

# G2M16 Algorithmics: Exam Paper, May 2001

## SECTION 1 (Compulsory)

- (1) (a) State Turing's Thesis. State without proof the relationship between Turing machines and **while**-programs, and between **while**-programs and a real programming language (such as C). **[4 marks]**
- (b) Let  $f(x)$  and  $g(x)$  be real-valued functions on  $\mathbb{R}$ . Define what is meant by:  $f(x) = \Theta(g(x))$ . Show that  $x^2 + 3x + 1 = \Theta(x^2)$ . **[4 marks]**
- (c) What does it mean to say that a problem is *tractable*? Program 1 has complexity  $n^4$  and Program 2 has complexity  $10^n$ . The largest problem which can be solved on a current computer in 1 hour by program 1 is  $n_1$ , and the largest problem which can be solved on a current computer in 1 hour by program 2 is  $n_2$ . When each of these programs is run on a computer 100 times faster, calculate what happens to the size of the largest 1-hour problem. **[7 marks]**
- (d) What is a *decision problem*? Describe the 'Hamiltonian cycle problem' and how it gives rise to a language. **[5 marks]**

## SECTION 2 (Answer 2 out of 4 questions)

- (2) In this question, you may use any standard macros discussed in the lectures.
- (a) Taking  $0 = F$  and  $1 = T$ , write macros in the language of **while**-programs to compute:  $Z := \neg Y$ ,  $Z := X \wedge Y$ , and  $Z := X \vee Y$ . where  $X$  and  $Y$  are assumed to have the values 0 or 1. In each case, you should show that your answer works. **[8 marks]**
- (b) Write a **while**-program to implement the following function and show that your program works:
- $$Bigger(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x > y \\ 2 & \text{if } x < y \end{cases}$$

**[7 marks]**

(3) Define what is meant by a *bit operation*; draw up an appropriate truth-table. **[5 marks]**

Determine an upper bound for the number of bit operations needed to multiply together two *k-bit* numbers using the standard multiplication algorithm. **[10 marks]**

(4) Let  $\phi = z_1 \vee \dots \vee z_k$  be a formula consisting of a disjunction of literals  $z_i$ . Show that there is a formula  $\phi'$  in 3-conjunctive normal form such that  $\phi$  is satisfiable if and only if  $\phi'$  is satisfiable. **[15 marks]**

(5) (a) Define the classes of languages P and NP. **[2 marks]**

(b) Define what it means for  $L_1$  to be *polynomial-time reducible* to  $L_2$ , denoted  $L_1 \leq_P L_2$ . Prove that if  $L_1 \leq_P L_2$ , and  $L_2 \leq_P L_3$ , then  $L_1 \leq_P L_3$ . **[7 marks]**

(c) Define what it means for a language to be *NP-complete*. Prove that if any one NP-complete language is in P then  $P = NP$ . **[6 marks]**

(Questions done: 1, 2, 4)

# G3M07 Combinatorial Group Theory: Exam Paper, May 2001

## **Answer 3 questions out of 5 (Questions Done: 2, 3, 5)**

- (1) Define what it means for a subgroup  $N$  of a group  $G$  to be normal. **[2 marks]**

A group  $G$  is given by the permutation:

$$G := \langle x, y \mid x^6 = e, yxy^{-1} = x^4, y^3 = e \rangle.$$

Prove that:

- (i) Any element of  $G$  can be written in at least one way in the form  $x^i y^j$  with  $0 \leq i \leq 5$  and  $0 \leq j \leq 2$ . **[8 marks]**
- (ii) The subgroup  $N = \langle x \rangle$ , generated by the element  $x$ , is a *normal* subgroup of  $G$ . **[7 marks]**
- (iii) The quotient group  $G/N$  is cyclic of order 3. **[3 marks]**
- (2) Define the four Tietze transformations on a group presentation  $\langle X \mid R \rangle$ , where  $X$  is the set of generators and  $R$  the set of relators. **[4 marks]**

Show, using Tietze transformations that the group with the presentation:

$$\langle a, b, c \mid a(bc^{-1})b, (cb^{-1})^2a \rangle$$

is cyclic, generated by  $x = bc^{-1}$ .

**[14 marks]**

Find the order of  $x$  in  $G$ .

**[2 marks]**

- (3) Using the Todd-Coxeter heuristic relative to the subgroup  $H = \langle a \rangle$ , find the order of the group with presentation:
- $$G := \langle a, b \mid a^4 = e, b^3 = e, ab^2a = b \rangle.$$
- [10 marks]**

Find the order of the subgroup  $\langle a^2b \rangle$ .

**[5 marks]**

Show that the subgroup  $\langle a^2, aba^{-1} \rangle$  contains the element  $b$  and hence show that this subgroup is normal. **[5 marks]**

- (4) (a) Suppose that  $\alpha: G \rightarrow G$  is a homomorphism such that for all  $g \in G$ ,  $\alpha(g) = g^2$ . Prove that  $G$  is abelian. **[5 marks]**
- (b) Let  $x$  be an element of order  $n$  in a group  $G$ . Suppose  $r$  is a natural number *coprime* to  $n$ . Prove that the  $r^{\text{th}}$  power,  $x^r$ , of  $x$  also has order  $n$ . **[9 marks]**
- (c) Let  $H$  be a subgroup of a group  $G$  and let  $N_G(H) = \{n \in G: n^{-1}Hn = H\}$ . Prove that  $N_G(H)$  is a subgroup of  $G$  and that  $H$  is a normal subgroup of  $N_G(H)$ . **[6 marks]**

- (5) Let  $G$  be any group. Define what it means for two elements  $g$  and  $g'$  of  $G$  to be *conjugate*. **[2 marks]**

Prove that *conjugacy* is an equivalence relation. **[7 marks]**

Now consider the dihedral group,  $D_n$ , which has presentation:

$$\langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1 \rangle.$$

This has  $2n$  elements all of which can be written in the form  $x^i y^j$  with  $i = 0, 1, \dots, n-1$  and  $j = 0, 1$ . Find the conjugacy classes of  $D_n$ . **[11 marks]**

## G3M80 Probability: Exam Paper, May 2001

### Answer 3 questions out of 5 (Questions Done: 1, 2, 4, 5)

- (1) (a) Define the upper limit  $\lim_{n \rightarrow \infty} \sup A_n$  and the lower limit  $\lim_{n \rightarrow \infty} \inf A_n$  of a sequence of sets. **[4 marks]**
- (b) Define a convergent sequence  $\{A_n\}$  in terms of its upper and lower limits. **[1 mark]**
- (c) Prove that  $\lim_{n \rightarrow \infty} \inf A_n \subset \lim_{n \rightarrow \infty} \sup A_n$ . **[5 marks]**
- (d) Find the upper and lower limits of the following sequences of events on the sample space  $\Omega$  and determine the convergence or otherwise of each sequence.
- (i)  $\{A_n\}$  where  $A_n = (1/n - 1, 1/n)$  for  $n = 1, 2, 3, \dots$
- (ii)  $\{A_n\}$  where  $A_n = [0, 1]$  for  $n$  even,  
 $A_n = \{0\}$  for  $n$  odd. **[10 marks]**
- (2) (a) Explain carefully what is meant by a Probability Space  $(S, F, P)$  and state its axioms. **[5 marks]**
- (b) Define a monotonic sequence of events  $\{A_n\}$  in the sample space  $S$  and state, without proof, the limit of a monotonic sequence distinguishing between the cases when the sequence is (a) increasing and (b) decreasing. **[5 marks]**
- (c) If  $\{A_n\}$  is a monotonically increasing sequence of events then prove that
- $$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right),$$
- indicating which axioms of the probability measure  $P$  you are using. **[10 marks]**
- (3) (a) Let  $X$  be a continuous random variable with probability density function  $f(x)$  and let  $y = g(x)$  be a strictly monotonic function of  $x$  such that  $dg/dx$  is continuous and non-zero in some open interval  $A$ . If  $g^{-1}(y)$  is the unique inverse of  $g(x)$  and  $B$  is the image of  $A$  in the range of  $Y$ , show that the random variable  $Y = g(X)$  is continuous and has probability density function  $h(y)$  for all  $y \in B$  given by
- $$h(y) = f(g^{-1}(y)) \left| \frac{d(g^{-1}(y))}{dy} \right|. \quad \text{[10 marks]}$$
- (b) A continuous random variable  $X$  has the probability density function
- $$f(x) = \begin{cases} kx^2 e^{-x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$
- Find the value of  $k$ . **[3 marks]**

(c) Find the probability density function  $h(y)$  of  $Y = X^2$ . **[5 marks]**

(d) Show that  $h(y) = \Gamma(\theta, \alpha; y)$  and determine the values of the parameters  $\theta, \alpha$  of the Gamma distribution. **[2 marks]**

(You may assume without proof that  $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ ).

(4) (a) A 2-dimensional continuous random variable  $(X, Y)$  has pdf  $f(x, y)$ . Define the marginal pdf's  $f_X(x), f_Y(y)$ . Give a necessary and sufficient condition in terms of  $f(x, y), f_X(x), f_Y(y)$  for  $X, Y$  to be independent. **[4 marks]**

(b) If  $(X, Y)$  has the bivariate normal distribution with pdf

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} \exp\left\{\frac{-1}{2(1-\rho_{XY}^2)}(x^2 - 2\rho_{XY}xy + y^2)\right\},$$

show that  $X, Y$  each has univariate normal distribution and that if  $\rho_{XY} = 0$  then  $X, Y$  are independent. **[16 marks]**

(You may assume without proof that  $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ ).

(5) (a) Define the moment generating function  $M_X(t)$  of a one-dimensional random variable  $X$ . **[2 marks]**

(b) Show that  $M_Z(t) = \exp(t^2/2)$  where  $Z$  denotes the standard normal random variable. **[4 marks]**

(c) Let  $X_1, X_2, \dots, X_n$  be independent random variables each having the same distribution with mean  $\mu$  and variance  $\sigma^2$ . If  $Z_n = \frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}$ , where  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ , show that  $M_{Z_n}(t) \rightarrow \exp(t^2/2)$  as  $n \rightarrow \infty$ . Deduce that (Central Limit Theorem)  $Z_n \rightarrow Z$  as  $n \rightarrow \infty$ . State without proof any theorems you use. **[14 marks]**

## G3M85 Operational Research 2: Exam Paper, May 2001

### Answer 3 questions out of 5 (Questions Done: 2, 3, 4)

(1) The LP-relaxation of the pure **IP** problem

$$P: \max(M = 5x_1 + 6x_2 + 7x_3 + 8x_4)$$

$$\text{such that } 3x_1 + 2x_2 + 4x_3 + 4x_4 \leq 52$$

$$4x_1 + 3x_2 + 2x_3 + 3x_4 \leq 47$$

$$6x_1 + 5x_2 + 4x_3 + 3x_4 \leq 59$$

$$x_1, x_2, x_3, x_4 \in \mathbf{Z}^{\geq 0}.$$

has solution tableau

M	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	s <sub>1</sub>	s <sub>2</sub>	s <sub>3</sub>	rhs	basic
10	33				11	11	1	1,148	M = 114.8
	3			10	1	6	-4	98	x <sub>4</sub> = 9.8
	11	10			-3	2	2	56	x <sub>2</sub> = 5.6
	-1		10		3	-7	3	4	x <sub>3</sub> = 0.4

(a) Apply the cutting plane algorithm to the second constraint (x<sub>2</sub>-row) in the solution tableau and solve the resulting LP-relaxation using the dual simplex method.

**[8 marks]**

(b) Let P<sub>1,3</sub> be the subproblem of P obtained by setting x<sub>1</sub> = x<sub>3</sub> = 0. Solve the LP-relaxation of P<sub>1,3</sub> graphically. Solve P<sub>1,3</sub> graphically.

**[10 marks]**

(c) State the best upper and lower bounds for the solution of P currently available.

**[2 marks]**

(2) (a) Let F1 be the linear fractional program

$$\max\left(\frac{p \cdot x + \alpha}{q \cdot x + \beta}\right) \text{ s.t. } \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{R}^m$ ,  $\alpha, \beta \in \mathbb{R}$ , A is an  $m \times n$  matrix, and  $\mathbf{q} \cdot \mathbf{x} + \beta \neq 0$  for all  $\mathbf{x}$  in the feasible region S.

(i) Explain why we may assume that  $\mathbf{q} \cdot \mathbf{x} + \beta > 0$  for all  $\mathbf{x} \in S$ .

(ii) Show that the transformation  $\mathbf{y} = \mathbf{zx}$ ,  $z^{-1} = \mathbf{q} \cdot \mathbf{x} + \beta$  converts F1 into a formally equivalent LP, F2:  $\max(\mathbf{p} \cdot \mathbf{y} + \alpha z)$  s.t.  $\mathbf{Ay} - \mathbf{zb} \leq \mathbf{0}$ ,  $\mathbf{q} \cdot \mathbf{y} + \beta z = 1$ ,

$$(\mathbf{y}, z) \geq \mathbf{0}.$$

**[10 marks]**

(b) Solve the linear fractional problem

$$F: \text{opt} \left( \frac{10 - x - 2y}{25 - 2x + y} \right) \text{ such that } \begin{cases} 3 \leq x + y \leq 9, \\ -5 \leq x - y \leq 7, \\ x, y \geq 0, \end{cases}$$

in the cases (i) **opt = min**; (ii) **opt = max**.

**[10 marks]**

(3) (a) Let  $P: \max(\mathbf{c} \cdot \mathbf{x}), \mathbf{A}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x}, \mathbf{s} \geq \mathbf{0}$  be a standard max problem, and let  $D(P): \min(\mathbf{b} \cdot \mathbf{u}), \mathbf{A}'\mathbf{u} - \mathbf{e} = \mathbf{c}, \mathbf{u}, \mathbf{e} \geq \mathbf{0}$  be the dual of  $P$ . Prove the following:

- (i) If  $\mathbf{x}_0, \mathbf{u}_0$  are feasible for  $P, D(P)$  respectively, then  $\mathbf{c} \cdot \mathbf{x}_0 \leq \mathbf{b} \cdot \mathbf{u}_0$ . [3 marks]
- (ii) If  $\mathbf{c} \cdot \mathbf{x}_0 = \mathbf{b} \cdot \mathbf{u}_0$ , then  $(\mathbf{x}_0, \mathbf{s}_0)$  and  $(\mathbf{u}_0, \mathbf{e}_0)$  are complementary slack. [3 marks]
- (iii) If  $P$  and  $D(P)$  are both feasible, then both have optimal solutions. [4 marks]

(b) Let  $P$  be the standard max LP problem

$$\begin{aligned}
 P: \max (M = 4x_1 + 3x_2 + 5x_3 + 6x_4) \\
 \text{such that } \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + s_1 = 9, \\
 & 2x_1 + 3x_2 + 3x_3 + 2x_4 + s_2 = 10, \\
 & 2x_1 + x_2 + 5x_3 + 3x_4 + s_3 = 8, \\
 & 3x_1 + 2x_2 + 5x_3 + 4x_4 + s_4 = 11, \\
 & \mathbf{x}, \mathbf{s} \geq \mathbf{0}.
 \end{aligned}$$

Write down the dual problem  $D(P)$ .

Test the following proposed vectors for optimality:

$$\text{(i) } \mathbf{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{(ii) } \mathbf{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{(iii) } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}. \quad [10 \text{ marks}]$$

(4) (a) A traveller wishes to buy some books to take on a journey. The estimated time taken to read each of the 6 books, and the purchase price of each book, are shown in the following table:

book	A	B	C	D	E	F
reading time (hours)	15	13	9	8	7	6
purchase price (£)	12	10	7	6	6	5

Which of these books should the traveller buy so as to give the maximum amount of reading time without spending more than £21? Use a branch and bound method to solve this problem, calculating the LP-relaxation solution at each stage. Show that your answer is unique. [12 marks]

(b) Jane estimates her chances of obtaining grades {A, B, C, D} as {20%, 50%, 20%, 10%} in module M.

Jane is indifferent between:  $L_1$ : with probability 1.0 she gains grade B;  
 and:  $L_2$ : with probability 0.4 she gains grade A,  
 with probability 0.6 she gains grade C;  
 and also between:  $L_3$ : with probability 1.0 she gains grade C;  
 and:  $L_4$ : with probability 0.5 she gains grade B,  
 with probability 0.5 she gains grade D.

Calculate the expected utility for module M.

[8 marks]

- (5) A rail traveller is planning a journey from Bangor to Cardiff with *either* a connection at Birmingham *or* a connection at Crewe (for the Shrewsbury and Hereford line). The traveller is considering the following trains:

Train No.	1	2	3	4	5	6
Bangor	09:20			09:50		
Crewe	↓			11:25	11:30	12:30
Birmingham	11:50	12:05	13:05		↓	↓
Cardiff		13:50	14:50		14:00	15:00

In each case, when a connection is missed, there is an alternative connection one hour later. Trains are assumed to arrive either *on time*, or *10 minutes late*, or *20 minutes late*, as detailed in the following table of probabilities:

train	arrival at	on time	10 min late	20 min late
1, 4	Bangor	1.0	0.0	0.0
1	Birmingham	0.2	0.4	0.4
2, 3	Birmingham	0.3	0.6	0.1
2, 3	Cardiff	0.2	0.5	0.3
4	Crewe	0.2	0.6	0.2
5, 6	Crewe	0.4	0.5	0.1
5, 6	Cardiff	0.1	0.6	0.3

Show that, when train 1 is taken, the traveller makes the connection at Birmingham with probability 0.88. What is the probability that, when train 4 is taken, the connection is made at Crewe? **[6 marks]**

Decide which train the traveller should catch from Bangor so as to minimise the expected journey time. **[10 marks]**

Train 1 is also scheduled to stop at Crewe at 10:50, allowing a connection with train 5. Discuss whether this extra possibility changes your decision. **[4 marks]**

## G3M86 Heuristic Optimisation Algorithms: QUIZ 1 (with answers)

Q: You have to design a *timetable* for 3 modules A,B,C. A & B are given **twice** a week, module C **3 times** a week. There is one *classroom* and 2 *timeslots* per day. Consider 2 representations for this problem. R1 is a **10-dimensional vector** corresponding to the 10 slots in one week. R2 is a **7-dimensional vector** corresponding to the time slots for one week, needed to cover *all the three modules*.

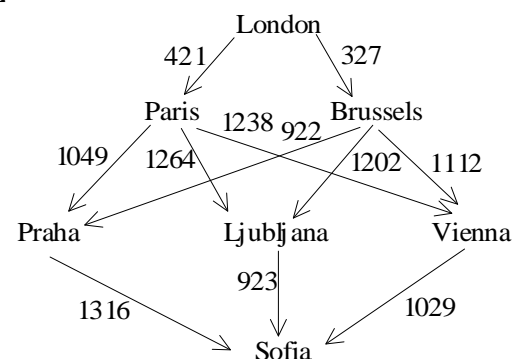
(a) What values can the vector components take in representation R1? Give an **example** of a feasible solution. A: Any *numerical* representation with 4 values e.g. 0,1,2,3. A feasible solution is  $S = [1 \ 2 \ 1 \ 2 \ 3 \ 0 \ 3 \ 0 \ 3 \ 0]^T$ . (b) What is the *cardinality of the search space* for representation R1? A:  $4^{10} = 1048576$ . (c) What values can the *vector components* take for R2? Given an **example** of a feasible solution. A: 1 to 10 for each of the *possible time slots* (no overlapping) e.g.  $S = [1 \ 3 \ 2 \ 4 \ 5 \ 7 \ 9]^T$ , the *same* solution as above.

(d) What is the *cardinality of the search space* of the feasible solutions with R2? A: All permutations of 7 out of 10 time slots where  $A_1$  and  $A_2$  are **indistinguishable**, so are  $B_1$  and  $B_2$ , and  $C_1, C_2, C_3$ . So  $|F| = \binom{10}{7} \cdot 7! / 2!2!3! = 25200$ . (e) If module C is given on **Mondays, Wednesdays** and **Fridays**, what is the cardinality of the search space of the *feasible solutions* with R2? A: There are  $2 \times 2 \times 2$  **choices** for  $C_1, C_2, C_3$ . **Therefore**,  $|F_{\text{restricted}}| = \binom{7}{4} \cdot 4! / 2!2! \cdot 2 \cdot 2 \cdot 2 = 1680$ .

(2) (Answers) Given the *starting permutation* T, the 2-opt algorithm for the TSP is **derivative free** (not derivative based); **numerical** (not analytical) and **deterministic** (not random). (b) The *Newton method* for finding the zeros of a function is **derivative based** (not derivative free), **numerical** (not analytical) and **fixed-point** (not bracketing). (c) The method called *Regula Falsi* is **derivative free** (not derivative based), **numerical** (not analytical), **deterministic** (not random) and **bracketing** (not fixed-point).

(d) What will happen to the *bisection algorithm* (the way it was given to you as a pseudo-code) if there are **2** zeros of the function  $f(x)$  in the *interval*  $[a,b]$ . A: It will always converge to *one of the zeros* (could be either one of them) *or b*.

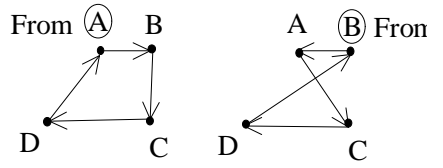
(3) Shown is some routes from **London** to **Sofia**. The distances in km are marked. (a) Find a solution of the shortest route problem *applying a greedy algorithm*. A: London - Brussels - Praha - Sofia ( $327+922+1316$ ) = 2565km. (b) Use **dynamic programming** to find the shortest route. A:  $f_4^*(\text{Sofia}) = 0$ .  $f_3^*(\text{Praha}) = f_3(\text{Praha}, \text{Sofia}) = 1316$   $f_3^*(\text{Ljubljana}) = f_3(\text{Ljubljana}, \text{Sofia}) = 923$ .  $f_3^*(\text{Vienna}) = f_3(\text{Vienna}, \text{Sofia}) = 1029$ .



$f_2(\text{Paris}, \text{Praha}) = 1049+316 = 2365$ .  $f_2(\text{Paris}, \text{Ljubljana}) = 1264+923 = 2187$ .  $f_2(\text{Paris}, \text{Vienna}) = 1238+1029 = 2267$ . So  $f_2^*(\text{Paris}) = 2187$  (*Ljubljana*).

$f_2(\text{Brussels, Praha}) = 922+1316 = 2238$ .  $f_2(\text{Brussels, Ljubljana}) = 1202+923 = 2125$ .  $f_2(\text{Brussels, Vienna}) = 1112+1029 = 2141$ . So  $f_2^*(\text{Brussels}) = 2125$  (*Ljubljana*).  $f_1(\text{London, Paris}) = 421+2187 = 2606$ .  $f_1(\text{London, Brussels}) = 327+2125 = 2452$ . So  $f_1^*(\text{London}) = 2452$ , the *optimal* cost. The **solution** is London - Brussels - Ljubljana - Sofia.

(4) Consider the **TSP** problem. Let A & B be the 2 *closest towns in the set*. If we start a tour from A and **another** tour from E using in both cases the *nearest-neighbour* greedy algorithm, will we **always** obtain the same tours (the direction does not matter)?



A: No, we cannot always *expect* the same tour. Example as shown, where  $A-B-C-D \neq A-C-D-B$ .

```

begin
  c ← 0.001
  a ← -4
  b ← -3
  while (b-a) < c, do
    begin
      t2 ← 8*sin(2*b-1)+b^3
      t1 ← 8*sin(2*a-1)+a^3
      b ← b-t2*(b-a)/(t2-t1)
      a ← b
    end
  end and return b

```

must be  $\geq c$

$e \leftarrow b$

(5) The **pseudo-code** shown implements one of the *methods* that you have studied in the course. (a) Which method is this and what is it **used** for? A: The **secant** method. Used to find a *zero of a function*. In this example it starts off with  $x_0 = -4$ ,  $x_1 = -3$ , and finds a *zero* of  $f(x) = 8\sin(2x-1)+x^3$ .

(b) There are 2 *mistakes in the pseudo-code*. Find them and fix the code as it should be (**mark** the corrections on the code). Explain what the mistakes were. A: **Mistake 1 (RED)**. The condition of the “while” loop is incorrect. It should be  $\geq \epsilon$ , otherwise we will *never enter the loop*. **Mistake 2 (BLUE)** The value of b has to be assigned to *another variable* (not a) before being changed. Then a gets the value that has been **stored**. (b) Explain what would happen if the *code was run as is*. A: The program will not execute the “while” loop, and will **return**  $b = -3$ .

(d) Explain what would happen if you *fix each of the mistakes* but leave the **other** one in. A: **Fix Mistake 1**. The program will enter the “while” loop, execute it once, and get  $a=b$  at the end. Since  $(b-a) < c$ , the loop will **not** be executed again and the program will terminate giving the value of  $x_2$  ( $x_0 = -4$ ,  $x_1 = -3$ ). **Fix Mistake 2**. The program will not execute the while loop and will *terminate returning*  $b = -3$

## G3M86 Heuristic Optimisation Algorithms: QUIZ 2 (with answers)

(1) [5 0 2 4 0 1 3 0] is a *tabu memory vector*. What can you say about the **horizon**? It is 5 ( $h = 5$ ). The vector does **not** signify either the beginning or the end of the algorithm. Q: If [0 1 1 0 1 1 0 1] is the *current* solution, what was the *previous* solution? A: [1 1 1 0 1 1 0 1] (bit 1 changed).

(2) You are optimising a function of **6** binary variables. Which of these are *legitimate* tabu memory vectors? [1 2 3 4 5 0], [0 1 0 0 0 0], [0 1 0 0 1 0], [5 2 3 4 0 0]. A: All but the **third** vector.

(3) Let  $\underline{x} \in \{0,1\}^4$ . The function to *maximise* is  $(y-3)^2$  where  $y$  is the **decimal** number corresponding to the binary string (For example, 0101 is  $1 \times 2^2 + 1 \times 2^0 = 5$ ). (a) Start with solution  $\underline{x}_0 = [0 0 0 0]$  and perform *5 steps of a tabu search* with horizon  $h = 2$  (find  $\underline{x}_5$ ). Start checking the bits from *left to right*. If there are **more** than one best solutions, move to the **first** one found. Give your *final* result.

A:  $\underline{x}_0 = [0 0 0 0]$ , TM = [0 0 0 0]. **Possible changes:** bits 1,2,3,4. Bit 1 gives [1 0 0 0],  $\underline{x} = 8$ ,  $f = 25$ . Bit 2 gives [0 1 0 0],  $\underline{x} = 4$ ,  $f = 1$ . Bit 3 gives [0 0 1 0],  $\underline{x} = 2$ ,  $f = 1$ . Bit 4 gives [0 0 0 1],  $\underline{x} = 1$ ,  $f = 4$ . **Choose** [1 0 0 0]. So  $\underline{x}_1 = [1 0 0 0]$ , TM = [2 0 0 0]. **Flipping** bit 2 gives [1 1 0 0],  $\underline{x} = 12$ ,  $f = 81$ . Bit 3 gives [1 0 1 0],  $\underline{x} = 10$ ,  $f = 49$ . Bit 4 gives [1 0 0 1],  $\underline{x} = 9$ ,  $f = 36$ . So **choose** [1 1 0 0].

$\underline{x}_2 = [1 1 0 0]$ , TM = [1 2 0 0]. Bit 3 gives [1 1 1 0],  $\underline{x} = 14$ ,  $f = 121$ . Bit 4 gives [1 1 0 1],  $\underline{x} = 13$ ,  $f = 100$ . **Choose** [1 1 1 0]. So  $\underline{x}_3 = [1 1 1 0]$ , TM = [0 1 2 0]. Bit 1 gives [0 1 1 0],  $\underline{x} = 6$ ,  $f = 9$ . Bit 4 gives [1 1 1 1],  $\underline{x} = 15$ ,  $f = 144$ . **Choose** [1 1 1 1]. So  $\underline{x}_4 = [1 1 1 1]$ , TM = [0 0 1 2]. Bit 1 gives [0 1 1 1],  $\underline{x} = 7$ ,  $f = 16$ . Bit 2 gives [1 0 1 1],  $\underline{x} = 11$ ,  $f = 64$ . **Choose** [1 0 1 1]. So  $\underline{x}_5 = [1 0 1 1]$ , TM = [0 2 0 1]. **Final** result:  $f = 144$  at  $\underline{x}_4 = [1 1 1 1]$ .

(b) What would the results be if you applied **hill-climbing** from the *same* initial solution  $\underline{x}_0$ ? A:  $\underline{x} = [1 1 1 1]$ ,  $f = 144$ .

(4) Let  $y \in \{-1, 0, 1, 2\}$ . Assume you are **minimising**  $f(y) = |y|$  using the SA algorithm. (a) Find the *equilibrium probability distribution* at  $c=5$ . A: As seen in the table on the **right**. (b) Show the *equilibrium probability distributions* for  $c \rightarrow 0$  and for  $c \rightarrow \infty$ . A: **When**  $y$  is -1, 0, 1, 2;  $q_0$  is 0, 1, 0, 0 and  $q_\infty$  is  $\frac{1}{4}$ ,  $\frac{1}{4}$ ,  $\frac{1}{4}$ ,  $\frac{1}{4}$ . (c) Find the **expected value** of the function  $f(y)$  at  $c=5$ . A:  $0.248 + 0.248 + (2 \times 0.202) = 0.90$ . (d) What is  $S_{\text{opt}}$ , the set of *optimal solutions* (states), for this problem? A:  $S_{\text{opt}} = \{0\}$ .

$y$	-1	0	1	2	
$f(y)$	1	0	1	2	
$-f(y)/c$	-0.2	0	-0.2	-0.4	
$\exp(\dots)$	0.819	1	0.819	0.670	$\Sigma = 3.308$
$q_5(y)$	0.248	0.302	0.248	0.202	

(5) Consider the *simulated annealing algorithm*. Knowing that for  $a \leq 0$ ,  $\lim_{x \rightarrow 0 (x > 0)} \exp(a/x) = 1$ , if  $a=0$ ; and 0 otherwise, prove that  $\lim_{c \rightarrow \infty} \langle f \rangle_c = 1/|S| \sum_{i \in S} f(i)$ , where  $S$  is the *space of possible states*  $i$  of the system,  $|S|$  is the **cardinality** of  $S$ , and  $\langle f \rangle_c$  is the *expected value* of  $c$  of the function being **minimised**. A:  $\lim_{c \rightarrow \infty} \langle f \rangle_c = \lim_{c \rightarrow \infty} \sum_{i \in S} f(i) \cdot q_c(i) = \lim_{c \rightarrow \infty} \sum_{i \in S} f(i) \cdot [\exp(-f(i)/c) / \sum_j \exp(-f(j)/c)] = \sum_{i \in S} f(i) \cdot [\lim_{c \rightarrow \infty} \exp(-f(i)/c)] / [\sum_j \lim_{c \rightarrow \infty} \exp(-f(j)/c)] = \sum_{i \in S} f(i) [1 / \sum_j 1] = 1/|S| \sum_{i \in S} f(i)$ . **QED.**

(6) Mark the following *statements* about SA as true or false.  $\partial / \partial c \langle f \rangle_c > 0$  (**True**). For any  $i \in S_{opt}$ ,  $\lim_{c \rightarrow 0} q_c(i) = 1/|S_{opt}|$  (**True**). For any  $i \in S_{opt}$ ,  $\lim_{c \rightarrow \infty} q_c(i) = 1/|S|$  (**True**). The *entropy* at equilibrium is a natural measure of **disorder** in the system (**True**). The *entropy* at equilibrium is a **decreasing function** of the **cooling** parameter  $c$  (**True**).

(7) Let  $\underline{x}_1 = [0 \ 1 \ 1 \ 0]$  and  $\underline{x}_2 = [1 \ 0 \ 1 \ 0]$  be *chromosomes* selected as **parents** in a genetic algorithm. The *fitness function* (to maximise) is the number of **ones** in the chromosome. (a) List all **common schemata** for the two parents. A: # # # #, # # # 0, # # 1 #, # # 1 0. (b) What is the *probability* that, after a one-point crossover and before mutation, there will be an offspring *better* than both parents.

A: We have  $0^1 1^2 1^3 0$  and  $1^1 0^2 1^3 0$ . So with *point 1* we have 4/1, with *point 2* we have 2/2 and with *point 3* we have 2/2. So there is a 1/3 chance for a **better** offspring. (c) What is the probability that the *two children* in the offspring set will **not** share the highest order schema common for  $\underline{x}_1$  and  $\underline{x}_2$  after **mutation** with **probability** 0.1?

A:  $P(2 \text{ children sharing } \# \# 1 0) = P(\text{1st child not mutating in the last 2 bits}) \times P(\text{2nd child not mutating in the last 2 bits}) = 0.9 \times 0.9 \times 0.9 \times 0.9$ . So  $P(2 \text{ children not sharing } \# \# 1 0) = 1 - 0.9^4 = 1 - 0.6561 = 0.3439$ .

(8) The *population set* in a GA has **fitness function** as follows: when  $i = 1, 2, 3, 4, 5$ ,  $f(\underline{x}_i)$  is 10, 20, 30, 40, 50. You are going to apply the *roulette wheel principle* to select a **mating set** of 5. What is the probability that the set will consist of *five different individuals*? A:  $P(\text{selecting 1}) = 10/10+...+50 = 10/150 = 1/15$ .  $P(\text{selecting 2}) = 2/15$ . ....  $P(\text{selecting 5}) = 5/15$ . So  $P(\text{selecting 12345 in this order}) = 1 \times 2 \times 3 \times 4 \times 5 / (15)^5$ . There are *5! permutations*, therefore  $P(\text{selecting 5 different}) = 5! \times 5! / (15)^5 = 64/3375 \approx 0.019$ .

(9) Guess my *age* (any number between  $-\infty$  and  $+\infty$ ). Bonus mark for the **correct** figure! A: ???!