G2M15 Automata Theory: Exam Paper, January 2001

SECTION 1 (Compulsory)

(1) (a) Let Σ be a fixed alphabet.
Define what is meant by a *recognisable language* over Σ. **[1 mark]**
Define what is meant by a *regular expression* over Σ. **[2 marks]**
State *Kleene's Theorem* without proof. **[1 mark]**

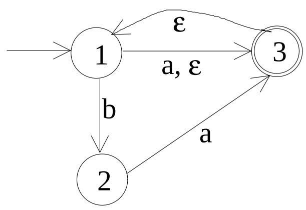
(b) Explain the algorithm for constructing an ε-automaton from a regular expression with reference to the regular expression (ab+a)*. **[6 marks]**

(c) Find a context-free grammar for the language $\{a^i b^j c^j a^i : i, j \geq 0\}$. Construct a leftmost derivation of the string ab^2c^2a and a corresponding derivation tree. **[5 marks]**

(d) Give the formal definition of a Turing machine. **[2 marks]**
Explain briefly what is meant by Turing's Thesis, you should make clear the relationship between 'algorithms' and Turing machines, and between Turing machines and real computers. **[3 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) This question refers to the ε-automaton **A** in the shown diagram.



(a) Convert **A** into a non-deterministic machine without ε-transitions **B** which recognises the same language as **A**. **[6 marks]**

(b) Convert **B** into a connected deterministic automaton **C** which recognises the same language as **B**. **[3 marks]**

(c) Write down the language equations associated with **C**. **[3 marks]**

(d) Solve the language equations in (c), and so find a regular expression for L(**A**). **[3 marks]**

(3) What is meant by a right-linear grammar? Prove that a language is generated by a right-linear grammar if and only if it is recognisable. **[15 marks]**

(4) This question concerns the context-free grammar **G** = (**N**, Σ, **P**, *S*) where **N** = {*S*, A}, Σ = {a,b} and **P** consists of the following productions:

1. $S \rightarrow AaA$
2. $A \rightarrow aA$
3. $A \rightarrow bA$
4. $A \rightarrow \varepsilon$

(a) Construct a leftmost derivation of the string a^3. **[2 marks]**
(b) Construct a pushdown automaton which recognises the language L(**G**). **[8 marks]**
(c) Show how your machine processes the string a^3 by tracing through the configurations assumed by your machine. **[5 marks]**

(5) A Turing machine is said to be *special* if it satisfies the following conditions:

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \#\}$
There is a unique start-state labelled 1, and a unique halt-state labelled 2.

It is a theorem that every recursively enumerable **Σ**-language can be recognised by a special Turing machine.

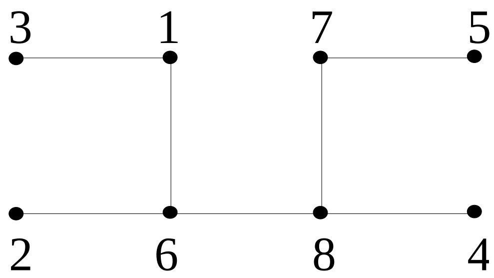
(a) Explain how every special Turing machine transition table can be encoded by means of a string from (a+b)*. **[5 marks]**
(b) Let $G \subseteq (a+b)^*$ be the set of all these strings *w* which encode a Turing machine transition table **A** and satisfy $w \in L(\mathbf{A})$. Prove that the complement of *G* is not recursively enumerable. **[10 marks]**

(Questions done: 1, 2, 4)

G3M04 Graphical Algorithms: Exam Paper, January 2001

Answer 3 questions out of 5 (Questions Done: 1, 3, 4)

(1) (a) Find the labelled tree with Prüfer code (7, 5, 3, 1, 7, 5) and the Prüfer code of the tree shown on the right. Comment on the significance of the procedures just illustrated for Cayley's formula for the number of labelled trees with n vertices. **[8 marks]**



(b) Let r_n be the number of simple, rooted trees with n vertices. Derive the recurrence relation

$$r_n = \begin{cases} r_{n-1} + \sum_{k=1}^{(n-2)/2} r_k r_{n-k-1} & \text{(n even)}, \\ r_{n-1} + \sum_{k=1}^{(n-3)/2} r_k r_{n-k-1} + \frac{1}{2} r_{(n-1)/2}(r_{(n-1)/2} + 1) & \text{(n odd)} \end{cases}$$

[7 marks]

(c) Find the number of simple rooted trees having 11 vertices and maximal sub-path containing 6 vertices. **[5 marks]**

(2) Let K_6 be the complete graph with vertices {A,B,C,D,E,F} and weights on edges given in the shown table.

	A	B	C	D	E	F
A	-	9	7	3	8	5
B	9	-	5	8	3	7
C	7	5	-	8	4	9
D	3	8	8	-	6	4
E	8	3	4	6	-	8
F	5	7	9	4	8	-

(a) Find a minimal spanning tree in K_6 using *both* of the following methods:

(i) Kruskal's greedy algorithm
(i) Prim's greedy algorithm.

Show all your working. **[10 marks]**

(b) Describe the travelling salesman problem, TSP. Obtain bounds for TSP on K_6 with weights as above. Improve these bounds until you obtain LB $\leq$ TSP $\leq$ UP with UB-LB $\leq$ 4. **[6 marks]**

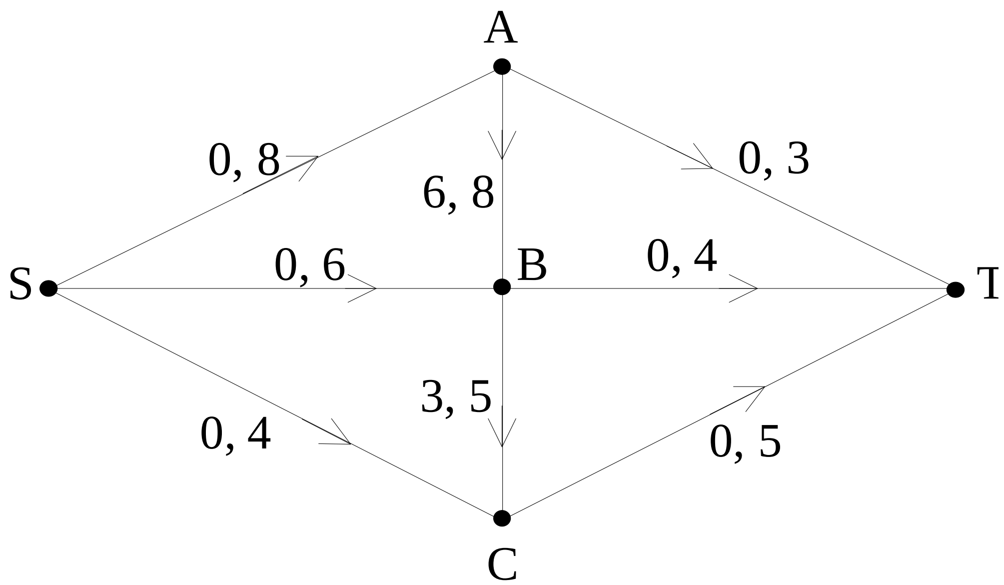
(c) The following procedure is a modification of Prim's algorithm:

- Choose an edge U—V of minimum weight and form the directed cycle C: U→V→U;
- Find a minimal edge X—Y with $X \in C$ and $Y \notin C$;
- Insert Y into C *immediately* after X:
  C: U→...→X→Z→....→U becomes C: U→...→X→Y→Z→...→U;
- Repeat until C contains all the vertices.

Implement this procedure for K_6 taking U—V to be A—D. What is the weight of the resulting cycle C? **[4 marks]**

(3) (a) Let N be a directed network with upper and lower capacities. Define the value of a cut in N. Prove that the value of a flow in N is less than or equal to the value of a cut. **[8 marks]**

(b) Let N be the following directed network with lower and upper capacities:
(Note that just two of the lower capacities are positive)



Illustrate the algorithm for constructing a feasible flow on N as follows:

(i) Construct the network N* with zero lower capacities;

(ii) Find *by inspection* an appropriate feasible flow F* in N*;

(iii) Write down the feasible flow F in N corresponding to F*.

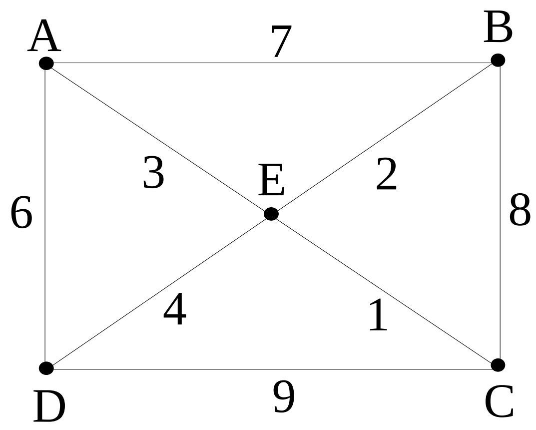
[9 marks]

Extend your flow F to a maximal flow in N, and obtain a minimum cut in N.

[3 marks]

(4) (a) Define the complete graph C(N) of an undirected, weighted network N. Prove that in any cycle of C(N) the smallest weight occurs at least twice. **[7 marks]**

(b) Find a maximal tree T in the complete graph C(N) where N is the undirected, weighted network



Hence find the maximal flows between all pairs of vertices of N. **[10 marks]**

Sketch a second maximal tree T' in C(N) such that T and T' are not isomorphic as trees when all the weights are removed. **[3 marks]**

(5) (a) Let G_1 be a connected graph such that every vertex has even degree. Prove that G1 is Eulerian. **[6 marks]**

Let G_2 be a connected graph with two vertices of odd degree and the rest even. Prove that G_2 has an Euler trail. **[3 marks]**

Let G_3 be a connected graph with four vertices w, x, y, z of odd degree, and the rest even. Prove that G_3 can be decomposed into two paths having w, x, y, z as the four end-points. **[3 marks]**

(b) Sketch the graph with the incidence matrix

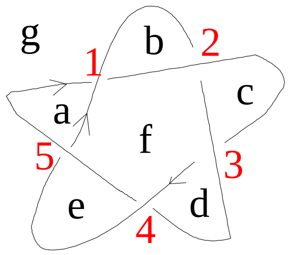
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Use the maze search algorithm to find an Eulerian trail in this graph. Show all your working. **[8 marks]**

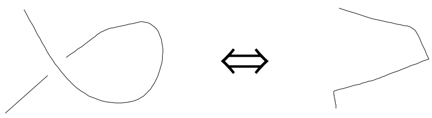
G3M24 Knots & Surfaces: Exam Paper, January 2001

Answer 3 questions out of 5 (Questions Done: 1, 2, 4)

(1) Describe Alexander's original method of calculating the Alexander polynomial of a knot, illustrating your description using the cinquefoi knot labelled and oriented as shown. **[14 marks]**

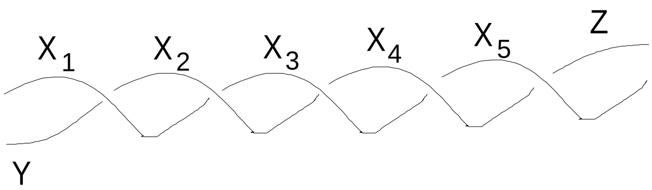


Prove that, **in general**, the Reidemeister move, R1 as shown on the right does not alter the normalised Alexander polynomial. **[6 marks]**



(2) Define what it means for a knot to be 5-colourable. **[2 marks]**

(i) The 5-crossing braid



is 5-coloured with colours $x_1,\ldots,x_5$, y, z. Prove that $z=x_1$ and $y=x_5$. **[10 marks]**

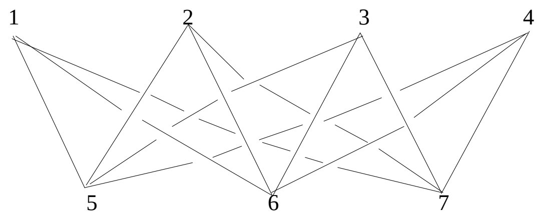
(ii) Prove that 5-colourability is invariant under the first and second Reidemeister moves. **[8 marks]**

(3) (a) Classify the surface with surface symbol $abf^{-1}dfb^{-1}a$ giving its normal form. **[6 marks]**

(b) Show how, by using the three elementary operations, one can replace a surface symbol of form $ABxCx^{-1}D$ by $AyCy^{-1}BD$, where y is a new symbol, not in the blocks A, B, C or D. **[6 marks]**

(c) Let T be a torus and P a projective plane. Two holes are cut in *each* surface and the surfaces are then glued together along the boundaries of these holes to yield a new surface, *S*. Classify the surface *S* in terms of orientability, genus, etc. (Note, since TWO holes are cut in each surface *S* is **not** the connected sum of T and P.) **[8 marks]**

(4) The graph $K_{3,4}$ is shown below.



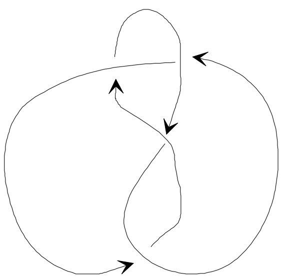
Using Edmonds algorithm, find a cell embedding of this graph in an oriented surface. Give the genus of the surface you have constructed. **[9 marks]**

Show that if $K_{3,4}$ has a cell embedding into S, an orientable surface of genus g, then $g \leq 3$. **[6 marks]**

Show that if in addition, the embedding can be chosen to be a regular one (i.e. so that each polygonal disc in the complement has the same number of sides) then $g = 2$ or 3. **[5 marks]**

(5) (a) If X is a topological space with R an equivalence relation on X, we will denote the set of equivalence classes by X/R and the quotient map sending each point to the equivalence class containing it by $q: X \to X/R$, $q(x) = [x]_R$. A subset $A \subseteq X/R$ is said to be *open* if $q^{-1}(A) = \{x \mid [x]_R \in A\}$ is open in the topology on X. Prove that the collection of open sets in this sense is a topology on X/R **[10 marks]**

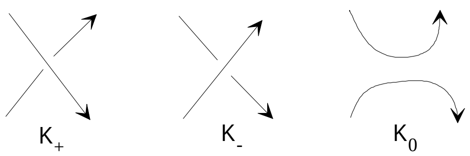
(b) Calculate the two variable (HOMFLY) polynomials $K(l,m)$ for the figure eight knot shown below.



You may assume that $K(l,m)$ is uniquely determined by the defining equation

$$lK_+(l,m) + l^{-1}K_-(l,m) + mK_0(l,m) = 0,$$

where K_+ etc. correspond, as usual, to changing a crossing in the picture of the knot as follows:



and the normalising condition, $K(l,m) = 1$ for the unknot, but all other results that you need should be derived from these. **[10 marks]**

G3M61 Numerical Analysis: Exam Paper, January 2001

Answer 3 questions out of 5 (Questions Done: 2, 4, 5)

(1) (a) Consider the first order ordinary differential equation $^{dy}/_{dt} = f(t,y)$ for $t > t_0$ with initial value $y(t_0) = y_0$. Write down the following numerical schemes for such an initial value problem using a uniform timestep Δt: (i) Euler, (ii) trapezoidal Euler, (iii) backward Euler. **[4 marks]**

(b) Write down the difference equations given by the three schemes when applied to the initial value problem $^{dy}/_{dt} = -\lambda y$ for $t>0$ with initial value $y(0) = 1$, where $\lambda > 0$ is a constant. **[4 marks]**

(c) Show that the exact solution $y = e^{-\lambda t}$ implies that $y(t_{n+1}) = e^{-\lambda\Delta t}y(t_n)$. Compare this exact expression with the approximations given by the three difference equations derived above. Hence comment on the stability of the three difference schemes. **[4 marks]**

(d) Illustrate your conclusions by comparing the exact solution with approximate solutions for $\lambda = 2$ using $\Delta t = ¼$ and $\Delta t = ^3/_2$ for 4 timesteps. **[4 marks]**

(e) Consider the logistic equation $^{dy}/_{dt} = y(1-y)$. Set up a predictor-corrector scheme using the Euler scheme for the predictor and the trapezoidal scheme for the corrector. Derive the difference equations, but do not attempt to solve them. **[4 marks]**

(2) The partial differential equation $^{\partial U}/_{\partial t} - ^{\partial^2 U}/_{\partial x^2} = 0$ is approximated at the point $x_j = j\Delta x$, $t_n = n\Delta t$ by the difference equation

$$\theta\left[\frac{u_{j,n+1}-u_{j,n-1}}{2\Delta t}\right] + (1-\theta)\left[\frac{u_{j,n}-u_{j,n-1}}{\Delta t}\right] - \frac{u_{j+1,n}-2u_{j,n}+u_{j-1,n}}{\Delta x^2} = 0.$$

(a) Expand U about this point in terms of Taylor series in time up to Δt^2 and Taylor series in space up to Δx^5. Hence show that the local truncation error at this point is $-½\Delta t(1-\theta)(^{\partial^2 U}/_{\partial t^2})_{j,n} - ^1/_{12}\Delta x^2(\frac{\partial^4 U}{\partial x^4})_{j,n} + O(\Delta t^2) + O(\Delta x^4)$. **[8 marks]**

(b) Show that setting $\theta = 1 + ^{\Delta x^2}/_{6\Delta t}$ reduces the local truncation error to one of order Δt^2 and Δx^4. **[3 marks]**

(c) With this value of θ and the choice $\Delta x^2 = 6\Delta t$, show by the Fourier method that the scheme is stable. **[9 marks]**

(3) Consider the initial value problem $\partial U/\partial t = \partial^2 U/\partial x^2$ with initial condition $U(x,0)$ given for $0 \le x \le 1$ and boundary conditions $U(0,t) = g(t)$, $U(1,t) = h(t)$ given for $t>0$.

Construct an explicit finite difference scheme for the problem using forward difference for the time derivative and central differences for the space derivative.

Write down the difference equations in matrix form and show that the stability of the scheme depends on the magnitude of the spectral radius of a certain matrix. Deduce that the scheme is stable when $\Delta t \le (\Delta x)^2/2$.

You may assume that the eigenvalues of the real symmetric $M \times M$ matrix

$$\begin{pmatrix} a & b & & & & & \\ b & a & b & & & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & b & a & b \\ & & & & & b & a \end{pmatrix}$$

are given by $\lambda_s = a + 2b(1 - 2\sin^2(s\pi/_{M+1}))$ for $s = 1, 2, ..., M$. **[20 marks]**

(4) The partial differential equation $\partial^2 U/\partial x^2 - U^2\, \partial^2 U/\partial y^2 = 2y$ is to be solved in the domain $x > 0$, $y > 1$. The initial curve is the line $y=1$ and the initial conditions are $U(x,1) = \partial U/\partial y(x,1) = x^2$.

Use the numerical method of characteristics to find a first approximation for U at the first characteristic grid point R between P = (1,1) and Q = (1.1, 1). **[16 marks]**

Assume that $U = x^2 y$ is the exact solution of the problem and compare your numerical answer with the exact solution. **[4 marks]**

You may assume that the partial differential equation $a\,\partial^2 U/\partial x^2 + b\,\partial^2 U/\partial x\partial y + c\,\partial^2 U/\partial y^2 + e = 0$, with the usual notation, reduces to $a\, d(\partial U/\partial x)\, dy/dx + c\, d(\partial U/\partial y) + e\, dy = 0$ on the characteristic curves given by the roots of $a(dy/dx)^2 - b(dy/dx) + c = 0$.

(5) Using the simplest central difference formulae and choosing a uniform mesh $\Delta x = \Delta y = \Delta$, write down a finite difference scheme for the partial differential equation $\partial^2 U/\partial x^2 + 3\partial^2 U/\partial y^2 = \alpha$, where α is a constant. **[5 marks]**

The equation is to be solved inside the square domain determined by the lines $x = \pm 1$, $y = \pm 1$, subject to the boundary conditions (i) $U(1,y) = 0$ for $-1 \le y \le 1$ and (ii) $U(x,1) = 1$ for $-1 < x < 1$, and the problem is symmetric with respect to Ox, Oy.

Consider the case $\alpha = -16$ and $\Delta = ¼$. Label the points with coordinates (0, 3/4), (1/4, 3/4), (1/2, 3/4), (3/4, 3/4), (0, 1/2), (1/4, 1/2), ... as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... etc. Show that it is necessary to solve sixteen finite difference equations which can be written in matrix form as A**u** = **b**, where **u** is a column vector whose transpose is $(u_1, u_2, ..., u_{16})$ and A is a matrix which can be written in partitioned form as

$$\begin{bmatrix} B & 3I & & \\ 3I & B & 3I & \\ & 3I & B & 3I \\ & & 6I & B \end{bmatrix}$$

where I is the 4 × 4 unit matrix. Find B and **b**. **[15 marks]**

G3M72 Quantum Mechanics: Exam Paper, January 2001

Answer 3 questions out of 5 (Questions Done: 1, 3, 4)

(1) (a) Show that for a conservative holonomic system with n degrees of freedom, Lagrange's equations and Hamilton's equations are equivalent. **[8 marks]**

(b) The Lagrangian function for a particle of mass m moving in the xy-plane is given by $L = (m/2)(\dot{r}^2 + (r\dot{\theta})^2) + (A/2)(r^2\dot{\theta})$ where r, θ are the polar coordinates of the particle and A is a constant.

Find expressions for the generalised momenta p_r, p_θ and show that the Hamiltonian function H is given by $H = (1/2m)(p_r^2 + (p_\theta/r)^2 + (Ar/2)^2 - Ap_\theta)$ and write down the canonical equations of motion. **[8 marks]**

If at t=0, r=a, $\dot{\theta}=0$, deduce that $\dot{\theta} = (A/2m)((a/r)^2-1)$, t>0. **[4 marks]**

(2) Write a short mathematical essay describing the evidence that a theory radically different from classical theory is necessary for the description of microphysical phenomena.
[20 marks]

(3) A particle of mass m moves in one dimension in a time-independent potential V(x). Apply the method of separation of variables to show that solutions $\psi_E(x,t)$ to the time-dependent Schrödinger equation may be found in the form $\psi_E(x,t) = U_E(x)T_E(t)$ where E denotes the separation constant, $U_E(x)$ denotes a solution to the time-independent Schrödinger equation and $T_E(t)$ is to be found. Give a physical interpretation of E. Hence write down the general solution to the time-dependent Schrödinger equation. **[10 marks]**

If V(x) is given by: V(x) = 0 for $0 < x < l$ and ∞ for $x \le 0$, $x \ge l$ show that $E = E_n = (\frac{1}{2m})(\frac{\hbar\pi}{l})^2 n^2$ for n = 1, 2, 3, ... **[10 marks]**

(4) A particle of mass m moves in one dimension in the potential given by $V(x) = V_0$ for $x < 0$ and $x > l$; 0 for $0 \le x \le l$. Show that there are only a finite number N of bound states of energy $E < V_0$, where E satisfies the equation
$\left(\frac{l}{\hbar}\right)\sqrt{2mE} = n\pi - 2\sin^{-1}\sqrt{E/V_0}$, n = 1, 2, 3, ..., N. **[18 marks]**

Contrast with the energy spectrum of a classical particle moving in the same potential.
[2 marks]

(5) The radial part of a bound state wave function of energy $E < 0$ of a hydrogen atom is given by: $R(\rho) = \exp(-\rho/2)F(\rho)$ where $\rho = \alpha r$, $\alpha^2 = (-8\mu E)/\hbar^2$ and $F(\rho)$ satisfies ${}^{d^2F}/_{d\rho^2} + [{}^{2}/_{\rho}-1]{}^{dF}/_{d\rho} + [{}^{(n-1)}/_{\rho} - {}^{l(l+1)}/_{\rho^2}]F = 0$, with $n = -\alpha(e^2/4\pi\varepsilon_0)/4E$, μ and e denoting respectively the reduced electronic mass and charge, and l taking on non-negative integral values. By considering a Frobenius series solution for F show that a physically acceptable solution for R is obtained only when n taken on positive integral values and find the corresponding energy levels.

[17 marks]

Show that the degeneracy of the n^{th} energy level is n^2. **[3 marks]**