

G2M16 Algorithmics: Exam Paper, May 2001

SECTION 1 (Compulsory)

- (1) (a) State Turing's Thesis. State without proof the relationship between Turing machines and **while**-programs, and between **while**-programs and a real programming language (such as C). **[4 marks]**
- (b) Let $f(x)$ and $g(x)$ be real-valued functions on \mathbb{R} . Define what is meant by: $f(x) = \Theta(g(x))$. Show that $x^2 + 3x + 1 = \Theta(x^2)$. **[4 marks]**
- (c) What does it mean to say that a problem is *tractable*? Program 1 has complexity n^4 and Program 2 has complexity 10^n . The largest problem which can be solved on a current computer in 1 hour by program 1 is n_1 , and the largest problem which can be solved on a current computer in 1 hour by program 2 is n_2 . When each of these programs is run on a computer 100 times faster, calculate what happens to the size of the largest 1-hour problem. **[7 marks]**
- (d) What is a *decision problem*? Describe the 'Hamiltonian cycle problem' and how it gives rise to a language. **[5 marks]**

SECTION 2 (Answer 2 out of 4 questions)

- (2) In this question, you may use any standard macros discussed in the lectures.
- (a) Taking $0 = F$ and $1 = T$, write macros in the language of **while**-programs to compute: $Z := \neg Y$, $Z := X \wedge Y$, and $Z := X \vee Y$. where X and Y are assumed to have the values 0 or 1. In each case, you should show that your answer works. **[8 marks]**
- (b) Write a **while**-program to implement the following function and show that your program works:
- $$Bigger(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x > y \\ 2 & \text{if } x < y \end{cases}$$

[7 marks]

(3) Define what is meant by a *bit operation*; draw up an appropriate truth-table. **[5 marks]**

Determine an upper bound for the number of bit operations needed to multiply together two *k-bit* numbers using the standard multiplication algorithm. **[10 marks]**

(4) Let $\phi = z_1 \vee \dots \vee z_k$ be a formula consisting of a disjunction of literals z_i . Show that there is a formula ϕ' in 3-conjunctive normal form such that ϕ is satisfiable if and only if ϕ' is satisfiable. **[15 marks]**

(5) (a) Define the classes of languages P and NP. **[2 marks]**

(b) Define what it means for L_1 to be *polynomial-time reducible* to L_2 , denoted $L_1 \leq_P L_2$. Prove that if $L_1 \leq_P L_2$, and $L_2 \leq_P L_3$, then $L_1 \leq_P L_3$. **[7 marks]**

(c) Define what it means for a language to be *NP-complete*. Prove that if any one NP-complete language is in P then $P = NP$. **[6 marks]**

(Questions done: 1, 2, 4)

G3M07 Combinatorial Group Theory: Exam Paper, May 2001

Answer 3 questions out of 5 (Questions Done: 2, 3, 5)

- (1) Define what it means for a subgroup N of a group G to be normal. **[2 marks]**

A group G is given by the permutation:

$$G := \langle x, y \mid x^6 = e, yxy^{-1} = x^4, y^3 = e \rangle.$$

Prove that:

- (i) Any element of G can be written in at least one way in the form $x^i y^j$ with $0 \leq i \leq 5$ and $0 \leq j \leq 2$. **[8 marks]**
- (ii) The subgroup $N = \langle x \rangle$, generated by the element x , is a *normal* subgroup of G . **[7 marks]**
- (iii) The quotient group G/N is cyclic of order 3. **[3 marks]**
- (2) Define the four Tietze transformations on a group presentation $\langle X \mid R \rangle$, where X is the set of generators and R the set of relators. **[4 marks]**

Show, using Tietze transformations that the group with the presentation:

$$\langle a, b, c \mid a(bc^{-1})b, (cb^{-1})^2a \rangle$$

is cyclic, generated by $x = bc^{-1}$.

[14 marks]

Find the order of x in G .

[2 marks]

- (3) Using the Todd-Coxeter heuristic relative to the subgroup $H = \langle a \rangle$, find the order of the group with presentation:
 $G := \langle a, b \mid a^4 = e, b^3 = e, ab^2a = b \rangle$. **[10 marks]**

Find the order of the subgroup $\langle a^2b \rangle$.

[5 marks]

Show that the subgroup $\langle a^2, aba^{-1} \rangle$ contains the element b and hence show that this subgroup is normal. **[5 marks]**

- (4) (a) Suppose that $\alpha: G \rightarrow G$ is a homomorphism such that for all $g \in G$, $\alpha(g) = g^2$. Prove that G is abelian. **[5 marks]**
- (b) Let x be an element of order n in a group G . Suppose r is a natural number *coprime* to n . Prove that the r^{th} power, x^r , of x also has order n . **[9 marks]**
- (c) Let H be a subgroup of a group G and let $N_G(H) = \{n \in G: n^{-1}Hn = H\}$. Prove that $N_G(H)$ is a subgroup of G and that H is a normal subgroup of $N_G(H)$. **[6 marks]**

- (5) Let G be any group. Define what it means for two elements g and g' of G to be *conjugate*. **[2 marks]**

Prove that *conjugacy* is an equivalence relation. **[7 marks]**

Now consider the dihedral group, D_n , which has presentation:

$$\langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1 \rangle.$$

This has $2n$ elements all of which can be written in the form $x^i y^j$ with $i = 0, 1, \dots, n-1$ and $j = 0, 1$. Find the conjugacy classes of D_n . **[11 marks]**

G3M80 Probability: Exam Paper, May 2001

Answer 3 questions out of 5 (Questions Done: 1, 2, 4, 5)

- (1) (a) Define the upper limit $\lim_{n \rightarrow \infty} \sup A_n$ and the lower limit $\lim_{n \rightarrow \infty} \inf A_n$ of a sequence of sets. **[4 marks]**
- (b) Define a convergent sequence $\{A_n\}$ in terms of its upper and lower limits. **[1 mark]**
- (c) Prove that $\lim_{n \rightarrow \infty} \inf A_n \subset \lim_{n \rightarrow \infty} \sup A_n$. **[5 marks]**
- (d) Find the upper and lower limits of the following sequences of events on the sample space Ω and determine the convergence or otherwise of each sequence.
- (i) $\{A_n\}$ where $A_n = (1/n - 1, 1/n)$ for $n = 1, 2, 3, \dots$
- (ii) $\{A_n\}$ where $A_n = [0, 1]$ for n even,
 $A_n = \{0\}$ for n odd. **[10 marks]**
- (2) (a) Explain carefully what is meant by a Probability Space (S, F, P) and state its axioms. **[5 marks]**
- (b) Define a monotonic sequence of events $\{A_n\}$ in the sample space S and state, without proof, the limit of a monotonic sequence distinguishing between the cases when the sequence is (a) increasing and (b) decreasing. **[5 marks]**
- (c) If $\{A_n\}$ is a monotonically increasing sequence of events then prove that
- $$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right),$$
- indicating which axioms of the probability measure P you are using. **[10 marks]**
- (3) (a) Let X be a continuous random variable with probability density function $f(x)$ and let $y = g(x)$ be a strictly monotonic function of x such that dg/dx is continuous and non-zero in some open interval A . If $g^{-1}(y)$ is the unique inverse of $g(x)$ and B is the image of A in the range of Y , show that the random variable $Y = g(X)$ is continuous and has probability density function $h(y)$ for all $y \in B$ given by
- $$h(y) = f(g^{-1}(y)) \left| \frac{d(g^{-1}(y))}{dy} \right|. \quad \text{[10 marks]}$$
- (b) A continuous random variable X has the probability density function
- $$f(x) = \begin{cases} kx^2 e^{-x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$
- Find the value of k . **[3 marks]**

(c) Find the probability density function $h(y)$ of $Y = X^2$. **[5 marks]**

(d) Show that $h(y) = \Gamma(\theta, \alpha; y)$ and determine the values of the parameters θ, α of the Gamma distribution. **[2 marks]**

(You may assume without proof that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$).

(4) (a) A 2-dimensional continuous random variable (X, Y) has pdf $f(x, y)$. Define the marginal pdf's $f_X(x), f_Y(y)$. Give a necessary and sufficient condition in terms of $f(x, y), f_X(x), f_Y(y)$ for X, Y to be independent. **[4 marks]**

(b) If (X, Y) has the bivariate normal distribution with pdf

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} \exp\left\{\frac{-1}{2(1-\rho_{XY}^2)}(x^2 - 2\rho_{XY}xy + y^2)\right\},$$

show that X, Y each has univariate normal distribution and that if $\rho_{XY} = 0$ then X, Y are independent. **[16 marks]**

(You may assume without proof that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$).

(5) (a) Define the moment generating function $M_X(t)$ of a one-dimensional random variable X . **[2 marks]**

(b) Show that $M_Z(t) = \exp(t^2/2)$ where Z denotes the standard normal random variable. **[4 marks]**

(c) Let X_1, X_2, \dots, X_n be independent random variables each having the same distribution with mean μ and variance σ^2 . If $Z_n = \frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}$, where $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, show that $M_{Z_n}(t) \rightarrow \exp(t^2/2)$ as $n \rightarrow \infty$. Deduce that (Central Limit Theorem) $Z_n \rightarrow Z$ as $n \rightarrow \infty$. State without proof any theorems you use. **[14 marks]**

G3M85 Operational Research 2: Exam Paper, May 2001

Answer 3 questions out of 5 (Questions Done: 2, 3, 4)

(1) The LP-relaxation of the pure **IP** problem

$$P: \max(M = 5x_1 + 6x_2 + 7x_3 + 8x_4)$$

$$\text{such that } 3x_1 + 2x_2 + 4x_3 + 4x_4 \leq 52$$

$$4x_1 + 3x_2 + 2x_3 + 3x_4 \leq 47$$

$$6x_1 + 5x_2 + 4x_3 + 3x_4 \leq 59$$

$$x_1, x_2, x_3, x_4 \in \mathbf{Z}^{\geq 0}.$$

has solution tableau

M	x ₁	x ₂	x ₃	x ₄	s ₁	s ₂	s ₃	rhs	basic
10	33				11	11	1	1,148	M = 114.8
	3			10	1	6	-4	98	x ₄ = 9.8
	11	10			-3	2	2	56	x ₂ = 5.6
	-1		10		3	-7	3	4	x ₃ = 0.4

(a) Apply the cutting plane algorithm to the second constraint (x₂-row) in the solution tableau and solve the resulting LP-relaxation using the dual simplex method.

[8 marks]

(b) Let P_{1,3} be the subproblem of P obtained by setting x₁ = x₃ = 0. Solve the LP-relaxation of P_{1,3} graphically. Solve P_{1,3} graphically.

[10 marks]

(c) State the best upper and lower bounds for the solution of P currently available.

[2 marks]

(2) (a) Let F1 be the linear fractional program

$$\max\left(\frac{p \cdot x + \alpha}{q \cdot x + \beta}\right) \text{ s.t. } \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\alpha, \beta \in \mathbb{R}$, A is an $m \times n$ matrix, and $\mathbf{q} \cdot \mathbf{x} + \beta \neq 0$ for all \mathbf{x} in the feasible region S.

(i) Explain why we may assume that $\mathbf{q} \cdot \mathbf{x} + \beta > 0$ for all $\mathbf{x} \in S$.

(ii) Show that the transformation $\mathbf{y} = \mathbf{zx}$, $z^{-1} = \mathbf{q} \cdot \mathbf{x} + \beta$ converts F1 into a formally equivalent LP, F2: $\max(\mathbf{p} \cdot \mathbf{y} + \alpha z)$ s.t. $\mathbf{Ay} - \mathbf{zb} \leq \mathbf{0}$, $\mathbf{q} \cdot \mathbf{y} + \beta z = 1$,

$$(\mathbf{y}, z) \geq \mathbf{0}.$$

[10 marks]

(b) Solve the linear fractional problem

$$F: \text{opt} \left(\frac{10 - x - 2y}{25 - 2x + y} \right) \text{ such that } \begin{cases} 3 \leq x + y \leq 9, \\ -5 \leq x - y \leq 7, \\ x, y \geq 0, \end{cases}$$

in the cases (i) **opt = min**; (ii) **opt = max**.

[10 marks]

(3) (a) Let $P: \max(\mathbf{c} \cdot \mathbf{x}), \mathbf{A}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x}, \mathbf{s} \geq \mathbf{0}$ be a standard max problem, and let $D(P): \min(\mathbf{b} \cdot \mathbf{u}), \mathbf{A}'\mathbf{u} - \mathbf{e} = \mathbf{c}, \mathbf{u}, \mathbf{e} \geq \mathbf{0}$ be the dual of P . Prove the following:

- (i) If $\mathbf{x}_0, \mathbf{u}_0$ are feasible for $P, D(P)$ respectively, then $\mathbf{c} \cdot \mathbf{x}_0 \leq \mathbf{b} \cdot \mathbf{u}_0$. [3 marks]
- (ii) If $\mathbf{c} \cdot \mathbf{x}_0 = \mathbf{b} \cdot \mathbf{u}_0$, then $(\mathbf{x}_0, \mathbf{s}_0)$ and $(\mathbf{u}_0, \mathbf{e}_0)$ are complementary slack. [3 marks]
- (iii) If P and $D(P)$ are both feasible, then both have optimal solutions. [4 marks]

(b) Let P be the standard max LP problem

$$\begin{aligned}
 P: \max (M = 4x_1 + 3x_2 + 5x_3 + 6x_4) \\
 \text{such that } \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + s_1 = 9, \\
 & 2x_1 + 3x_2 + 3x_3 + 2x_4 + s_2 = 10, \\
 & 2x_1 + x_2 + 5x_3 + 3x_4 + s_3 = 8, \\
 & 3x_1 + 2x_2 + 5x_3 + 4x_4 + s_4 = 11, \\
 & \mathbf{x}, \mathbf{s} \geq \mathbf{0}.
 \end{aligned}$$

Write down the dual problem $D(P)$.

Test the following proposed vectors for optimality:

$$\text{(i) } \mathbf{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{(ii) } \mathbf{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{(iii) } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}. \quad [10 \text{ marks}]$$

(4) (a) A traveller wishes to buy some books to take on a journey. The estimated time taken to read each of the 6 books, and the purchase price of each book, are shown in the following table:

book	A	B	C	D	E	F
reading time (hours)	15	13	9	8	7	6
purchase price (£)	12	10	7	6	6	5

Which of these books should the traveller buy so as to give the maximum amount of reading time without spending more than £21? Use a branch and bound method to solve this problem, calculating the LP-relaxation solution at each stage. Show that your answer is unique. [12 marks]

(b) Jane estimates her chances of obtaining grades {A, B, C, D} as {20%, 50%, 20%, 10%} in module M.

Jane is indifferent between: L_1 : with probability 1.0 she gains grade B;
 and: L_2 : with probability 0.4 she gains grade A,
 with probability 0.6 she gains grade C;
 and also between: L_3 : with probability 1.0 she gains grade C;
 and: L_4 : with probability 0.5 she gains grade B,
 with probability 0.5 she gains grade D.

Calculate the expected utility for module M. [8 marks]

- (5) A rail traveller is planning a journey from Bangor to Cardiff with *either* a connection at Birmingham *or* a connection at Crewe (for the Shrewsbury and Hereford line). The traveller is considering the following trains:

Train No.	1	2	3	4	5	6
Bangor	09:20			09:50		
Crewe	↓			11:25	11:30	12:30
Birmingham	11:50	12:05	13:05		↓	↓
Cardiff		13:50	14:50		14:00	15:00

In each case, when a connection is missed, there is an alternative connection one hour later. Trains are assumed to arrive either *on time*, or *10 minutes late*, or *20 minutes late*, as detailed in the following table of probabilities:

train	arrival at	on time	10 min late	20 min late
1, 4	Bangor	1.0	0.0	0.0
1	Birmingham	0.2	0.4	0.4
2, 3	Birmingham	0.3	0.6	0.1
2, 3	Cardiff	0.2	0.5	0.3
4	Crewe	0.2	0.6	0.2
5, 6	Crewe	0.4	0.5	0.1
5, 6	Cardiff	0.1	0.6	0.3

Show that, when train 1 is taken, the traveller makes the connection at Birmingham with probability 0.88. What is the probability that, when train 4 is taken, the connection is made at Crewe? **[6 marks]**

Decide which train the traveller should catch from Bangor so as to minimise the expected journey time. **[10 marks]**

Train 1 is also scheduled to stop at Crewe at 10:50, allowing a connection with train 5. Discuss whether this extra possibility changes your decision. **[4 marks]**

G3M86 Heuristic Optimisation Algorithms: QUIZ 1 (with answers)

Q: You have to design a *timetable* for 3 modules A,B,C. A & B are given **twice** a week, module C **3 times** a week. There is one *classroom* and 2 *timeslots* per day. Consider 2 representations for this problem. R1 is a **10-dimensional vector** corresponding to the 10 slots in one week. R2 is a **7-dimensional vector** corresponding to the time slots for one week, needed to cover *all the three modules*.

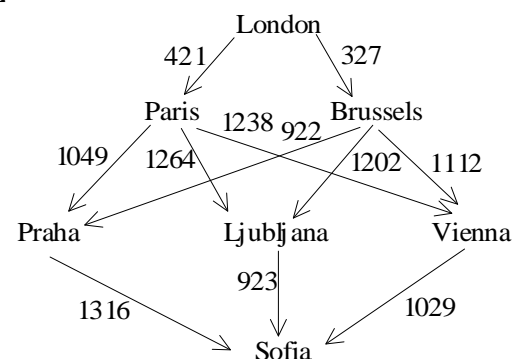
(a) What values can the vector components take in representation R1? Give an **example** of a feasible solution. A: Any *numerical* representation with 4 values e.g. 0,1,2,3. A feasible solution is $S = [1\ 2\ 1\ 2\ 3\ 0\ 3\ 0\ 3\ 0]^T$. (b) What is the *cardinality of the search space* for representation R1? A: $4^{10} = 1048576$. (c) What values can the *vector components* take for R2? Given an **example** of a feasible solution. A: 1 to 10 for each of the *possible time slots* (no overlapping) e.g. $S = [1\ 3\ 2\ 4\ 5\ 7\ 9]^T$, the *same* solution as above.

(d) What is the *cardinality of the search space* of the feasible solutions with R2? A: All permutations of 7 out of 10 time slots where A_1 and A_2 are **indistinguishable**, so are B_1 and B_2 , and C_1, C_2, C_3 . So $|F| = \binom{10}{7} \cdot 7! / 2!2!3! = 25200$. (e) If module C is given on **Mondays, Wednesdays** and **Fridays**, what is the cardinality of the search space of the *feasible solutions* with R2? A: There are $2 \times 2 \times 2$ **choices** for C_1, C_2, C_3 . **Therefore**, $|F_{\text{restricted}}| = \binom{7}{4} \cdot 4! / 2!2! \cdot 2 \cdot 2 \cdot 2 = 1680$.

(2) (Answers) Given the *starting permutation* T, the 2-opt algorithm for the TSP is **derivative free** (not derivative based); **numerical** (not analytical) and **deterministic** (not random). (b) The *Newton method* for finding the zeros of a function is **derivative based** (not derivative free), **numerical** (not analytical) and **fixed-point** (not bracketing). (c) The method called *Regula Falsi* is **derivative free** (not derivative based), **numerical** (not analytical), **deterministic** (not random) and **bracketing** (not fixed-point).

(d) What will happen to the *bisection algorithm* (the way it was given to you as a pseudo-code) if there are **2** zeros of the function $f(x)$ in the *interval* $[a,b]$. A: It will always converge to *one of the zeros* (could be either one of them) *or* b .

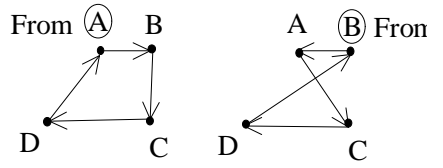
(3) Shown is some routes from **London** to **Sofia**. The distances in km are marked. (a) Find a solution of the shortest route problem *applying a greedy algorithm*. A: London - Brussels - Praha - Sofia ($327+922+1316$) = 2565km. (b) Use **dynamic programming** to find the shortest route. A: $f_4^*(\text{Sofia}) = 0$. $f_3^*(\text{Praha}) = f_3(\text{Praha}, \text{Sofia}) = 1316$. $f_3^*(\text{Ljubljana}) = f_3(\text{Ljubljana}, \text{Sofia}) = 923$. $f_3^*(\text{Vienna}) = f_3(\text{Vienna}, \text{Sofia}) = 1029$.



$f_2(\text{Paris}, \text{Praha}) = 1049+1316 = 2365$. $f_2(\text{Paris}, \text{Ljubljana}) = 1264+923 = 2187$. $f_2(\text{Paris}, \text{Vienna}) = 1238+1029 = 2267$. So $f_2^*(\text{Paris}) = 2187$ (*Ljubljana*).

$f_2(\text{Brussels, Praha}) = 922+1316 = 2238$. $f_2(\text{Brussels, Ljubljana}) = 1202+923 = 2125$. $f_2(\text{Brussels, Vienna}) = 1112+1029 = 2141$. So $f_2^*(\text{Brussels}) = 2125$ (*Ljubljana*). $f_1(\text{London, Paris}) = 421+2187 = 2606$. $f_1(\text{London, Brussels}) = 327+2125 = 2452$. So $f_1^*(\text{London}) = 2452$, the *optimal* cost. The **solution** is London - Brussels - Ljubljana - Sofia.

(4) Consider the **TSP** problem. Let A & B be the 2 *closest towns in the set*. If we start a tour from A and **another** tour from E using in both cases the *nearest-neighbour* greedy algorithm, will we **always** obtain the same tours (the direction does not matter)?



A: No, we cannot always *expect* the same tour. Example as shown, where $A-B-C-D \neq A-C-D-B$.

```

begin
  c ← 0.001
  a ← -4
  b ← -3
  while (b-a) < c, do
    begin
      t2 ← 8*sin(2*b-1)+b^3
      t1 ← 8*sin(2*a-1)+a^3
      b ← b-t2*(b-a)/(t2-t1)
      a ← b
    end
  end and return b

```

must be $\geq c$

$e \leftarrow b$

(5) The **pseudo-code** shown implements one of the *methods* that you have studied in the course. (a) Which method is this and what is it **used** for? A: The **secant** method. Used to find a *zero of a function*. In this example it starts off with $x_0 = -4$, $x_1 = -3$, and finds a *zero* of $f(x) = 8\sin(2x-1)+x^3$.

(b) There are 2 *mistakes in the pseudo-code*. Find them and fix the code as it should be (**mark** the corrections on the code). Explain what the mistakes were. A: **Mistake 1 (RED)**. The condition of the “while” loop is incorrect. It should be $\geq \epsilon$, otherwise we will *never enter the loop*. **Mistake 2 (BLUE)** The value of b has to be assigned to *another variable* (not a) before being changed. Then a gets the value that has been **stored**. (b) Explain what would happen if the *code was run as is*. A: The program will not execute the “while” loop, and will **return** $b = -3$.

(d) Explain what would happen if you *fix each of the mistakes* but leave the **other** one in. A: **Fix Mistake 1**. The program will enter the “while” loop, execute it once, and get $a=b$ at the end. Since $(b-a) < c$, the loop will **not** be executed again and the program will terminate giving the value of x_2 ($x_0 = -4$, $x_1 = -3$). **Fix Mistake 2**. The program will not execute the while loop and will *terminate returning* $b = -3$

G3M86 Heuristic Optimisation Algorithms: QUIZ 2 (with answers)

(1) [5 0 2 4 0 1 3 0] is a *tabu memory vector*. What can you say about the **horizon**? It is 5 ($h = 5$). The vector does **not** signify either the beginning or the end of the algorithm. Q: If [0 1 1 0 1 1 0 1] is the *current* solution, what was the *previous* solution? A: [1 1 1 0 1 1 0 1] (bit 1 changed).

(2) You are optimising a function of **6** binary variables. Which of these are *legitimate* tabu memory vectors? [1 2 3 4 5 0], [0 1 0 0 0 0], [0 1 0 0 1 0], [5 2 3 4 0 0]. A: All but the **third** vector.

(3) Let $\underline{x} \in \{0,1\}^4$. The function to *maximise* is $(y-3)^2$ where y is the **decimal** number corresponding to the binary string (For example, 0101 is $1 \times 2^2 + 1 \times 2^0 = 5$). (a) Start with solution $\underline{x}_0 = [0 0 0 0]$ and perform *5 steps of a tabu search* with horizon $h = 2$ (find \underline{x}_5). Start checking the bits from *left to right*. If there are **more** than one best solutions, move to the **first** one found. Give your *final* result.

A: $\underline{x}_0 = [0 0 0 0]$, TM = [0 0 0 0]. **Possible changes:** bits 1,2,3,4. Bit 1 gives [1 0 0 0], $\underline{x} = 8$, $f = 25$. Bit 2 gives [0 1 0 0], $\underline{x} = 4$, $f = 1$. Bit 3 gives [0 0 1 0], $\underline{x} = 2$, $f = 1$. Bit 4 gives [0 0 0 1], $\underline{x} = 1$, $f = 4$. **Choose** [1 0 0 0]. So $\underline{x}_1 = [1 0 0 0]$, TM = [2 0 0 0]. **Flipping** bit 2 gives [1 1 0 0], $\underline{x} = 12$, $f = 81$. Bit 3 gives [1 0 1 0], $\underline{x} = 10$, $f = 49$. Bit 4 gives [1 0 0 1], $\underline{x} = 9$, $f = 36$. So **choose** [1 1 0 0].

$\underline{x}_2 = [1 1 0 0]$, TM = [1 2 0 0]. Bit 3 gives [1 1 1 0], $\underline{x} = 14$, $f = 121$. Bit 4 gives [1 1 0 1], $\underline{x} = 13$, $f = 100$. **Choose** [1 1 1 0]. So $\underline{x}_3 = [1 1 1 0]$, TM = [0 1 2 0]. Bit 1 gives [0 1 1 0], $\underline{x} = 6$, $f = 9$. Bit 4 gives [1 1 1 1], $\underline{x} = 15$, $f = 144$. **Choose** [1 1 1 1]. So $\underline{x}_4 = [1 1 1 1]$, TM = [0 0 1 2]. Bit 1 gives [0 1 1 1], $\underline{x} = 7$, $f = 16$. Bit 2 gives [1 0 1 1], $\underline{x} = 11$, $f = 64$. **Choose** [1 0 1 1]. So $\underline{x}_5 = [1 0 1 1]$, TM = [0 2 0 1]. **Final** result: $f = 144$ at $\underline{x}_4 = [1 1 1 1]$.

(b) What would the results be if you applied **hill-climbing** from the *same* initial solution \underline{x}_0 ? A: $\underline{x} = [1 1 1 1]$, $f = 144$.

(4) Let $y \in \{-1, 0, 1, 2\}$. Assume you are **minimising** $f(y) = |y|$ using the SA algorithm. (a) Find the *equilibrium probability distribution* at $c=5$. A: As seen in the table on the **right**. (b) Show the *equilibrium probability distributions* for $c \rightarrow 0$ and for $c \rightarrow \infty$. A: **When** y is -1, 0, 1, 2; q_0 is 0, 1, 0, 0 and q_∞ is $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$. (c) Find the **expected value** of the function $f(y)$ at $c=5$. A: $0.248+0.248+(2 \times 0.202) = 0.90$. (d) What is S_{opt} , the set of *optimal solutions* (states), for this problem? A: $S_{\text{opt}} = \{0\}$.

y	-1	0	1	2	
$f(y)$	1	0	1	2	
$-f(y)/c$	-0.2	0	-0.2	-0.4	
$\exp(\cdot)$	0.819	1	0.819	0.670	$\Sigma = 3.308$
$q_5(y)$	0.248	0.302	0.248	0.202	

(5) Consider the *simulated annealing algorithm*. Knowing that for $a \leq 0$, $\lim_{x \rightarrow 0 (x > 0)} \exp(a/x) = 1$, if $a=0$; and 0 otherwise, prove that $\lim_{c \rightarrow \infty} \langle f \rangle_c = 1/|S| \sum_{i \in S} f(i)$, where S is the *space of possible states* i of the system, $|S|$ is the **cardinality** of S , and $\langle f \rangle_c$ is the *expected value* of c of the function being **minimised**. A: $\lim_{c \rightarrow \infty} \langle f \rangle_c = \lim_{c \rightarrow \infty} \sum_{i \in S} f(i) \cdot q_c(i) = \lim_{c \rightarrow \infty} \sum_{i \in S} f(i) \cdot [\exp(-f(i)/c) / \sum_j \exp(-f(j)/c)] = \sum_{i \in S} f(i) \cdot [\lim_{c \rightarrow \infty} \exp(-f(i)/c)] / [\sum_j \lim_{c \rightarrow \infty} \exp(-f(j)/c)] = \sum_{i \in S} f(i) [1 / \sum_j 1] = 1/|S| \sum_{i \in S} f(i)$. **QED**.

(6) Mark the following *statements* about SA as true or false. $\partial / \partial c \langle f \rangle_c > 0$ (**True**). For any $i \in S_{opt}$, $\lim_{c \rightarrow 0} q_c(i) = 1/|S_{opt}|$ (**True**). For any $i \in S_{opt}$, $\lim_{c \rightarrow \infty} q_c(i) = 1/|S|$ (**True**). The *entropy* at equilibrium is a natural measure of **disorder** in the system (**True**). The *entropy* at equilibrium is a **decreasing function** of the **cooling** parameter c (**True**).

(7) Let $\underline{x}_1 = [0 \ 1 \ 1 \ 0]$ and $\underline{x}_2 = [1 \ 0 \ 1 \ 0]$ be *chromosomes* selected as **parents** in a genetic algorithm. The *fitness function* (to maximise) is the number of **ones** in the chromosome. (a) List all **common schemata** for the two parents. A: # # # #, # # # 0, # # 1 #, # # 1 0. (b) What is the *probability* that, after a one-point crossover and before mutation, there will be an offspring *better* than both parents.

A: We have $0^1 1^2 1^3 0$ and $1^1 0^2 1^3 0$. So with *point 1* we have 4/1, with *point 2* we have 2/2 and with *point 3* we have 2/2. So there is a 1/3 chance for a **better** offspring. (c) What is the probability that the *two children* in the offspring set will **not** share the highest order schema common for \underline{x}_1 and \underline{x}_2 after **mutation** with **probability** 0.1?

A: $P(2 \text{ children sharing } \# \# 1 0) = P(\text{1st child not mutating in the last 2 bits}) \times P(\text{2nd child not mutating in the last 2 bits}) = 0.9 \times 0.9 \times 0.9 \times 0.9$. So $P(2 \text{ children not sharing } \# \# 1 0) = 1 - 0.9^4 = 1 - 0.6561 = 0.3439$.

(8) The *population set* in a GA has **fitness function** as follows: when $i = 1, 2, 3, 4, 5$, $f(\underline{x}_i)$ is 10, 20, 30, 40, 50. You are going to apply the *roulette wheel principle* to select a **mating set** of 5. What is the probability that the set will consist of *five different individuals*? A: $P(\text{selecting 1}) = 10/10+...+50 = 10/150 = 1/15$. $P(\text{selecting 2}) = 2/15$ $P(\text{selecting 5}) = 5/15$. So $P(\text{selecting 12345 in this order}) = 1 \times 2 \times 3 \times 4 \times 5 / (15)^5$. There are *5! permutations*, therefore $P(\text{selecting 5 different}) = 5! \times 5! / (15)^5 = 64/3375 \approx 0.019$.

(9) Guess my *age* (any number between $-\infty$ and $+\infty$). Bonus mark for the **correct** figure! A: ???!