^{1}/_{3}[(t-1)(4-t) + (6-t)(t-3)]N_{30} + {}^{1}/_{6}(6-t)^2N_{40} = {}^{1}/_{6}(t-1)^2N_{20} + {}^{1}/_{3}[-2t^2+14t-22]N_{30} + {}^{1}/_{6}(6-t)^2N_{40}$. The *sketch* is as shown on the *left*. The derivative $N'_{2,2}(t)$ is given by ${}^{1}/_{3}(t-1)$ for $t \in [1, 3)$, ${}^{1}/_{3}[-4t+14]$ for $t \in [3, 4)$, and $-{}^{1}/_{3}(6-t)$ for $t \in [4, 6)$. At t = 3, it is ${}^{2}/_{3}$ or ${}^{2}/_{3}$, OK; *and* at t = 4, it is $-{}^{2}/_{3}$ or $-{}^{2}/_{3}$, OK.



Exam Paper: January 2002

Answer 3 questions out of 5 (Questions Done: 1, 4, 5)

(1) Prove that the relation on $\mathbf{R}^3\backslash\{\mathbf{0}\}$ defined by
$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \Leftrightarrow (x_1, y_1, z_1) = r(x_2, y_2, z_2)$ for some $r \neq 0$
is an *equivalence relation*.

Explain how this equivalence is used to define the projective plane $\mathbf{P}^2$ and how *affine transformations* of $\mathbf{R}^2$ may be represented by *linear transformations* of $\mathbf{P}^2$. **[10 marks]**

Obtain the matrices associated to the following affine transformations:
(i) *Rotation* through an angle θ about the point (5, 7);
(ii) *Reflection* in the line $y = x + 3$. **[10 marks]**

(2) Let $\mathbf{v} = (l, m, n)$ be a unit vector in $\mathbf{R}^3$. Show that the *Householder matrix*
$H_\mathbf{v} = I_3 - 2\mathbf{v}^t\mathbf{v}$
is *symmetric* and *orthogonal*. Show also that $H_\mathbf{v}$ is the reflection matrix when $\mathbf{R}^3$ is reflected in the plane Π through $\mathbf{0}$ with normal $\mathbf{v}$. **[8 marks]**
Hence obtain the projection matrix for the *parallel projection* of $\mathbf{R}^3$ onto Π. **[4 marks]**
Calculate the product $H_\mathbf{u}H_\mathbf{v}$ when $\mathbf{u} = (^1/_3, ^2/_3, -^2/_3)$ and $\mathbf{v} = (-^2/_3, ^2/_3, ^1/_3)$.
Comment on the results of your calculation. **[8 marks]**

(3) Describe the *viewing pipeline* by which a three-dimensional object may be **visualised** on a two-dimensional display device. **[5 marks]**

When $\mathbf{n}.\mathbf{X} = lX+mY+nZ+pW = 0$ is the equation of a *plane* Π in $\mathbf{P}^3$, and when $\mathbf{V} = (X_0, Y_0, Z_0, W_0)$ is a *viewpoint* also in $\mathbf{P}^3$, the *perspective projection matrix* for the mapping of $\mathbf{P}^3$ to Π from $\mathbf{V}$ is given by $M = \mathbf{n}^t\mathbf{V} - (\mathbf{n}.\mathbf{V})I_4$ (you are *not* required to prove this result!).

Use this formula to find the matrix M corresponding to the *projection* from $P \equiv (3, 3, 3) \in \mathbf{R}^3$ onto the plane Π with equation $x+y+z-1 = 0$. Calculate the *images* A', B', C' under this projection of the vertices of the right-angled triangle ABC in $\mathbf{R}^3$, where $A \equiv (2, 2, 2)$, $B \equiv (1, 2, 2)$ and $C \equiv (2, 1, 2)$, and find the *angle* between A'B' and A'C'. **[8 marks]**

Suggest a suitable *transformation* from Π to a screen with 400×300 pixels which displays A'B'C' near the **centre** of the screen. **[7 marks]**

(4) (a) The *Bernstein polynomials* are defined by
$B_{i,n}(t) = (^n_i)(1-t)^{n-i}t^i$ for $0 \le i \le n$ and $0 \le t \le 1$; and $B_{i,n}(t) = 0$ otherwise.

Prove the following:

(i) $\Sigma_{i=0}^{n} B_{i,n}(t) = 1$,

(ii) $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$. **[8 marks]**

(b) A *quartic* Bezier curve B = B(t) has control points
$\mathbf{b}_0 \equiv (0, 2)$, $\mathbf{b}_1 \equiv (4, 2)$, $\mathbf{b}_2 \equiv (6, 6)$, $\mathbf{b}_3 \equiv (8, 2)$, $\mathbf{b}_4 \equiv (12, 2)$.

Evaluate the point B(½) in two ways:

(i) Substituting t = ½ into the *defining equation* of the Bezier curve,

(ii) Applying the *de Casteljau* algorithm.

Make a **sketch** illustrating the points derived when applying the de Casteljau algorithm, including your *best attempt* at B(t).

If the curve B is **subdivided** at t = ½ into two quartic segments B_{left} and B_{right}, what are the *control points* for these two segments? **[12 marks]**

(5) (a) Let $T = [t_0, t_1, ..., t_m]$ be a knot vector with $t_0 < t_1 < .. < t_m$.
The B-spline *basis functions* $N_{i,d}$ of degree d determined by T are defined recursively by

$N_{i,0}(t) = 1$ when $t_i \le t < t_{i+1}$, and $N_{i,0}(t) = 0$ otherwise;
$N_{i,j}(t) = ((t-t_i)/(t_{i+j}-t_i))N_{i,j-1}(t) + ((t_{i+j+1}-t)/(t_{i+j+1}-t_{i+1}))N_{i+1,n-1}(t)$
for all $0 \le j \le d$ and $0 \le i \le m-1-j$.

Prove, for j > 0, that $N_{i,j}(t) = 0$ for all $t \in (-\infty, t_i] \cup [t_{i+j+1}, \infty)$. **[10 marks]**

(b) Let d = 2 and let T = [0, 1, 2, 3, 4, 5, 6, 7].
Evaluate the basis function $N_{0,2}(t)$ and verify that it is *differentiable* at t = 1 and at t = 2. Sketch the graph of $N_{0,2}(t)$. **[10 marks]**