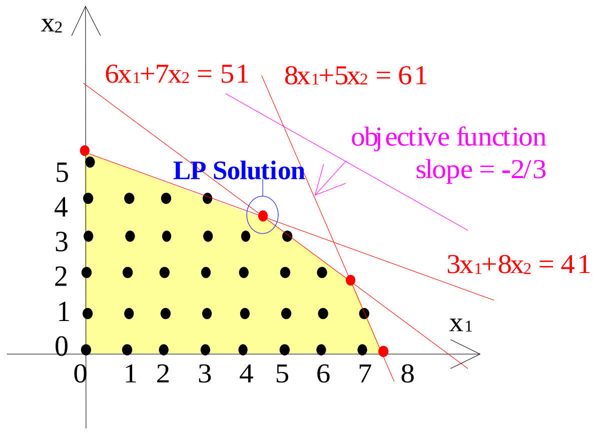
Revision of G2M85

Example: *Maximise* M = $8x_1+9x_2+5x_3$ subject to $x_1+x_2+2x_3 \le 2$, $2x_1+3x_2+4x_3 \le 3$, and $6x_1+6x_2+2x_3 \le 8$, with $x_1, x_2, x_3 \geq 0$. We *set out the solution* as shown in the table on the right (remember to take the **largest -ve entry** in the objective function row, and the **smallest positive ratio** to get the pivot position). At the end, all the *objective row coefficients* are ≥ 0, so the solution is $x_1 = 1$, $x_2 = {}^1/_3$, $x_3 = 0$, $s_1 = {}^2/_3$, $s_2 = 0$, $s_3 = 0$, and M = 11. *Note* that we went from (0,0,0) to (0,1,0), and then to $(1, {}^1/_3, 0)$.

	M	x_1	x_2	x_3	s_1	s_2	s_3	RHS	basic	ratio
	1	-8	-9	-5	0	0	0	0	= M	
	0	1	1	2	1	0	0	2	= s_1	2
	0	2	3	4	0	1	0	3	= s_2	1
	0	6	6	2	0	0	1	8	= s_3	4/3
R_1+3R_3	1	-2	0	7	0	3	0	9	= M	
$3R_1-R_2$	0	1	0	2	3	-1	0	3	= $3s_1$	3
	0	2	3	4	0	1	0	3	= $3x_2$	3/2
R_3-2R_2	0	2	0	-6	0	-2	1	2	= s_3	1
R_1+R_4	1	0	0	1	0	1	1	11	= M	
$2R_2-R_4$	0	0	0	10	6	0	-1	4	= $6s_1$	
R_3-R_4	0	0	3	10	0	3	-1	1	= $3x_2$	
	0	2	0	-6	0	-2	1	2	= $2x_1$	



The Integer Programming (IP) problem is to maximise M with some *constraints* and with $x_1, x_2, x_3 \in \mathbf{Z}^{\geq 0}$. At (0,0,0), M = 0; at (1,0,0), M = 8; and at (0,1,0), M = 9. There are *only 3 integer points in this feasible region*. Consider a more **complex** case: **Maximise** M = $2x_1+3x_2$ subject to $3x_1+8x_2 \le 41$, $6x_1+7x_2 \le 51$, and $8x_1+5x_2 \le 61$, with $x_1, x_2 \geq 0$. From the *graph*, we see that the LP solution is at the intersection of the red lines (at position $({}^{121}/_{27}, {}^{93}/_{27})$), with M approximately 19.3. Taking the *slope down*, we see that there are **two** possible IP solutions: at (5,3), where M = 19; and at (3,4), where M = 18. It follows that *(5,3) is the IP solution*.

Definition: An IP is an *optimisation problem* where (1) we attempt to maximise or minimise a linear function (the **objective** function) in a set of *decision variables;* (2) the values of the variables are required to satisfy a set of **linear constraints** ($\le$, =, $\geq$); (3) for each decision variable x_i, *either* $x_i \geq 0$, *or* x_i is unrestriced in sign (**note**: LP to here); and (4) *one or more* of the variables is required to be an integer.

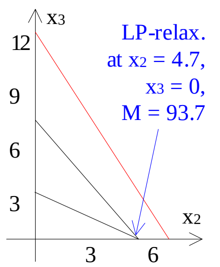
An IP is **pure** if all $x_i \in$ **Z**, and is **mixed** otherwise. It is 0-1 if all $x_i \in \{0,1\}$. The *LP-relaxation* of an IP is obtained by removing all the "$x_i \in$ **Z**" and "0-1" *requirements*. **Example:** *Maximise* M = $7x_1+11x_2+9x_3$ *subject to* $40x_1+20x_2+10x_3 \le 364$ and $10x_1+30x_2+60x_3 \le 201$, with $x_1, x_2, x_3 \in \mathbf{Z}^{\geq 0}$. We take on *trust* that the LP relaxation has solution as shown on the right.

BASIC NON-BASIC

$$\begin{bmatrix} M \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10.0 & 0.1 & 0.3 \\ -0.9 & 0.03 & -0.02 \\ 2.3 & -0.01 & 0.04 \end{bmatrix} \begin{bmatrix} x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 96.7 \\ 6.9 \\ 4.4 \end{bmatrix}$$

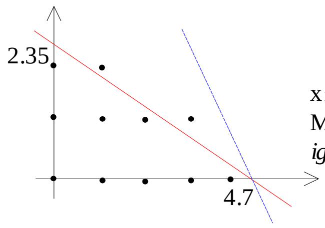
i.e. in tableau form

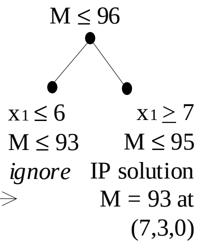
M	x_1	x_2	x_3	s_1	s_2	rhs
1			10	0.1	0.3	96.7
	1		-0.9	0.03	-0.02	6.9
		1	2.3	-0.01	0.04	4.4

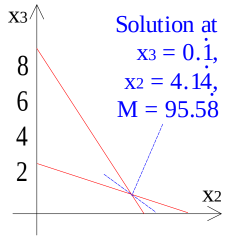


Consider *2 cases*: (i) $x_1 \le 6$, and (ii) $x_1 \geq 7$. Find the *LP solutions* when (i) $x_1 = 6$, and (ii) $x_1 = 7$. This is easy to do *graphically* (it is now a **2-D** problem). P(i): M = $11x_2+9x_3+42$ s.t. $20x_2+10x_3 \le 124$ and $30x_2+60x_3 \le 141$. The solution is shown on the *left*.

P(ii): $M = 11x_2+9x_3+49$ s.t. $20x_2+10x_3 \le 84$ *and* $30x_2+60x_3 \le 131$. The solution is shown *on the right*. By **inspection**, P(ii) has an IP solution at $x_3 = 0$, $x_2 = 4$, $x_1 = 7$, and $M = 93$. P(i) has a *solution* as shown on the left, at $x_1 = 6$, $x_2 = 4$, $x_3 = 0$, and $M = 86$. So using the <u>branch and bound method</u>, the *solution* at this stage is in {93, 94, 95}.







5th February 2001

Previously, we solved the problem of maximising $M = 2x_1+3x_2$ *subject to* $3x_1+8x_2 \le 41$, $6x_1+7x_2 \le 51$, and $8x_1+5x_2 \le 61$ ($x_i \ge 0$) *graphically*. Note that this example is special so that the **three** integer points (3,4), (5,3) and (7,1) lie on constraints. In this case, sliding the objective function in a *south-west* direction, the first integer point met is (5,3), where M = 19. Solving by the **tableau** method, we arrive at the final tableau (as *shown*), with solution $M = 19^8/_{27}$ at $x_1 = 4^{13}/_{27}$, $x_2 = 3^4/_9$, and $x_3 = 0$.

M	x_1	x_2	s_1	s_2	s_3	rhs	basic
1	0	0	4/27	7/27	0	$19\frac{8}{27}$	M
0	0	1	2/9	-1/9	0	$3\frac{4}{9}$	x_2
0	1	0	-7/27	8/27	0	$4\frac{13}{27}$	x_1
0	0	0	26/27	$-1\frac{22}{27}$	1	$7\frac{25}{27}$	s_3

<u>The Dual Simplex Algorithm</u>

This is *applicable* when (1) the **objective** row has *non-negative* coefficients only, and (2) at least one RHS/basic variable is negative. This may occur when an **additional** constraint is added to a solution tableau. The *algorithm goes as follows:* <u>Iterate</u>: (1) Find the **most negative** basic variable (this determines the <u>pivot</u> row).

(2) For each *negative coefficient* in this row (if there are none, there is **no** solution), calculate the *negative* ratio (row 0 entry in *same* column) / (negative coefficient). (3) Choose a column with the **smallest negative ratio** (this determines the *pivot* column). (4) Pivot on this row/column entry, and make this column **basic**.

<u>Primal and Dual</u>. A primal problem, P: $\max(M = \underline{c}.\underline{x})$ s.t. $A\underline{x} \le b$, $\underline{x} \ge 0$, has **dual** D(P) = Q: $\min(N = \underline{b}.\underline{u})$ s.t. $A^t\underline{u} \ge \underline{c}$, $\underline{u} \ge 0$; and this dual has *max form* $\max(-N = (-\underline{b}.\underline{u}))$ s.t. $(-A^t)\underline{u} \le (-\underline{c})$, $\underline{u} \ge 0$. The *dual of our example is to* **minimise** $N = 41u_1+51u_2+61u_2$ subject to $3u_1+6u_2+8u_3 \ge 2$, and $8u_1+7u_2+5u_3 \ge 3$ ($u_i \ge 0$).

Max form: maximise $N' = -41u_1-51u_2-61u_3$ subject to $-3u_1-6u_2-8u_3 \le -2$, and $-8u_1-7u_2-5u_3 \le -3$ ($u_i \ge 0$). Tableau as shown *on the right*. **<u>Solution</u>**: $N' = -19^8/_{27}$ at $u_1 = {}^4/_{27}$, $u_2 = {}^7/_{27}$, and $u_3 = 0$. Compare the *entries in this solution tableau* to the entries in the solution tableau of the **primal** problem.

N'	u_1	u_2	u_3	s_1	s_2	rhs	basic	
1	41	51	61	0	0	0	N' = 0	
0	-3	-6	-8	1	0	-2	$s_1 = -2$	
0	-8	-7	-5	0	1	-3	$s_2 = -3$	←
ratio:	$-5^1/_8$	$-7^2/_7$	$-12^1/_5$					
8	0	121	283	0	41	-123	N' = -15.4	
0	0	-27	-49	8	-3	-7	$s_1 = -^7/_8$	←
0	8	7	5	0	-1	3	$u_1 = {}^3/_8$	
ratio:		-4.48	-5.78		-13.6			
216	0	0	1712	968	744	-4168	N' = -19.3	
0	0	27	49	-8	3	7	$u_2 = {}^7/_{27}$	
0	216	0	-208	56	-48	32	$u_1 = {}^4/_{27}$	

Converting to *fractions*

N'	u_1	u_2	u_3	s_1	s_2	rhs	basic
1	0	0	$7^{25}/_{27}$	$4^{13}/_{27}$	$3^4/_9$	$-19^8/_{27}$	N'
0	0	1	$1^{22}/_{27}$	$-^8/_{27}$	$^1/_9$	$^7/_{27}$	u_2
0	1	0	$-^{26}/_{27}$	$^7/_{27}$	$-^2/_9$	$^4/_{27}$	u_1

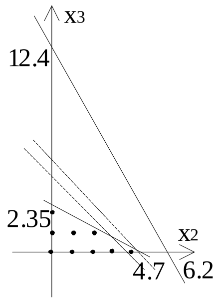
	M	x_1	x_2	s_1	s_2	s_3	s_2	rhs
R_4	0	1	1	0	0	0	1	7
	0	0	0	1/27	-5/27	0	1	-25/27

(new row, basic s_4 = -25/27)

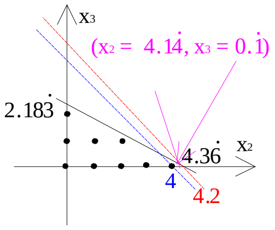
Now consider the solution tableau of the *original* problem. At the **solution,** $x_1+x_2 = 7^{25}/_{27}$. Add a *constraint* $x_1+x_2 \le 7$. So we have a new **row** as shown, and x_1 and x_2 are now no longer *unit vectors*. Subtract R_1 and R_2 from R_4 to rectify this, and we get a **new** row as shown. So the solution is M = 18 at (3,4).

$$\begin{pmatrix} M \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 10.0 & 0.10 & 0.30 \\ -0.9 & 0.03 & -0.02 \\ 2.3 & -0.01 & 0.04 \end{pmatrix} \begin{pmatrix} x_3 \\ s_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 96.7 \\ 6.9 \\ 4.4 \end{pmatrix}$$

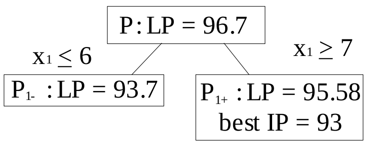
IP Problem: *maximise* M = $\underline{c}.\underline{x}$ *subject to* A$\underline{x} \le \underline{b}$, $\underline{x} \ge \underline{0}$, *where* $\underline{c} = \begin{pmatrix} 7 \\ 11 \\ 9 \end{pmatrix}$, A = $\begin{pmatrix} 40 & 20 & 10 \\ 10 & 30 & 60 \end{pmatrix}$, b = $\begin{pmatrix} 364 \\ 201 \end{pmatrix}$, and x = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. (i) Find the *LP-relaxation* solution of the initial problem, P: this goes through as normal using the **simplex** method. The LP-relaxation of P has optimal solution as shown above, i.e. M = 96.7 at $x_1 = 6.9$, $x_2 = 4.4$, and $x_3 = 0$.



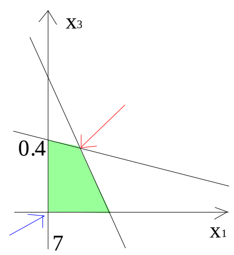
Now *add the constraint* $x_1 = 6$. Solve **graphically**: M = $11x_2+9x_3+42$ subject to $20x_2+10x_3 \le 124$, and $30x_2+60x_3 \le 141$. When $x_1 = 6$, the *LP-relaxation* of P_{1-} has **solution** M = 93.7 at $x_1 = 6$, $x_2 = 4.7$, and $x_3 = 0$. The *IP Solution* at $x_1 = 6$ is M = 86 at $x_1 = 6$, $x_2 = 4$, and $x_3 = 0$. Now solve P_{1-} using the *dual simplex algorithm* (add a **constraint** $x_1+a_1 = 6$). The a_1 row is the *only row* with a **negative** RHS. The only *negative* entry on the LHS of this row is in the s column, so pivot at *this position*. Again, the LP-relaxation P_{1-} has **solution** M = 93.7, at $x_1 = 6$, $x_2 = 4.7$, and $x_3 = 0$.



Now with problem P_{1+}, add the *constraint* $x_1 = 7$. Solve **graphically**: M = $11x_2+9x_3+49$ subject to $20x_2+10x_3 \le 84$, and $30x_2+60x_3 \le 131$. The *LP-relaxation* of P_{1+} has solution M = 95.5888... at $x_1 = 7$, $x_2 = 4.1444...$, and $x_3 = 0.1111...$. The **IP solution** of P_{1+} is M = 93 at $x_1 = 7$, $x_2 = 4$, and $x_3 = 0$. Note that we could solve P_{1+} using the *dual simplex algorithm* as well.



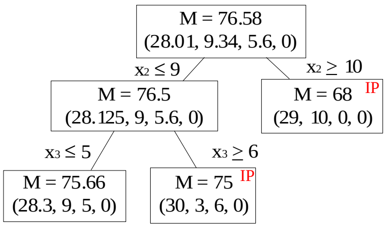
What we know so far is *summarised in the diagram on the left.* It follows that the IP solution is 93, 94 or 95. On the P_{1-} side, the **best** possible IP is 93. The next step is to *refine* P_{1+} using $x_2 \le 4$, and $x_2 \ge 5$. So P_{1+2-} is $x_1 \ge 7$ and $x_2 = 4$: M = $7x_1+9x_3+44$ subject to $x_1 \ge 7$, $40x_1+10x_3 \le 284$, and $10x_1+60x_3 \le 81$. We have a *very small feasible region*, as shown on the **right**. The red arrow points to the *LP relaxation solution*, where M is approximately 94.96, at $x_1 = 7.0565$, $x_2 = 4$, and $x_3 = 0.174$. The blue arrow is the IP solution (M = 93) as before. For P_{1+2+}, we have $x_1 \ge 7$, and $x_2 = 5$. The *second constraint* is $10x_1+60x_3 \le 51$, which is *incompatible* with $x_1 \ge 7$, so there is **no** solution. **<u>Conclusion</u>**: The IP solution is M = 93, at $\underline{x}$ = (7, 4, 0).



When doing these kinds of problems, get an *initial* solution first, e.g. M = 42.42857 at x_1 = 14.14, $x_2 = 0$, $x_3 = 28.28$, and $x_4 = 0$. Then add in more **constraints** to get an *integer solution*, e.g. branch out by letting $x_1 \le 14$, and $x_1 \ge 15$. If needed, go on to constrain *more* variables, and then later on, it may be necessary to **re-constrain** some variables, e.g. set $x_1 \le 0$, or set $x_1 \ge 1$. The use of **Lindo** in attempting these problems is always recommended. Draw a <u>branch and bound diagram</u> to keep track of the *important* variables.

8th February 2001

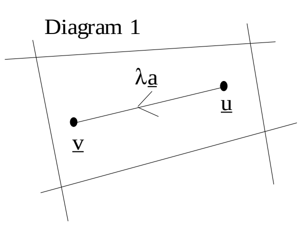
A Simpler Example

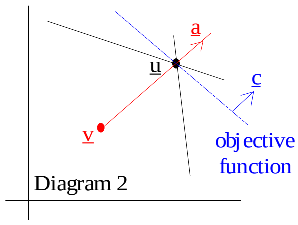


Maximise $2x_1+x_2+2x_3+x_4$ *subject* to $3x_1+x_2+x_3+5x_4 \leq 99$, $2x_1+x_2+6x_3+x_4 \leq 99$, and $x_1+7x_2+x_3+x_4 \leq 99$. We have a **depth-first** tree of LP problems, as shown on the *left*. Afterwards, we have an IP solution of 75, and there is <u>no need</u> to go further. An *interesting* example: $M = x_1+x_2+x_3+x_4$ *subject* to $x_1+2x_2+3x_3+4x_4 \leq 99$, $3x_1+2x_2+2x_3+3x_4 \leq 99$, and $5x_1+3x_2+x_3+x_4 \leq 99$. The *LP-relaxation solution from Lindo* is $M = 42.43$ at $(14^1/_7, 0, 28^2/_7, 0)$.

Now $(1,-2,1,0)$ is *perpendicular* to $(1,2,3,4)$, $(5,3,1,1)$, and $(1,1,1,1)$. There is a **line segment of points** at the boundary of the feasible region, **all** <u>giving</u> the LP max: $(k, 28^2/_7-2k, 14^1/_7+k, 0)$, with $k \in [0, 14^1/_7]$, so that $x_1+x_2+x_3+x_4 = 3\times14^1/_7 = 42^3/_7$. *Similarly*, there are 29 points at which the **IP** solution is obtained: $M = 42$ at $(0,27,15,0)+k(1,-2,1,0)$, for $k \in \{0,1,...,13\}$; and $(0,28,14,0)+k(1,-2,1,0)$, for $k \in \{0,1,...,14\}$.

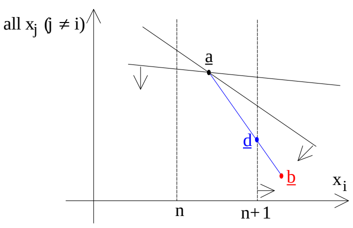
Consider the **situation** as shown in <u>Diagram 1</u>. Suppose that $\underline{v} = \underline{u}+\lambda\underline{a}$. Then $M_{\underline{u}} = \underline{c}\bullet\underline{u}$, and $M_{\underline{v}} = \underline{c}\bullet\underline{v} = \underline{c}\bullet\underline{u} + \lambda\underline{c}\bullet\underline{a}$; $M_{\underline{v}} = M_{\underline{u}}+\lambda(\underline{c}\bullet\underline{a})$. We can *choose* <u>a</u> so that $\underline{c}\bullet\underline{a} \geq 0$ (if **not**, take -<u>a</u> instead). *Suppose that* $M_{\underline{u}}$ is the *LP-max*. When $\lambda > 0$, $M_{\underline{v}} > M_{\underline{u}}$ *when* $\underline{c}\bullet\underline{a} \neq 0$, so that <u>v</u> is <u>**outside**</u> the feasible region. For $\lambda < 0$, $M_{\underline{v}} < M_{\underline{u}}$ when $\underline{c}\bullet\underline{a} \neq 0$, and <u>v</u> <u>**may**</u> be *inside* the feasible region.





<u>**Proposition**</u> (Winston, page 514): Let S be a *subproblem* of an IP problem P with solution at <u>a</u> in which the <u>basic</u> variable x_i has value $a_i \in (n, n+1)$. Let S_1 be the **subproblem** of S obtained by *adding* the constraint $x_i \geq n+1$. Then S_1 has an **optimal** solution to its LP-relaxation, with $x_i = n+1$.

<u>**Proof by Contradiction**</u>. Let <u>b</u> be a point where the *LP-max* of S_1 is obtained, with $x_i = b_i > n+1$. Let $k = (n+1-a_i/b_i-a_i)$ $(0 < k < 1)$, and let $\underline{d} = k\underline{b}+(1-k)\underline{a}$, so that $d_i = kb_i+(1-k)a_i = a_i+(n+1-a_i/b_i-a_i)(b_i-a_i) = n+1$. By **convexity**, <u>d</u> is **in** the feasible region for S, and *so* for S_1. The **values** of the objective function satisfy $M_{\underline{b}} \leq M_{\underline{d}} \leq M_{\underline{a}}$. Since $M_{\underline{b}}$ is *optimal* for S_1, it **follows** that $M_{\underline{b}} = M_{\underline{d}}$. **So** (either $\underline{b} = \underline{d}$ or) <u>d</u> is an *alternative optimal point* for S_1. <u>**End of Proof**</u>. Note: this was the case in the "*interesting*" example.



12th February 2001

Special Types of IP

Knapsack Problems: $\max(c_1x_1+...+c_nx_n)$ subject to $a_1x_1+...+a_nx_n \leq b$, with $x_i = 0$ or 1. The *significant feature* is that there is a single constraint. <u>**Interpretation**</u>: items 1...n may (or may not) be carried in a *knapsack*; c_i is the value (benefit) of item i; and a_i is the weight (cost) of item i. These *problems* may be solved by <u>branch-and-bound</u> techniques, *without* IP.

Fixed charge problems: max($c_1x_1+...+c_nx_n-d_1y_1-...-d_ny_n$) *subject to* $a_{11}x_1+...+a_{1n}x_n \le b_1$, ..., $a_{m1}x_1+...+a_{mn}x_n \le b_m$; $x_1 \le M_1y_1$, ..., $x_n \le M_ny_n$, $x_i \in \mathbf{Z}^{\ge 0}$, and $y_j = 0$ or 1, where the M_i are *large positive numbers*. **Interpretation**: x_i is the number of items of *product* i produced; d_i is a **fixed** charge incurred if $x_i > 0$; and the *constraints* $x_i \le M_iy_i$ force $y_i = 1$ when $x_i > 0$.

Location problems: min($x_1+...+x_n$) *subject to* $a_{11}x_1+...+a_{1n}x_n \ge 1$, ..., $a_{n1}x_1+...+a_{nn}x_n \ge 1$, where $x_i = 0$ or 1, and $a_{ij} = 0$ or 1. *Interpretation*: service centres are to be built in certain of the locations 1....n so that every location has a **service** centre at most d miles away. The constraint coefficient is $a_{ij} = 1$ if distance(location i, location j) $\le$ d. Again, there are *combinatorial* algorithms for this problem.

Example: A county has 6 cities, and needs to build *fire stations* so that at least one fire station is within 15 minutes driving time of each city. The times to drive are as shown on the right. Formulate an **IP** problem to solve this problem. A: For each city, define variables x_1, ..., x_6, where $x_i = 1$ if a fire station is *built* in city i, and zero **otherwise**. The total number of fire stations that are built is given by $x_1+x_2+x_3+x_4+x_5+x_6$, and we want to **minimise** this, the objective function.

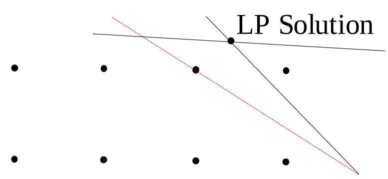
	1	2	3	4	5	6
1	0	10	20	30	30	20
2	10	0	25	35	20	10
3	20	25	0	15	30	20
4	30	35	15	0	15	25
5	30	20	30	15	0	14
6	20	10	20	25	14	0

City **1** can reach *cities 1 and 2* within 15 minutes. For city **2**, it is 1, 2 and 6. **3**: 3,4. **4**: 3,4,5. **5**: 4,5,6. **6**: 2,5,6. To ensure that *at least one fire station* is within 15 minutes of city 1, we add the constraint $x_1+x_2 \ge 1$. This constraint ensures that $x_1 = x_2 = 0$ is impossible, so that at least **one** fire station will be built within 15 minutes of city 1. *Similarly*, the constraint $x_1+x_2+x_6 \ge 1$ ensures that at least one fire station will be located within 15 minutes of city 2. We have **four** other similar constraints, and combining these six constraints with the *objective* function gives us our 0-1 IP. One optimal solution to this IP is as follows: z = 2 at $x_2 = x_4 = 1$, and $x_1 = x_3 = x_5 = x_6 = 0$. Thus we build *fire stations* in cities 2 and 4 to solve the problem.

Piecewise-linear function: If a piecewise linear function, f(x), with breaks at $r_1,...,r_k$, occurs in *place* of a linear function in an LP (as the *objective* function or as a constraint), add the variables $z_1,....,z_k$ and $y_1,....,y_{k-1}$; *replace* f(x) by $z_1f(r_1)+...+z_kf(r_k)$; and add the *constraints* $z_1 \le y_1$, $z_2 \le y_1+y_2$, ..., $z_{k-1} \le y_{k-2}+y_{k-1}$, $z_k \le y_{k-1}$, $y_1+...+y_{k-1} = 1$, $z_1+...+z_k = 1$, $y_i \in \mathbf{Z}^{\ge 0}$, $z = z_1r_1+...+z_kr_k$, $y_i = 0$ or 1, and $z_j \ge 0$.

12th February 2001

The Cutting Plane Algorithm



The *geometrical idea* is as shown on the right. The new "cut" is an **extra** constraint which excludes the LP-relaxation solution **Method:** Take each row (in turn) of the *solution* tableau which has a fractional basic variable. Rewrite each coefficient in turn as $a_{ij} = \lfloor a_{ij} \rfloor + (a_{ij} - \lfloor a_{ij} \rfloor)$, where $\lfloor a_{ij} \rfloor \in \mathbf{Z}$, and $(a_{ij} - \lfloor a_{ij} \rfloor) \in [0,1)$, e.g. -5¾ = -6 +¼. Transfer the **integers** to the LHS, and the **fractions** to the RHS. Add the *constraint* RHS ≤ 0.

Example 1: $M = 2x_1+3x_2$ *subject to* $3x_1+8x_2 \le 41$, $6x_1+7x_2 \le 51$, and $8x_1+5x_2 \le 61$. The final solution tableau is as shown on the right. **Rewrite** R: $x_1 + (-1+{}^{20}/_{27})s_1 + (0+{}^{8}/_{27})s_2 = 4+{}^{13}/_{27} \Rightarrow x_1-s_1\ (+0s_2) - 4 = -{}^{20}/_{27}s_1 - {}^{8}/_{27}s_2 + {}^{13}/_{27}$. RHS ≤ 0, so that $-20s_1-8s_2 \le -13$. We can't *plot this immediately* in the (x_1,x_2)-plane, but, eliminating s_1 and s_2, we find that this constraint is *equivalent* to $x_1+2x_2 \le 11^1/_4$. *Eliminating* gives $M = 19.2$ at $x_1 = 4.65$, $x_2 \approx 3.35$, and $s_3 = 7.3$. Similarly for R_1: $x_2-s_2-3 = {}^4/_9-{}^2/_9s_1-{}^5/_8s_2$ gives $M = 19$ at $(5, 3)$, $s_1 = 2$, and $s_3 = 6$. Now R_3's *constraint* is $-7s_2+s_3-7 = {}^{25}/_{27}-{}^{26}/_{27}s_1-{}^7/_8s_2$. So $s_3-7s_2-7 = -{}^{26}/_{27}s_1-{}^7/_8s_2+{}^{25}/_{27}$: **add in** $-208s_1-189s_2 \le -200$.

	M	x_1	x_2	s_1	s_2	s_3	a_2	rhs	basic
R_0	1			$^4/_{27}$	$^7/_8$			$19^8/_{27}$	
R_1			1	$^6/_{27}$	$-^3/_8$			$3^{12}/_{27}$	
R_2		1		$-^7/_{27}$	$^8/_{27}$			$4^{13}/_{27}$	
R_3				$^{26}/_{27}$	$-^{49}/_8$	1		$7^{25}/_{27}$	
				-20	-8		1	-13	
				$-^1/_{135}$	$-^7/_{65}$			ratios	

Past Paper Question: The LP-relaxation of the IP problem P: max $(3x_1+4x_2)$ subject to $2x_1+5x_2 \le 30$, $4x_1+3x_2 \le 50$, and $5x_1+2x_2 \le 60$, with $x_1, x_2 \ge 0$, has solution tableau as shown on the right. Apply the *cutting plane algorithm* to the x_2-row, and solve the **resulting** LP-relaxation using the *simplex* method. A: The x_2-row gives $x_2+{}^2/_7s_1-{}^1/_7s_2 = {}^{10}/_7$, so that $x_2-s_2-1 = {}^3/_7-{}^2/_7s_1-{}^6/_7s_2$. Therefore, $3-2s_1-6s_2 \le 0$; $-2s_1-6s_2 \le -3$. Add this *constraint* to the tableau (as shown in yellow), and solve as shown to give *solution* $M = 39^3/_4$ at $x_1 = 11^1/_4$, $x_2 = 1^1/_2$, $s_1 = 0$, $s_2 = ½$, $s_3 = 5^1/_4$, and $a_2 = 0$.

M	x_1	x_2	s_1	s_2	s_3	a_2	rhs	basic
2			1	1			80	M = 40
		7	2	-1			10	$x_2 = {}^{10}/_7$
	14		-3	5			160	$x_1 = {}^{80}/_7$
			29	-95	70		320	$s_3 = {}^{32}/_7$
			-2	-6		1	-3	
			$-^1/_2$	$-^1/_6$				ratio
12	0	0	4	0	0	1	477	$6R_0+R_4$
0	0	42	14	0	0	-1	63	$6R_1-R_4$
0	84	0	-28	0	0	5	945	$6R_2+5R_4$
0	0	0	364	0	420	-95	2205	$6R_3-95R_4$
0	0	0	2	6	0	-1	3	$-R_4$

15th February 2001

Knapsack Problems

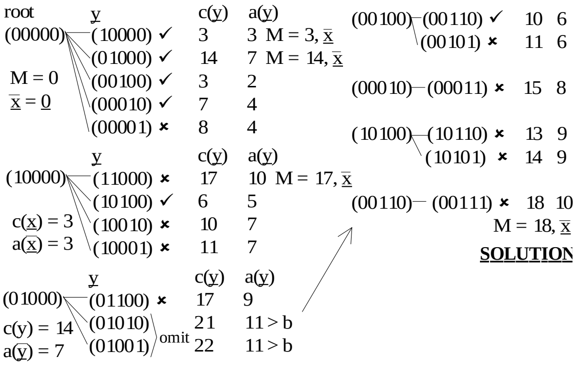
Items x_i, costs a_i, values c_i, and $x_i = 0$ or 1 according to if the i^{th} item has or has not been taken. **Problem**: max $M = \Sigma c_i x_i$ s.t. $\Sigma a_i x_i \le b$. **Method**: Represent a selection by a *vector* $\underline{x} = (x_1,...,x_n) \in \{0,1\}^n$. Let $l(\underline{x})$ denote the *position* of the last 1 in $\underline{x}$, e.g. $l(1011000) = 4$, and $l(0000000) = 0$. **Set** $M = 0$, and set $\bar{x} = 0$. Construct a tree with *root 0* and vertices $\underline{x}$.

Pick the **first** vertex $\underline{x}$ not marked with a "✓" or a "×". Add edges from $\underline{x}$ down to the $(n-l(x))$ vertices $\underline{y}$ obtained by replacing 0 with 1 in *positions* $j \in \{l(x)+1,...,n\}$, e.g. $\underline{x} = (1011000)$ *goes* to (1011100), (1011010), and (1011001).

Calculate the **value** $c(\underline{y}) = c(\underline{x})+c_j = \Sigma c_i y_i$, and the *cost* $a(\underline{y}) = a(\underline{x})+a_j = \Sigma a_i y_i$. Mark $\underline{y}$ with a **cross** if $a(\underline{y}) > b$. If $c(\underline{y}) > M$, and if $a(\underline{y}) \le b$, then a **better** solution has been found, so set $M = c(\underline{y})$, and set $\bar{x} = \underline{y}$. (Or, if $c(\underline{y}) = M$, and if $a(\underline{y}) < a(\bar{x})$). Now mark the **vertex** $\underline{y}$ with a **cross** if $l(\underline{y}) = n$, or if $b-a(\underline{y}) \le \min(a_j)$ for $l(\underline{y}) < j \le n$. Mark $\underline{x}$ with a *tick* (which says that $\underline{x}$ has been **processed**). Repeat until *all vertices* are marked.

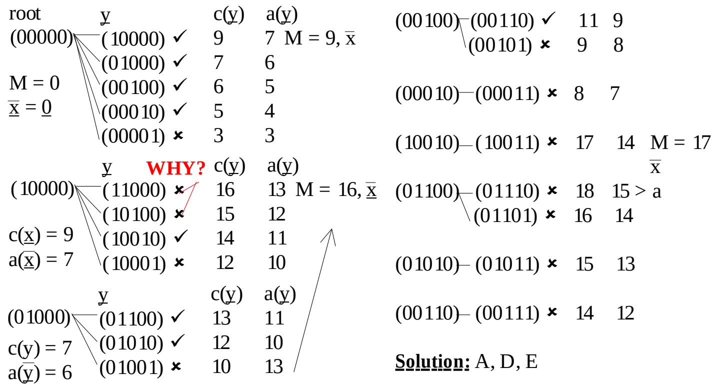
Example: A machine in a factory makes items A-E. The costs a_i and values c_i are shown in the table opposite, with b = *days of possible operation* = 10. We start at the **root** (00000), adding branches to (10000): (01000), (00100), (00010), and (00001). We analyse the added costs and values (as shown), and show if a **better** value is found. At the end, we have *solution*: make items C, D and E using 10 days, with value £18. <u>IP formulation</u>: $\max(3x_1+14x_2+3x_3+7x_4+8x_5)$ s.t. $3x_1+7x_2+2x_3+4x_4+4x_5 \leq 10$, with $x_i \in \{0,1\}$, and $1 \leq i \leq 5$.

i	A	B	C	D	E
a_i	3	7	2	4	4
c_i	3	14	3	7	8



Past Paper Question: A machine makes items A-E as shown in the table. The machine is available for 14 days, and *at most one* of each item is made by the machine. Use a <u>branch and bound</u> method to **maximise** the value of the items produced. A: as shown on the right.

item	A	B	C	D	E
time	7	6	5	4	3
value	9	7	6	5	3



19th February 2001

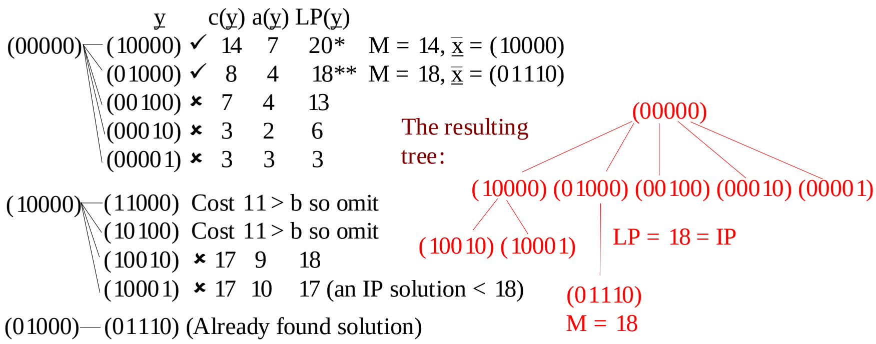
Theorem: Let P be a *knapsack IP problem*, with n items x_i having **costs** c_i and **values** v_i, ordered so that $v_1/c_1 \geq v_2/c_2 \geq \ldots \geq v_n/c_n$, and with *constraint* $\Sigma_{i=1}^{n} c_i x_i \leq b$. The *associated LP-relaxation* has optimal solution $v_1+v_2+\ldots+v_{i-1}+\lambda v_i$, where $c_1+\ldots+c_{i-1}+\lambda c_i = b$ $(0 \leq \lambda < 1)$.

<u>The Modified Knapsack Algorithm</u>

Render items by *decreasing* $({}^{\text{value}}/_{\text{cost}})$. Calculate the *LP-relaxation solution* at each vertex. Mark a vertex with a **cross** if the LP-relaxation solution is ≤ the stored optimal value. If the LP-relaxation is an **IP** better than the stored value, then store M and $\bar{\mathbf{x}}$. **<u>The calculation of the LP-relaxation solution:</u>** Take the sum of the values *chosen so far*; keep taking the next item while cost allows; and if the **next** item is too expensive, then take that *fraction* of the next item to use up the maximum cost.

#	1	2	3	4	5
label	B	E	D	C	A
cost a_i	7	4	4	2	3
value c_i	14	8	7	3	3
c_i/a_i	2	2	$1^3/_4$	$1^1/_2$	1

Example: Re-order the items from a *previous* question as shown on the left. Start at M = 0, $\bar{\mathbf{x}}$ = 0, and the vector (00000) as *normal*.



For the first part of the tree, at *, $7+{}^3/_4(4) = 10$ (*max cost*), and $14+{}^3/_4(8) = 20$ (*LP solution*). At **, 4+4+2 = 10, and 8+7+3 = 18.

Knapsack Problems: Multiple Items

Type	A	B	C
Weight	10	8	5
Nutrition	24	20	11
Max #	4	3	9

Rank	1	2	3
Type	B	A	C
Weight	8	10	5
Nutrition	20	24	11
Max #	3	4	9

It may happen that *several copies of the same item* are available. **Example**: A hill walker has supplies of *3 types of food* (as shown on the left). Select the **most** nutritious items, up to a maximum weight of 50. **IP formualtion**: x_i = the number of type i taken, c_i = weight, v_i = nutrition. Problem: $\max(24x_A+20x_B+11x_C)$ subject to $10x_A+8x_B+5x_C \leq 50$, $x_A \leq 4$, $x_B \leq 3$, and $x_C \leq 9$, where $x_i \in \mathbf{Z}^{\geq 0}$. Compare ratios: ($v_i/c_i$, or c_i/a_i). For *type A*, the ratio is 2.4 (**2nd** place). For *type B*, the ratio is 2.5 (**1st** place). For *type C*, the ratio is 2.2 (**3rd** place). So we *reorder* as shown in the second table. Now construct the usual tree, with nodes (x_1, x_2, x_3). As before, *each LP max* is easily calculated as shown on the **right**. Note: ***** if IP. For the *initial* problem, 3×8+2.6×10 = 50, so that LP_{max} = 3×20+2.6×24 = 60+62.4 = 122.4. **Solution**: take *2 of A, 3 of B, and 1 of C*, so that the weight is 20+24+5 = 49, and that the nutrition is 48+60+11 = 119. You *can* draw a tree.

	Vertex	Processed	Cost	Value	LPmax
(000)	(100)	✓	8	20	122.4
	(010)	×	10	24	96+22 = 118 * (0 4 2)
	(001)	×	5	11	99
(100)	(200)	✓	16	40	122.4
	(110)	✓	18	44	20+96+(2/5)11 = 120.4
	(101)	×	13	31	20+8.4×11 = 112.4
(200)	(300)	✓	24	60	122.4
	(210)	✓	26	64	40+3.4×24 = 121.6
	(201)	×	21	51	40+6.8×11 = 114.8
(300)	(310)	✓	34	84	122.4
	(301)	×	29	71	60+5.2×11
(310)	(320)	✓	44	108	122.4
	(311)	✓	39	95	95+2.2×11 = 119.2
(320)	(321)	×	49	119	121.2
	etc.				

22nd February 2001

Past Paper Question

The table shown shows the *estimated time from town i to town j*. We want to minimise the number of depots needed in order that any site can be **reached** within 30 minutes. Formulate an IP for this problem, and, by inspection, write down a solution when *(i) 3 depots are needed, and (ii) 2 depots are needed*. Now (b) formulate an IP for an alternative problem: Estimates for the number n_i of incidents per year in the 8 districts are given in the **second** table. We want to build 4 depots so as to *minimise the average response time*. The problem is to determine which towns should be used.

r_{ij}	1	2	3	4	5	6	7	8
1	10	45	15	20	70	30	60	65
2	45	5	55	30	25	25	15	40
3	15	55	5	30	60	40	65	70
4	20	30	30	10	65	15	40	45
5	70	25	60	65	5	50	30	55
6	30	25	40	15	50	5	25	30
7	60	15	65	40	30	25	10	25
8	65	40	70	45	55	30	25	10

i	1	2	3	4	5	6	7	8
n_i	17	2	11	14	8	5	18	16

A: Let x_i = 1 if *there is a depot in town i*, and let x_i = 0 otherwise ($1 \leq i \leq 8$). The 8 constraints are as follows: $x_1+x_3+x_4+x_6 \geq 1$, $x_2+x_4+x_5+x_6+x_7 \geq 1$, $x_1+x_3+x_4 \geq 1$, $x_1+x_2+x_3+x_4+x_6 \geq 1$, $x_2+x_5+x_7 \geq 1$, $x_1+x_2+x_4+x_6+x_7+x_8 \geq 1$, $x_2+x_5+x_6+x_7+x_8 \geq 1$, and $x_6+x_7+x_8 \geq 1$. There is *some redundancy*: (8) ⇒ (6) and (7); (5) ⇒ (2); and (3) ⇒ (1) and (4). **Problem**: Minimise $\Sigma_{i=1}^{8} x_i$ such that $x_i \in \{0,1\}$, with $1 \leq i \leq 8$, and such that $x_1+x_3+x_4 \geq 1$, $x_2+x_5+x_7 \geq 1$, and $x_6+x_7+x_8 \geq 1$. *3 depots are needed:* lots of solutions, e.g. $x_1 = x_2 = x_6 = 1$, the rest 0. *2 depots are needed:* 3 solutions: $x_1 = x_7 = 1$, $x_3 = x_7 = 1$, and $x_4 = x_7 = 1$. (b) Let y_{ij} =1 if town i *responds to town j*, and let y_{ij} = 0 otherwise. The **annual total response time** is given by $R = \Sigma_{i=1}^{8}\Sigma_{j=1}^{8} n_i r_{ij} y_{ij}$. We want to *minimise R*. Now $y_{ij} = 1 \Rightarrow x_i = 1$, so that $y_{ij} \leq x_i$. So a **suitable** IP is as follows: min. R s.t. $\Sigma x_i = 4$, $\Sigma y_{ij} = 1$ (8 *conditions*), $y_{ij} \leq x_i$ (64 *conditions*), and x_i, $y_{ij} \in \{0,1\}$, so that we have 72 **variables**, and 64+(2×(1+8)) = 82 **constraints** ("u = v" is *implemented* as $u \leq v$ and $u \geq v$).

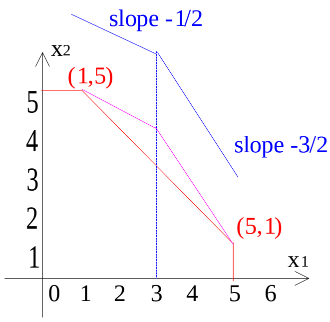
Plumber Problem

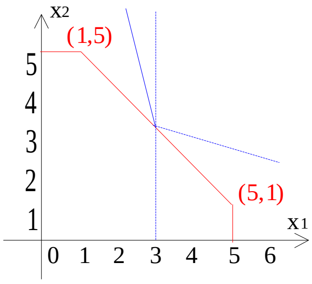
Consider that pipes are *available in 6' and 12' lengths*. We **need** 8 at 4', 5 at 5', and 3 at 7' — and want to minimise "waste". *Cutting* patterns: (1) 6' into 4' and 2' waste. (2) 6' into 5' and 1' waste. (3) 12 = 5+7, (4) 12 = 4+7 (+1 waste), (5) 12 = 5+5 (+2 waste), (6) 12 = 4+5 (+3 waste), and (7) 12 = 4+4+4. Let x_i be the number of times pattern (i) is used ($1 \le i \le 7$). The *waste function* is $2x_1+x_2+x_4+2x_5+3x_6$. The number of 4' *lengths* is $x_1+x_4+x_6+3x_7$, 8 needed. The number of 5' *lengths* is $x_2+x_3+2x_5+x_6$, 5 needed. The number of 7' *lengths* is x_3+x_4, 3 needed. **IP 1:** Min. waste s.t. $x_1+x_4+x_6+3x_7 \ge 8$, $x_2+x_3+2x_5+x_6 \ge 5$, and $x_3+x_4 \ge 3$, with $x_i \in \mathbf{Z}^{\ge 0}$. *Solution*: $x_3 = 5$, $x_7 = 3$, the rest 0. **IP 2**: min $6(x_1+x_2) + 12(x_3+x_4+x_5+x_6+x_7)$ = *the total length bought*, with the **same** constraints. Solution: $x_3 = 3$, $x_5 = 1$, and $x_7 = 3$: 84' used with 2' **waste** and an extra 4' *length*.

26th February 2001

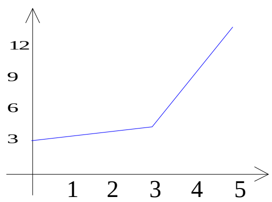
Piecewise-Linear Objective Function

Example: Maximise $M = x_1+2x_2+3$ when $x_1 \le 3$, and $3x_1+2x_2-3$ when $x_1 \ge 3$, such that $x_1 \le 5$, $x_2 \le 5$, and $x_1+x_2 \le 6$. (*Note*: $x_1, x_2 \ge 0$, and $M = 2x_2+6$ when $x_1 = 3$). The **graphical** solution is as shown on the right, where M is *maximum* at both (1,5) and (5,1) — with M = 14.



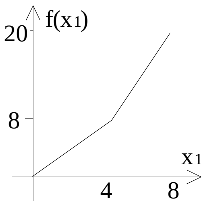


Alternative Example: $M = 3x_1+2x_2-3$ when $x_1 \le 3$, and x_1+2x_2+3 when $x_1 \ge 3$, with the *same constraints*. Maximum: M = 12 at (3,3) as shown on the left. This shows that M **need not occur** at a corner of a feasible region, but at the *intersection of the boundary* with (in this case) $x_1 = 3$. When the number of variables is > 2, **convert** the problem to a mixed IP.

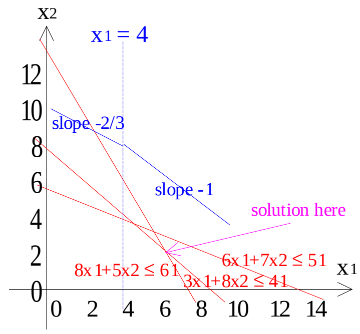


Step 1: Find the *upper bound for* x_1. In the example, when $x_2 = 0$, the constraints are $x_1 \le 5$, and $x_1 \le 6$, so that $0 \le x_1 \le 5$. **Step 2**: Write M as $f(x_1)+2x_2$, where $f(x_1) = x_1+3$ when $x_1 \le 3$, and $3x_1-3$ when $x_1 \ge 3$. **Step 3** Introduce *variables* z_1, z_2, z_3, y_1, y_2 (**all** ≥ 0), with $y_1, y_2 \in \{0,1\}$, $y_1+y_2 = 1$, $z_1 \le y_1$, $z_2 \le y_1+y_2$, $z_3 \le y_2$, and $z_1+z_2+z_3 = 1$. It *follows* that either (A) $y_1 = 1$, $y_2 = 0$, ($\Rightarrow$) $z_3 = 0$, $0 \le z_1,z_2 \le 1$, and $z_1+z_2 = 1$; or (B) $y_1 = 0$, $y_2 = 1$, ($\Rightarrow$) $z_1 = 0$, $0 \le z_2,z_3 \le 1$, and $z_2+z_3 = 1$. **Step 4:** *Write* $x_1 = 0z_1+3z_2+5z_3 \in [0,5]$, *and* $f(x_1) = f(0)z_1+f(3)z_2+f(5)z_3 = 3z_1+6z_2+12z_3$. *Check*: in case (A), $x_1 = 3z_2 \in [0,3]$; $f(x_1) = 3(z_1+z_2)+3z_2 = 3+x_1$. **OK**. In *case* (B), $x_1 = 3z_2+5z_3 \in [3,5]$; $f(x_1) = 6z_2+12z_3 = 6+6z_3 = 3(3+2z_3)-3 = 3x_1-3$. **OK**. **Conclusion**: We have a new mixed IP: max $M = 3z_1+6z_2+12z_3+2x_2$ *subject to* $x_1 \le 5$, $x_2 \le 5$, $x_1+x_2 \le 6$, $z_1 \le y_1$, $z_2 \le y_1+y_2$, $z_3 \le y_2$, $y_1+y_2 = 1$, $z_1+z_2+z_3 = 1$, $x_1 = 0z_1+3z_2+5z_3$, x_1, x_2, (y_1, y_2), and $z_1, z_2, z_3 \ge 0$, *with* $y_1, y_2 \in \{0,1\}$.

Tutorial: *Maximise* ($\underline{c}\bullet\underline{x}$) such that $A\underline{x} \le \underline{b}$, *where* $\underline{c}\bullet\underline{x} = 2x_1+3x_2$ **when** $x_1 \le 4$, and $3x_1+3x_2-4$ *when* $x_1 \ge 4$, with $A = (^3{}_6{}_8\ ^8{}_7{}_5)$, and $b = (^{41}{}_{51}{}_{61})$. So we have *constraints* $3x_1+8x_2 \le 41$, $6x_1+7x_2 \le 51$, and $8x_1+5x_2 \le 61$. **Step 1:** The *constraints* are $3x_1 \le 41$, or $x_1 \le 13^2/_3$; $6x_1 \le 51$, or $x_1 \le 8^1/_2$; and $8x_1 \le 61$, or $x_1 \le 7^5/_8$. Therefore, $0 \le x_1 \le 7^5/_8$. But *take an integer upper bound*, so that $0 \le x_1 \le 8$.



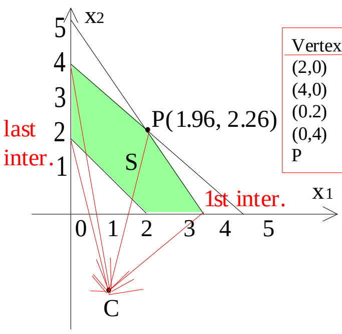
Step 2: $M = f(x_1)+3x_2$, where $f(x_1) = 2x_1$ when $x_1 \le 4$, and $3x_1-4$ *when* $x_1 \geq 4$. **Step 3:** Introduce *Variables* .... (exactly as before). It follows that .... **Step 4:** $x_1 = 0z_1+4z_2+8z_3$; $f(x_1) = 0z_1+8z_2+20z_3$; and $M = 8z_2+20z_3+3x_2$, *subject to* $3x_1+8x_2 \le 41$, etc., $y_1+y_2 = 1$, $z_1+z_2+z_3 = 1$, $z_1 \le y_1$, $z_2 \le y_1+y_2$, $z_3 \le y_2$, $y_1, y_2 \in \{0,1\}$, and $x_1 = 4z_2+8z_3$. Note that we can **write** this as 2 LPs: (A) *with* $y_1 = 1$, $y_2 = 0$, *and* (B) with $y_1 = 0$, $y_2 = 1$.



We can also solve this problem **graphically** (as shown on the right). Intuitively, we move the slopes down vertically, and find *two candidate solutions*. A solution is at the intersection of two lines. We find the *co-ordinates* by usual methods — and *then* find M.

1st March 2001

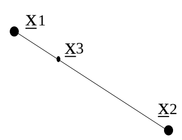
Linear Fractional Programming



Vertex	M
(2,0)	1
(4,0)	0.81
(0.2)	1.86
(0,4)	1.162
P	1.134

An LFP has a *natural linear objective function*, linear *constraints*, and some *non-negative decision variables*. **Example**: *maximise* $(3x_1+4x_2+5/5x_1+3x_2+1)$ *subject to* $8x_1+9x_2 \le 36$, $7x_1+5x_2 \le 25$, and $x_1+x_2 \geq 2$. We see the diagrammatic representation on the left. Put $z = 1/5x_1+3x_2+1$ (*), $y_1 = x_1z$, and $y_2 = x_2z$. The objective *function becomes* $M = (3x_1+4x_2+5)z = 3y_1+4y_2+5z$. *Note* that, on S, $x_1 \geq 0$, and $x_2 \geq 0$, so that $z \geq 0$, $y_1 \geq 0$, and $y_2 \geq 0$. Now $z \geq 0 \Rightarrow$ the *first* constraint can be written as $8x_1z + 9x_2z \le 36z$, i.e. $8y_1+9y_2-36z \le 0$. *Similarly for constraints 2 and 3.* **Further**, (*) can be written as $5y_1+3y_2+z = 1$. Conclusion: The LFP can be *converted* to an LP: $\max(3y_1+4y_2+5z)$ s.t. $8y_1+9y_2-36z \le 0$, $7y_1+5y_2-25z \le 0$, $y_1+y_2-2z \geq 0$, and $5y_1+3y_2+z = 1$.

General Case: $\text{Max}(f = \underline{p}.\underline{x}+\alpha/\underline{q}.\underline{x}+\beta)$ subject *to* $A\underline{x} \le b$, and $\underline{x} \geq 0$. Note that $q.\underline{x}+\beta \neq 0$ on S, *and that* the matrix $(^{\mathbf{p}}_{\mathbf{q}}\ {}^{\alpha}_{\beta})$ has *rank* 2. Convert to an LP: Put $z = 1/\underline{q}.\underline{x}+\beta$, and put $\underline{y} = z\underline{x}$, so that $\underline{q}.\underline{y}+\beta z = 1$. The *objective function becomes* $f = (\underline{p}.\underline{x}+\alpha)z = \underline{p}.\underline{y}+\alpha z$, and the i^{th} *constraint* becomes $\Sigma_{j=1}^{n} a_{ij}x_jz \le b_iz$; $\Sigma_{j=1}^{n} a_{ij}y_j - b_iz \le 0$. So the LP is *max*$(\underline{p}.\underline{y}+\alpha z)$ *subject to* $A\underline{y}-z\underline{b} \le 0$, $\underline{y} \geq 0$, $z \geq 0$, and $\underline{q}.\underline{y}+\beta z = 1$.



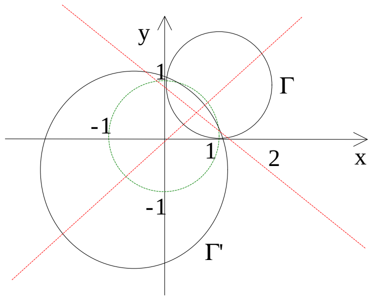
Lemma: *Either* $\underline{q}.\underline{x}+\beta > 0$ on S, or $\underline{q}.\underline{x}+\beta < 0$ on S. **Proof:** $\underline{x}_1, \underline{x}_2 \in S$, with $\underline{q}.\underline{x}_1+\beta = c_1 > 0$, and $\underline{q}.\underline{x}_2+\beta = c_2 < 0$. **Let** $\underline{x}_3 = (c_2/c_2-c_1)\underline{x}_1 - (c_1/c_2-c_1)\underline{x}_2$. Now $\underline{q}.\underline{x}_3 + \beta = (c_2/c_2-c_1)(c_1-\beta) - (c_1/c_2-c_1)(c_2-\beta) + \beta = 0$. By *convexity*, $\underline{x}_3 \in S$, so we have a **contradiction**. If $\underline{q}.\underline{x}+\beta < 0$ for all $\underline{x} \in S$, *replace* f by $-(\underline{p}.\underline{x}+\alpha)/-(\underline{q}.\underline{x}+\beta)$ — so that we can assume that the *denominator* is always +ve.

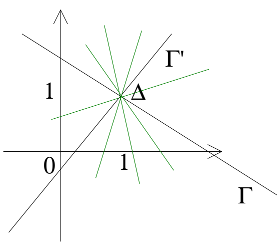
Proposition: if the LFP is *rewritten* as an LP, and if $\bar{\mathbf{y}}$ and $\bar{z}$ gives an *optimal* solution for the LP, then $\bar{\mathbf{x}} = (1/\bar{z})\bar{\mathbf{y}}$ is an *optimal* solution for the LFP. **Proof:** $(\bar{\mathbf{y}}, \bar{z})$, with $\bar{z} > 0$ *optimal* for LP $\Rightarrow \underline{p}.\underline{y}+\alpha z \le \underline{p}.\bar{\mathbf{y}}+\alpha\bar{z}$ for *all* $(\underline{y}, z)$ feasible. **Hence** $(\underline{p}.\underline{x}+\alpha/\underline{q}.\underline{x}+\beta) \le (\underline{p}.\bar{\mathbf{x}}+\alpha/\underline{q}.\bar{\mathbf{x}}+\beta)$ for all *feasible* $\underline{x}$, where $\underline{x} = \underline{y}/z$, and $\bar{\mathbf{x}} = \bar{\mathbf{y}}/\bar{z}$. **Therefore,** $\bar{\mathbf{x}} = \bar{\mathbf{y}}/\bar{z}$ is *optimal* for the LFP. Graphical Method: Solve $3x_1+4x_2+5 = 0$; $5x_1+3x_2+1 = 0$, i.e. $x_1 = 1$, $x_2 = -2$ (C). Refer *back to the diagram.*

5th March 2001

One-Parameter Families of Curves and Surfaces

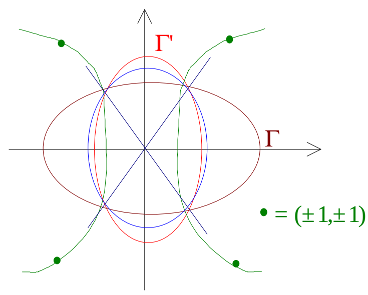
Let Γ and Γ' be surfaces (*curves*) with equations $f(\underline{x}) = 0$, and $f'(\underline{x}) = 0$ ($\underline{x} \in \mathbf{R}^n$). The surface $\Gamma_{\lambda/\mu}$ with *equation* $\lambda f(\underline{x}) + \mu f'(\underline{x}) = 0$ passes through the *surface* Δ (of *curves* or *points*) in which Γ and Γ' intersect. The **surfaces** $\Gamma_{\lambda/\mu}$ are said to form a "<u>pencil of surfaces</u>". **<u>Example 1</u>**: A pencil of *circles:* Γ: $(x-1)^2+(y-1)^2 = 1$; Γ': $(x+1)^2+(y+1)^2 = 5$. Now Γ and Γ' *intersect* at Δ = {(1,0), (0,1)}. $\Gamma_{\lambda/\mu}$ has *equation* $\lambda[(x-1)^2+(y-1)^2-1] + \mu[(x+1)^2+(y+1)^2-5] = 0$, which can be *arranged* as $\{x-({}^{\lambda-\mu}/_{\lambda+\mu})\}^2 + \{y-({}^{\lambda-\mu}/_{\lambda+\mu})\}^2 = {}^{\lambda^2-2\lambda\mu+5\mu^2}/_{(\lambda+\mu)^2}$, which is a circle with *centre* on the line y = x, and this line is **perpendicular** to the line joining (1,0) and (0,1).

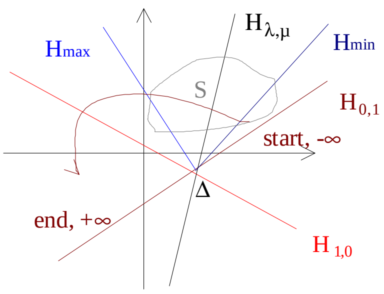




<u>Notes</u>: $\Gamma_{\lambda/0} = \Gamma$; $\Gamma_{0/\mu} = \Gamma'$; $\Gamma_{1/1}$ has *equation* $x^2+y^2 = 1$; and $\Gamma_{-1/1}$ is *undefined* (or is the **line** x+y = 1). **<u>Example 2</u>**: Γ is the line 3x+4y = 7, and Γ' is the line 4x-3y = 1. Therefore, $\Gamma_{\lambda/\mu}$ is $(3\lambda+4\mu)x + (4\lambda-3\mu)y = 7\lambda+\mu$. **<u>Example 3</u>**: Γ and Γ' are *ellipses*: $x^2+3y^2 = 4$, and $3x^2+y^2 = 4$. The **pencil** is set the set of conics $\lambda(x^2+3y^2-4) + \mu(3x^2+y^2-4) = 0$, or $(\lambda+3\mu)x^2 + (3\lambda+\mu)y^2 = 4(\lambda+\mu)$.

Consider the *curve* through (3,5): $\lambda(9+75-4) + \mu(27+25-4) = 0$; $80\lambda+48\mu = 0$; $5\lambda+3\mu = 0$. *Choose* λ = 3. Now μ = -5, so that $-12x^2+4y^2 = -8$, or $3x^2-y^2 = 2$. Γ: $({}^x/_2)^2+({}^y/_{2/\sqrt{3}})^2 = 1$ (dark green in the diagram). Find a member of the pencil *which is a circle*. Here, we need $\lambda+3\mu = 3\lambda+\mu \Rightarrow \mu = \lambda$. This gives $4\lambda x^2+4\lambda y^2 = 8\lambda$; $x^2+y^2 = 2$ (blue). Now find a member *passing through (0,0):* $0x^2+0y^2 = 4(\lambda+\mu) \Rightarrow \mu = -\lambda$, so that $-2\lambda x^2+2\lambda y^2 = 4(0)$; $x^2 = y^2$ (dark blue).



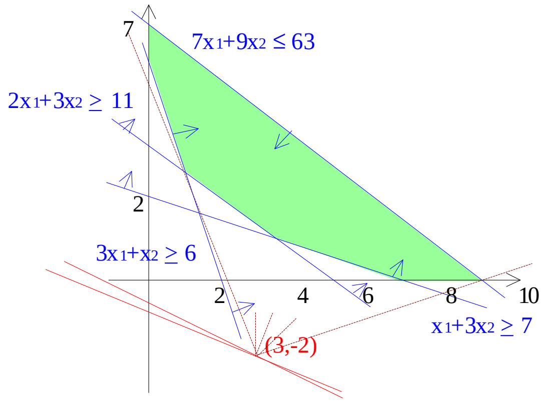


<u>Apply this to our LFP:</u> (F1): max(f: $\underline{p}.\underline{x}+\alpha/\underline{q}.\underline{x}+\beta$) ($\underline{p}$ and $\underline{q}$ are *Linearly Independent*) such that $\underline{x} \geq \underline{0}$, $A\underline{x} \leq \underline{b}$, and $\underline{q}.\underline{x}+\beta > 0$ on S. The *equations* $\underline{p}.\underline{x}+\alpha = 0$ and $\underline{q}.\underline{x}+\beta = 0$ are equations of *hyperplanes* (H and H') of **dimension** (n-1) when $\underline{x} \in \mathbf{R}^n$. They intersect in a *hyperplane* Δ on dimension (n-2), and $S \cap \Delta = \phi$. The *pencil of hyperplanes* $H_{\lambda,\mu}$: $\lambda(\underline{p}.\underline{x}+\alpha) + \mu(\underline{q}.\underline{x}+\beta) = 0$ all *contain* Δ.

The **objective function** f has *constant* value on $H_{\lambda,\mu}$, namely $-({}^\mu/_\lambda)$. Sketch a *section* of $\mathbf{R}^n$ **perpendicular** to Δ: as shown above. **As** $(-{}^\mu/_\lambda)$ *increases* from -∞ to +∞, $H_{\lambda,\mu}$ **first** meets S at some H_{min}, and **last** meets S at some H_{max}. These *two values* of $-{}^\mu/_\lambda$ provide the *optimal* values for f. Put μ = 1, and λ = +ε gives $-{}^\mu/_\lambda \doteq -\infty$. Further, λ = -ε gives $\doteq +\infty$. So λ goes from +ε to +∞, and then from -∞ to -ε.

<u>Exercise Sheet C1:</u> Convert the *linear fractional program* (F1): **opt** ($f = 2x_1+5x_2+4/3x_1+7x_2+5$) *subject to* $7x_1+9x_2 \leq 63$, $3x_1+x_2 \geq 6$, $2x_1+3x_2 \geq 11$, $x_1+3x_2 \geq 7$, and $x_1, x_2 \geq 0$ into an equivalent *linear* program, (F2). Solve (F1) **graphically** when (i) **opt** = **min**, and (ii) **opt** = **max**. Solve (F2) using *Lindo*.

A: Put $z = (3x_1+7x_2+5)^{-1}$, $y_1 = x_1z$, $y_2 = x_2z$, and $y_3 = x_3z$, then (F1) is *equivalent* to (F2): opt($2y_1+5y_2+4z$) s.t. $7y_1+9y_2-63z \leq 0$, $3y_1+y_2-6z \geq 0$, $2y_1+3y_2-11z \geq 0$, $y_1+3y_2-7z \geq 0$, $3y_1+7y_2+5z = 1$, $y_1, y_2, z \geq 0$, and $z \neq 0$, where the *fifth constraint* comes from the definition of z. *Graphical* Solution: **Solve** $2x_1+5x_2+4$ and $3x_1+7x_2+5$ *simultaneously* to get $x_1 = 3$, and $x_2 = -2$. This is where the two lines meet. Now draw in the two lines and **all** the constraints to get what is shown on the *right*. Draw a pencil of lines through (3, -2): it **hits** (9,0) first, where $f = {}^{22}/_{34} = 0.6875$ here, a *minimum*; and it hits (1,3) last, where $f = {}^{21}/_{29} = 0.7241$ here, a *maximum*. Because the **numerator** of f is almost parallel to the **denominator** of f, the problem is *ill-conditioned*, and sensitivity analysis is indicated.



C2: By considering the *equivalent linear program* and its dual, **show** that $f = 8x_1+6x_2-10/4x_1-2x_2+40$ s.t. $x_1+x_2 \leq 10$, $3x_1-5x_2 \leq 6$, and $x_1, x_2 \geq 0$ has *extreme* values at (0,0) and (0,10). Confirm the results **graphically**. **Hints**: If S is the feasible region, *use the first constraint* to show that $4x_1-2x_2+40 > 0$ on S. **Compute** f(0,0) and f(0,10) to decide which point should be considered as the *local maximum*, and which as the local *minimum*. Consider the maximum and minimum problems **separately**.

A: Put $z = (4x_1-2x_2+40)^{-1}$, $y_1 = x_1z$, $y_2 = x_2z$, and $y_3 = x_3z$. Then we *have the linear program* opt($8y_1+6y_2-10z$) **subject** to $y_1+y_2-10z \leq 0$, $3y_1-5y_2-6 \leq 0$, $4y_1-2y_2+40z = 1$, $y_1, y_2, z \geq 0$, and $z \neq 0$.

8th March 2001

Weak and Strong Duality

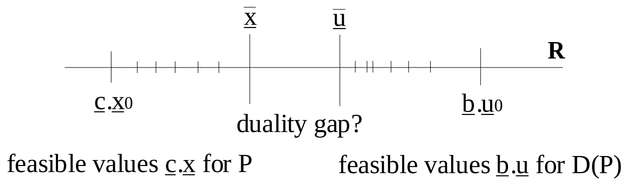
Example: Q: Min ($N = 4u_1+7u_2$) subject to $u_1+2u_2 \geq 11$, $u_1+3u_2 \geq 13$, and $2u_1+5u_2 \geq 18$ ($\underline{u} \geq \underline{0}$), with $B = [^{1}1_{2}\ ^{2}3_{5}]$, and $\underline{e} = [^{11}13_{18}]$. Q' is the *max form of Q*: max ($M = -4u_1-7u_2$) s.t. $-u_1-2u_2 \leq -11$, $-u_1-3u_2 \leq -13$, and $-2u_1-5u_2 \leq -18$ ($\underline{u} \geq \underline{0}$).

Here is a *max problem:* P: Max ($M = 11x_1+13x_2+18x_3$) s.t. $x_1+x_2+2x_3 \leq 4$, and $2x_1+3x_2+5x_3 \leq 7$ ($\underline{x} \geq \underline{0}$), where $\underline{c} = (11, 13, 18)$, $A = (^{1}{}_{2}\,^{1}{}_{3}\,^{2}{}_{5})$, and $\underline{b} = (^{4}{}_{7})$. P has *dual* D(P) = Q as above, a **standard** min problem. Convert Q to *max form* Q' (using *variables* v_1, v_2, v_3, say): R: Min ($-11v_1-13v_2-18v_3$) s.t. $-v_1-v_2-2v_3 \geq -4$, and $-2v_1-3v_2-5v_3 \geq -7$ ($\underline{v} \geq \underline{0}$). R' is the max *form* of R: Max ($11v_1+13v_2+8v_3$) s.t. $v_1+v_2+2v_3 \leq 4$, and $2v_1+3v_2+5v_3 \leq 7$ ($\underline{v} \geq \underline{0}$).

Duality: Any LP can be regarded as a *max* or *min* problem, as max M = min(-M). Also, *linear constraints can be written* as $A\underline{x} \leq \underline{b}$ ($\underline{x} \geq 0$), or $B\underline{u} \geq \underline{c}$ ($\underline{u} \geq \underline{0}$). We *define* 2 standard problems: (1) A standard **max** problem: P: max($M = \underline{c}.\underline{x}$) *subject* to $A\underline{x} \leq \underline{b}$, with $\underline{x} \geq \underline{0}$; (2) A standard *min* problem: Q: min($N = \underline{f}.\underline{u}$) *subject* to $B\underline{u} \geq \underline{g}$, with $\underline{u} \geq \underline{0}$. Any LP can be *expressed* in these forms. In particular, the max form of Q is max($-N$) = $(-\underline{f}).\underline{u}$ s.t. $(-B)\underline{u} \leq (-g)$, with $\underline{u} \geq \underline{0}$.

Dual of an LP: When the *primal* problem P is P: max($M = \underline{c}.\underline{x}$) s.t. $A\underline{x} \le \underline{b}$, with $\underline{x} \ge \underline{0}$, the *dual*, D(P), is Q: min($N = \underline{b}.\underline{u}$) s.t. $A^t\underline{u} \ge \underline{c}$, with $\underline{u} \ge \underline{0}$. Therefore, the *dual* of a standard max LP is a standard **min** LP. **Theorem 1.1:** D(D(P)) is *equivalent* to P. **Proof:** Let P be "max $\underline{c}.\underline{x}$ s.t. $A\underline{x} \le \underline{b}$, with $\underline{x} \ge \underline{0}$". Then D(P) is "min $\underline{b}.\underline{u}$ s.t. $A^t\underline{u} \ge \underline{c}$, with $\underline{u} \ge \underline{0}$", which has a *standard max form* "Q = max$(-\underline{b}).\underline{u}$ s.t. $(-A^t)\underline{u} \le (-\underline{c})$, with $\underline{u} \ge \underline{0}$". Further, D(Q) = "min$(-\underline{c}).\underline{v}$ s.t. $(-A^t))^t\,\underline{v} \ge (-\underline{b})$, with $\underline{v} \ge \underline{0}$", which has *standard max form* "R = max$(-(-\underline{c}).\underline{v})$ s.t. $(-(-A^t)^t)\underline{v} \le -(-\underline{b})$, with $\underline{v} \ge \underline{0}$" = "max $\underline{c}.\underline{v}$ s.t. $A\underline{v} \le \underline{b}$, with $\underline{v} \ge \underline{0}$". **QED**.

Weak Duality. **Theorem 2.1:** Given P: max $\underline{c}.\underline{x}$ s.t. $A\underline{x} \le \underline{b}$, with $\underline{x} \ge \underline{0}$; and D(P): min $\underline{b}.\underline{u}$ s.t. $A^t\underline{u} \ge \underline{c}$, with $\underline{u} \ge \underline{0}$; if $\underline{x}_0$ and $\underline{u}_0$ are feasible for P and D(P) *respectively*, then $\underline{c}.\underline{x}_0 \le \underline{u}_0^tA\underline{x}_0 \le \underline{b}.\underline{u}_0$. **Proof:** $\underline{x}_0$ *feasible* for P $\Rightarrow \underline{u}_0^tA\underline{x}_0 \le \underline{u}_0^t\underline{b} = \underline{b}.\underline{u}_0$. And $\underline{u}_0$ *feasible* for D(P) $\Rightarrow \underline{x}_0^tA^t\underline{u}_0 \ge \underline{x}_0^t\underline{c} = \underline{c}.\underline{x}_0$. The result *follows* since $\underline{x}_0^tA^t\underline{u}_0 = (\underline{u}_0^tA\underline{x}_0)^t$.



Theorem 2.2: If P and D(P) are *both feasible*, then both have optimal solutions. **Proof:** Let F P be the *set of feasible values* $\underline{c}.\underline{x}$, and let F D(P) be the set of *feasible values* $\underline{b}.\underline{u}$. The **previous** result shows that every $\underline{b}.\underline{u} \in$ F D(P) is an *upper* bound for F P, and that *every* $\underline{c}.\underline{x} \in$ F P is a *lower* bound for F D(P). The two *functions* $\mathbf{R}^n \to \mathbf{R}$, $\mathbf{x} \mapsto \mathbf{c}.\mathbf{x}$; and $\mathbf{R}^m \to \mathbf{R}$, $\mathbf{u} \mapsto \mathbf{b}.\mathbf{u}$, are *continuous*, so that $\bar{\mathbf{x}} \in \mathbf{R}^n$ and $\bar{\mathbf{u}} \in \mathbf{R}^m$ exist, with $\mathbf{c}.\bar{\mathbf{x}}$ = l.u.b. F P, and $\mathbf{b}\bar{\mathbf{u}}$ = g.l.b. F D(P). Since the feasible regions are *compact*, these bounds are **attained**.

Strong Duality. **Theorem 3.1:** If $\underline{x}_0$ is *feasible* for P, if $\underline{u}_0$ is feasible for D(P), and if $\underline{c}.\underline{x}_0 = \underline{b}.\underline{u}_0$ then $\underline{x}_0$ is *optimal* for P, and $\underline{u}_0$ is *optimal* for D(P). **Proof:** We *know* that $\underline{c}.\underline{x}_0$ is a *lower* bound for **both** optP and optD(P), and that $\underline{b}.\underline{u}_0$ is an *upper* bound for *both* optD(P) and optP. In this case, a **lower** bound is equal to an **upper** bound, so that the *optimal values are equal* (and the duality gap is <u>empty</u>). **Corollary 3.2:** (1) If P is *feasible*, and if D(P) is *infeasible*, then F P is <u>unbounded</u> above. (2) If P is *infeasible*, and if D(P) is *feasible*, then F D(P) is <u>unbounded</u> below. **Theorem 3.3:** If $\underline{x}_0$ and $\underline{u}_0$ are *feasible* for P and D(P) respectively, then **optimal** solutions $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ *exist*, with $\underline{c}.\bar{\mathbf{x}} = \underline{b}.\bar{\mathbf{u}}$ (in **fact**, if <u>one</u> optimal solution exists, then *so does the other).*

Proof: The following proof is based on the *properties of the simplex algorithm* (other **algebraic** proofs are possible). **Start** with P: max ($M = \underline{c}.\underline{x}$) *subject to* $A\underline{x} \le \underline{b}$, with $\underline{x} \ge \underline{0}$. Add *slack* and *artificial* variables to get $A\underline{x}+\underline{s} = \underline{b}$. An arbitrary *linear* combination of the n **constraints** may be written as $\underline{l}^tA\underline{x} + \underline{l}^t\underline{s} = \underline{l}^t\underline{b}$ for *some* $\underline{l} \in \mathbf{R}^n$. The *initial* tableau has the form of the first table on the right (the **third** row is an *arbitrary combination of rows 1 to n,* <u>corresponding</u> to constraint $\underline{l}^tA\underline{x} + \underline{l}^t\underline{s} = \underline{l}^t\underline{b}$). In general, we need to *apply Phase 1 so as to obtain an initial solution.* At the end of Phase 1, we remove any **artificial** variables. The new top row is the original row plus an arbitrary linear combination of the constraints. As the iteration proceeds, <u>further</u> linear combinations are added. By *weak* duality, we know that an optimal solution exists, so that eventually we will obtain a **top** row as shown in the second table, where $(-\underline{c}^t+\underline{l}^tA)$ and $(+\underline{l}^t)$ have entries which are *all non-negative* (so that the **solution** is thereby obtained).

M	**x** columns	**s** columns	r.h.s.
1	$-\mathbf{c}^t$	$\mathbf{0}^t$	0
0	A	I	**b**
$\boldsymbol{l}^t\mathbf{0}$	$\boldsymbol{l}^tA$	$\boldsymbol{l}^t$	$\boldsymbol{l}^t\mathbf{b}$

M	**x** columns	**s** columns	r.h.s
1	$-\mathbf{c}^t+\boldsymbol{l}^tA$	$+\boldsymbol{l}^t$	$+\boldsymbol{l}^t\mathbf{b}$

The **optimal** M is $\underline{l}^t\underline{b}$ (at $\bar{\mathbf{x}}$, say). *From* the $\underline{x}$- and $\underline{s}$- columns, $-\underline{c}^t+\underline{l}^tA \geq \underline{0}^t \Rightarrow A^t\underline{l} \geq \underline{c}$; $+\underline{l}^tI \geq \underline{0}^t \Rightarrow \underline{l} \geq \underline{0}$. **Now** we have D(P): $\min(\underline{b}.\underline{u})$ s.t. $A^t\underline{u} \geq \underline{c}$, with $\underline{u} \geq 0$, so we have that $\underline{l}$ is a **feasible** solution to D(P), and that $\underline{b}.\underline{l}$ is an *optimal* value for P. By Theorem 3.1, $\underline{x}$ is *optimal* for P, and $\underline{l}$ is optimal for D(P). **Furthermore**, after adding excess variables, the *dual constraints* have the form $A^t\underline{u}-\underline{e} = \underline{c} \Rightarrow A^t\underline{u}-\underline{c} = \underline{e} \Rightarrow -\underline{c}^t+\underline{u}^tA = \underline{e}^t$. Thus the *vector* in row 0 and the *$\underline{x}$-columns* of the **final** tableau, $(-\underline{c}^t+\underline{l}^tA)$, contains the *optimal* values for the **excess** variables of D(P). Note that the *final top row* (when the dual simplex algorithm is **applied** to D(P)) is as shown on the right.

N	**u** columns	**e** columns	r.h.s
1	$+\underline{b}^t - \bar{\mathbf{x}}^tA^t$	$+\bar{\mathbf{x}}^t$	$-\bar{\mathbf{x}}^t\mathbf{c}$

12th March 2001

Complementary Slackness

Consider P: $\max(\underline{c}.\underline{x})$ s.t. $A\underline{x} \leq \underline{b}$, with $\underline{x} \geq \underline{0}$; and D(P): $\min(\underline{b}.\underline{u})$ s.t. $A^t\underline{u} \geq \underline{c}$, with $\underline{u} \geq \underline{0}$. Now the *canonical/normal* form for the constraints is as follows: $A\underline{x}+\underline{s} = \underline{b}$, with $\underline{x}, \underline{s} \geq \underline{0}$; and $A^t\underline{u}-\underline{e} = \underline{c}$, with $\underline{u}, \underline{e} \geq \underline{0}$. Analysis: A is an $m\times n$ matrix, with $\underline{x}, \underline{c}, \underline{e} \in \mathbf{R}^n$, and $\underline{b}, \underline{s}, \underline{u} \in \mathbf{R}^m$. *Solutions* $(\underline{x},\underline{s})$ to P, and $(\underline{u},\underline{e})$ to D(P), are said to be *complementary slack* (written "$(\underline{x},\underline{s})$ c.s. $(\underline{u},\underline{e})$") if $\underline{x}.\underline{e} = \underline{u}.\underline{s} = 0$, i.e. (*****) $u_i = 0$ or $s_i = 0$ *for* $1 \leq i \leq m$, and $x_j = 0$ or $e_j = 0$ *for* $1 \leq j \leq n$. **<u>Theorem 4.1:</u>** $\underline{c}.\underline{x} = \underline{b}.\underline{u} \Leftrightarrow (\underline{x},\underline{s})$ c.s. $(\underline{u},\underline{e})$. **<u>Proof:</u>** $\underline{c}.\underline{x} = (A^t\underline{u}-\underline{e}).\underline{x} = (\underline{u}^tA\underline{x})^t-\underline{e}.\underline{x}$; $\underline{b}.\underline{u} = (A\underline{x}+\underline{s}).\underline{u} = \underline{u}^tA\underline{x}+\underline{s}.\underline{u}$. Now $\underline{c}.\underline{x} = \underline{b}.\underline{u} \Rightarrow -\underline{e}.\underline{x} = \underline{s}.\underline{u}$; $\underline{e}.\underline{x} = 0 = \underline{s}.\underline{u}$ when $\underline{x}, \underline{s}, \underline{u}, \underline{e} \geq \underline{0} \Rightarrow (\underline{x},\underline{s})$ c.s. $(\underline{u},\underline{e})$. **<u>Corollary 4.2:</u>** $(\underline{x},\underline{s})$ *feasible* for P, $(\underline{u},\underline{e})$ feasible for D(P), and $(\underline{x},\underline{s})$ c.s. $(\underline{u},\underline{e}) \Rightarrow (\underline{x},\underline{s})$ and $(\underline{u},\underline{e})$ are optimal for P and D(P) respectively. **<u>Proof:</u>** (4.1) $\Rightarrow \underline{c}.\underline{x} = \underline{b}.\underline{u}$, and (3.1) $\Rightarrow \underline{x}$ and $\underline{u}$ are both *optimal*. **Algorithm 4.3** to test if $\underline{x}_0$ is optimal for P: (1) *Calculate* $\underline{s}_0 = \underline{b}-A\underline{x}_0$. If $\underline{s}_0 \geq \underline{0}$, then $\underline{x}_0$ **is** feasible. (2) Try to *find* $(\underline{u}_0, \underline{e}_0)$ c.s. $(\underline{x}_0, \underline{s}_0)$ *using* (*). If $(\underline{u}_0, \underline{e}_0)$ *exists*, with $\underline{u}_0, \underline{e}_0 \geq \underline{0}$, then $\underline{x}_0$ (and $\underline{u}_0$) are *optimal*. **<u>Example</u>**: P = max $\underline{c}.\underline{x}$ s.t. $A\underline{x} \leq \underline{b}$, where $\underline{c} = ({}^{11}13_{18})$, $A = ({}^{1}2_{3}\ {}^{1}3_{1}\ {}^{2}1_{1})$, and $\underline{b} = ({}^{4}7_{5})$. Q: Test for *optimality* $\underline{x}_0 = ({}^{0}0_{0})$, $\underline{x}_1 = ({}^{0}1_{4})$, $\underline{x}_2 = ({}^{1}1_{1})$, $\underline{x}_3 = ({}^{1}0_{1½})$, and $\underline{x}_4 = ({}^{0}2_{1})$. A: P is as follows: maximise $M = 11x_1+13x_2+18x_3$ s.t. $x_1+x_2+2x_3 \leq 4$, $2x_1+3x_2+x_3 \leq 7$, and $3x_1+x_2+x_3 \leq 5$, with $\underline{x} \geq \underline{0}$. D(P) is as follows: minimise $N = 4u_1+7u_2+5u_3$ s.t. $u_1+2u_2+3u_3 \geq 11$, $u_1+3u_2+u_3 \geq 13$, and $2u_1+u_2+u_3 \geq 18$, with $\underline{u} \geq \underline{0}$. $(\underline{x}_0)$. **Feasible** OK. $\underline{s}_0 = \underline{b}$. (*) $\Rightarrow \underline{u} = 0$ (and *no information* for $\underline{e}$). $\underline{u} = 0 \Rightarrow \underline{e} = -\underline{c}$. We have *found* $(\underline{0}, -\underline{c})$ c.s. $(\underline{0}, \underline{b})$. But $-\underline{c}$ is not $\geq \underline{0}$ — so $\underline{x}_0$ is **not** optimal. $(\underline{x}_1)$. **Feasible:** $\underline{s}_1 = ({}^{4}7_{5}) - ({}^{9}7_{5}) = ({}^{-5}0_{0})$, which is **not** ≥ 0 — so $\underline{x}_1$ is *not* feasible. $(\underline{x}_2)$. $\underline{s}_2 = ({}^{4}7_{5}) - ({}^{4}6_{5}) = ({}^{0}1_{0}) \geq \underline{0}$, OK. (*) $\Rightarrow u_2 = 0$ and $\underline{e} = \underline{0}$. The three *dual* equations are as follows: $u_1+3u_3 = 11$, $u_1+u_3 = 13$, and $2u_1+u_3 = 18$: *inconsistent*, so $\underline{x}_2$ is **not** optimal. $(\underline{x}_3)$. $\underline{s}_3 = ({}^{4}7_{5})-({}^{4}3½_{4½}) = ({}^{0}3½_{½})$, so $\underline{x}_3$ is *feasible*. (*) $\Rightarrow e_1 = e_3 = 0$, and $u_2 = u_3 = 0$. The three *dual* equations are as follows: $u_1 = 11$, $u_1-e_2 = 13$, and $u_1 = 18$: *inconsistent*. $(\underline{x}_4)$. $\underline{s}_4 = ({}^{4}7_{5})-({}^{4}7_{3}) = ({}^{0}0_{2})$. (*) $\Rightarrow e_2 = e_3 = 0$, and $u_3 = 0$. The three *dual* equations are as follows: $u_1+2u_2-e_1 = 11$, $u_1+3u_2 = 13$, and $2u_1+u_2 = 18$. *Solve* (by a matrix) to get $u_2 = {}^{8}/_{5}$, $u_1 = {}^{41}/_{5}$, and $u_3 = 0$. Now $e_1 = {}^{2}/_{5}$, so that $\underline{u}_4 = ({}^{41}/_{5}\ {}^{8}/_{5}\ 0)^t$, *and* that $\underline{e}_4 = ({}^{2}/_{5}\ 0\ 0)^t$, *both* ≥ 0 — so $\underline{x}_4$ and $\underline{u}_4$ are *optimal*. **<u>Check</u>**: $M = 11(0)+13(2)+18(1) = 44$, and $N = 4(8{}^{1}/_{5})+7(1{}^{3}/_{5})+5(0) = 44$.

<u>Tutorial:</u> Let $\underline{c}^t = (4,4,3,5)$, $\underline{b}^t = (7,7,9)$, and $A = ({}^{1}2_{2}\ {}^{1}1_{3}\ {}^{2}3_{1}\ {}^{3}1_{2})$. *Test* $\underline{x}_0^t = (0,2,1,1)$, $\underline{x}_1^t = (3,1,0,0)$, $\underline{x}_2^t = ({}^{13}/_{4}, 0, 0, {}^{5}/_{4})$, $\underline{x}_3^t = ({}^{11}/_{3}, 0, {}^{5}/_{3}, 0)$, and $\underline{x}_4^t = ({}^{8}/_{3}, {}^{1}/_{3}, 0, {}^{4}/_{3})$ for *optimality*. A: $(\underline{x}_0^t)$. $\underline{s}_0 = (7\ 7\ 9)^t - (7\ 6\ 9)^t = (0\ 1\ 0)^t$, OK. (*) $\Rightarrow e_2 = e_3 = e_4 = 0$, and $u_2 = 0$. The four *dual equations are as follows*: $u_1+2u_3-e_1 = 4$, $u_1+3u_3 = 4$, $2u_1+u_3 = 3$, and $3u_1+2u_3 = 5$. Solve the *3rd and 4th equations* to get $u_1 = u_3 = 1$, which **satisfy** the 2nd equation. But the *1st equation* $\Rightarrow e_1 = -1$, which is ≤ 0 — so **NOT** optimal.

($\underline{x}_1^t$). $\underline{s}_1 = (7\ 7\ 9)^t - (4\ 7\ 9)^t = (3\ 0\ 0)^t$, **OK**. (*) $\Rightarrow e_1 = e_2 = 0$, and $u_1 = 0$. The four *dual equations are as follows*: $2u_2+2u_3 = 4$, $u_2+3u_3 = 4$, $3u_2+u_3-e_3 = 3$, and $u_2+2u_3-e_4 = 5$. Solve the *1st and 2nd equations* to get $u_2 = u_3 = 1$. Therefore, $e_3 = 1$, and $e_4 = -2$, so that x_1 is not optimal. ($\underline{x}_2^t$). $\underline{s}_2 = (7\ 7\ 9)^t - (7\ 7^3/_4\ 9)^t = (0\ \text{-ve}\ 0)^t$: **not** optimal.

($\underline{x}_3^t$). $\underline{s}_4 = (7\ 7\ 9)^t - (7\ 12^1/_3\ 9)^t = (0\ \text{-ve}\ 0)^t$: **not** optimal. ($\underline{x}_4^t$). $\underline{s}_2 = (7\ 7\ 9)^t - (7\ 7\ 9)^t = (0\ 0\ 0)^t$. Unusually, *all 3 constraints* are exactly satisfied, so that $\underline{x}_4$ is *optimal* with zero slack. (*) $\Rightarrow e_1 = e_2 = e_4 = 0$. The four *dual equations are as follows:* $u_1+2u_2+2u_3 = 4$, $u_1+u_2+3u_3 = 4$, $2u_1+3u_2+u_3-e_3 = 3$, and $3u_1+u_2+2u_3 = 5$. Solve the *1st, 2nd and 4th equations* to get $\underline{u}_2 = (^8/_9, ^7/_9, ^7/_9)^t$. **Check**: $M_2 = ^{32}/_3+^4/_3+0 = 18^2/_3$, and $N_2 = ^{56}/_9+^{49}/_9+^{63}/_9 = 18^2/_3$. $\underline{x}_2$ is therefore **optimal**.

Exercise: Solve P (on the *previous* page) by the *simplex method*, and D(P) by the *dual* method. A: Solve P as shown on the *right*. After the second iteration, we get $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $s_1 = 0$, $s_2 = 0$, $s_3 = 2$, and $M = 44$. Now form the *dual program* of D(P), but this is just P — which we have **already** solved! We read off the solution by looking at the *coefficients in the objective row of the final tableau*. So $u_1 = ^{41}/_5$, $u_2 = ^8/_5$, $u_3 = 0$, $e_1 = ^2/_5$, $e_2 = 0$, $e_3 = 0$, and $N = 44$. These answers *agree* with previous calculations.

	M	x1	x2	x3	s1	s2	s3	RHS	Ratio
	1	-11	-13	-18	0	0	0	0	
	0	1	1	(2)	1	0	0	4	2 ←
	0	2	3	1	0	1	0	7	7
	0	3	1	1	0	0	1	5	5
R1+9R2	1	-2	-4	0	9	0	0	36	
	0	1	1	2	1	0	0	4	4
R3-½R2	0	3/2	(5/2)	0	-1/2	1	0	5	2 ←
R4-½R2	0	5/2	1/2	0	-1/2	0	1	3	6
R1+8/5R3	1	2/5	0	0	41/5	8/5	0	44	
R2-2/5R3	0	2/5	0	2	6/5	-2/5	0	2	
	0	3/2	5/2	0	-1/2	1	0	5	
R4-1/5R3	0	11/5	0	0	-2/5	-1/5	1	2	

Now solve the *tutorial* question by the **simplex** method (P) and the **dual simplex** method (D(P)). A: P is *max* $M = 4x_1+4x_2+3x_3+5x_4$ s.t. $x_1+x_2+2x_3+3x_4+s_1 = 7$, $2x_1+x_2+3x_3+x_4+s_2 = 7$, and $2x_1+3x_2+x_3+2x_4+s_3 = 9$, with $x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$. D(P) is *min* $N = 7u_1+7u_2+9u_3$ s.t. $u_1+2u_2+2u_3-e_1 = 4$, $u_1+u_2+3u_3-e_2 = 4$, $2u_1+3u_2+u_3-e_3 = 3$, and $3u_1+u_2+2u_3-e_4 = 5$, with $u_1, u_2, u_3, e_1, e_2, e_3, e_4 \geq 0$. The calculations are as shown **below**.

M	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs	ratio
1	-4	-4	-3	-5	0	0	0	0	
0	1	1	2	**3**	1	0	0	7	$2^1/_3$
0	2	1	3	1	0	1	0	7	7
0	2	3	1	2	0	0	1	9	4½
3	-7	-7	1	0	5	0	0	35	
0	1	1	2	3	1	0	0	7	7
0	5	2	7	0	-1	3	0	14	7
0	4	**7**	-1	0	-2	0	3	13	$1^6/_7$
1	-1	0	0	0	1	0	1	16	
0	1	0	5	7	3	0	-1	12	12
0	**9**	0	17	0	-1	7	-2	24	$2^2/_3$
0	4	7	-1	0	-2	0	3	13	$3^1/_4$
9	0	0	17	0	8	7	7	168	
0	0	0	4	9	4	-1	-1	12	
0	9	0	17	0	-1	7	-2	24	
0	0	9	-11	0	-2	-4	5	3	

-N	u_1	u_2	u_3	e_1	e_2	e_3	e_4	rhs	basic
1	7	7	9	0	0	0	0	0	-N = 0
0	-1	-2	-2	1	0	0	0	-4	$e_1 = -4$
0	-1	-1	-3	0	1	0	0	-4	$e_2 = -4$
0	-2	-3	-1	0	0	1	0	-3	$e_3 = -3$
0	**-3**	-1	-2	0	0	0	1	-5	$e_4 = -5$
	$-^7/_3$	-7	-4.5						*ratio*
3	0	14	13	0	0	0	7	-35	$-N = -11^2/_3$
0	0	-5	-4	3	0	0	-1	-7	$e_1 = -^7/_3$
0	0	-2	**-7**	0	3	0	-1	-7	$e_2 = -^7/_3$
0	0	-7	1	0	0	3	-2	1	$e_3 = ^1/_3$
0	-3	-1	-2	0	0	0	1	-5	$u_1 = ^5/_3$
		-7	-1.85						*ratio*
7	0	24	0	0	13	0	12	-112	-N = -16
0	0	**-9**	0	7	-4	0	-1	-7	$e_1 = -1$
0	0	-2	-7	0	3	0	-1	-7	$u_3 = 1$
0	0	-17	0	0	1	7	-5	0	$e_4 = 0$
0	-7	-1	0	0	-2	0	3	-7	$u_1 = 1$
		-2.6			-3.25		-12		*ratio*
3	0	0	0	8	1	0	4	-56	$-N = -18^2/_3$
0	0	-9	0	7	-4	0	-1	-7	$u_2 = ^7/_9$
0	0	0	-9	-2	5	0	-1	-7	$u_3 = ^7/_9$
0	0	0	0	-17	11	9	-4	17	$e_3 = 1^8/_9$
0	-9	0	0	-1	-2	0	4	-8	$u_1 = ^8/_9$

As we can see, *from the first table*, after completing the **simplex** method, we get the answer $M = 18^2/_3$, $x_1 = 2^2/_3$, $x_2 = {}^1/_3$, $x_3 = 0$, $x_4 = 1^1/_3$, and $s_1 = s_2 = s_3 = 0$. After *completing the second table* (by the **dual** simplex method), we get the solution $N = 18^2/_3$, $u_1 = {}^8/_9$, $u_2 = u_3 = {}^7/_9$, $e_1 = e_2 = e_4 = 0$, and $e_3 = 1^8/_9$.

15th March 2001

Decision Making Under Uncertainty

State of the World Decision Model

Notation: DM = *Decision Maker;* $A = \{a_1,a_2,\ldots,a_k\}$, a set of *actions;* $S = \{s_1,s_2,\ldots,s_n\}$, *states of the world; and* r_{ij} = the **reward** which the DM receives if action a_i is chosen when the *world state* is s_j. **Example:** You sell newspapers at the *entrance to the science museum* each day. Determine the number of newspapers to order. You **pay** 20p for each paper, and **sell** them for 25p (unsold papers are **worthless)**. Each day, you sell *between 26 and 30 papers*, each value equally likely.

Model: DM = "Me"; A = {26,27,28,29,30} = the number **ordered;** S = {26,27,28,29,30} = the number **sold** / demand; and r_{ij} = the profit when a_i are *ordered* and s_j *are sold:* $r_{ij} = 25j-20i$ when $i \geq j$, and $r_{ij} = 5i$ when $i \leq j$.

a_i \ s_j	26	27	28	29	30	maximin	maximax	$\bar{r}_i$
26	130	130	130	130	130	130	130	130
27	110	135	135	135	135	110	135	130
28	90	115	140	140	140	90	140	125
29	70	95	120	145	145	70	145	115
30	50	75	100	125	150	50	150	100

We have 4 criteria: (1) **Maximin** (*least* adventurous, *most* secure): choose the a_i with the **largest** $\min_{s_j\in S} r_{ij}$. In the *example*, we order 26 papers. (2) **Maximax** (*most* adventurous, *least* secure): choose the a_i with the **largest** $\max_{s_j\in S} r_{ij}$. In the *example*, we order 30 papers. (3) **Minimax Regret**. For each j, find an **action** $a_{i^*(j)}$ that *maximises* r_{ij} (where $a_{i^*(j)}$ is the *best possible action* in state j). For each (i,j), calculate the "**opportunity loss of regret**" (the o.l.o.r.) to be $r_{i^*(j),j}-r_{ij}$ = how much the DM loses when a_i is chosen *instead* of $a_{i^*(j),j}$ in s_j. For example, **when** $s_j = 27$, the **best** value is 135 in row 27, so that $i^*(27) = 27$. If we **choose** $a_i = 26$, the o.l.o.r. is $a_{27,27}-a_{26,27}$ = 135-130 = 5p. The *table of regrets* is as shown on the left. We select a row which **minimises** the max regret. In this case, *order* 26 or 27 papers.

	26	27	28	29	30	max
26	0	5	10	15	20	20
27	20	0	5	10	15	20
28	40	20	0	5	10	40
29	60	40	20	0	5	60
30	80	60	40	20	0	80

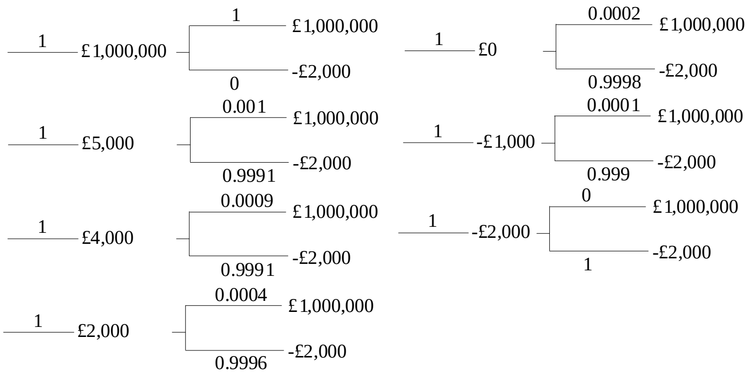
(4) **Expected Value Criterion.** The probability that s_j *occurs* is $p_j = {}^1/_5$ ($26 \leq j \leq 30$). The *expected reward for action* a_i is $\bar{r}_i = \Sigma r_{ij}p_j = {}^1/_5\Sigma_j r_{ij}$. *Choose* $\max_i \bar{r}_i$, and we again get *26 or 27.*

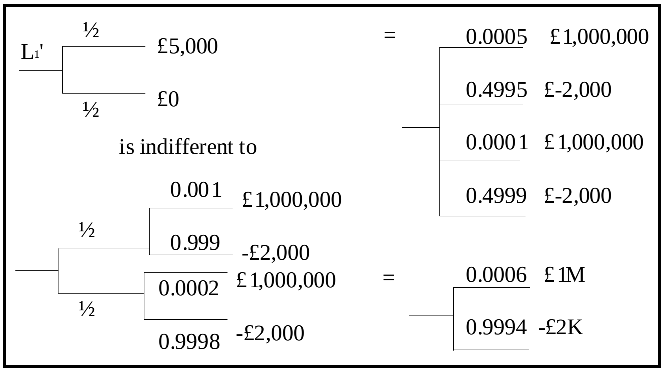
19th March 2001

Von Neumann / Morgenstern Utility Theory

Lottery: n rewards r_i; and n *probabilities* p_i for n events, written $(p_1, r_1; p_2, r_2; \ldots; p_n, r_n)$, or in *diagrammatic* format. We must have $\Sigma p_i = 1$. **Example 1:** Suppose that L_1 is a lottery with *probability* 1 that £0 is won, and that L_2 is a lottery such that with *probability* ½ we win £1000, and with *probability* ½ we lose £1000. We would **expect** L_1 and L_2 to be *equivalent*. **Example 2**: Suppose that Lottery 1 has probability ½ of *winning £5000*, and probability ½ of *winning £0*; L_2: probability 1 of winning £2000; L_3: 0.01 to win £1,000,000, 0.99 to win -£1000; and L_4: ½ to win £4000, $^1/_3$ to win £2000, and $^1/_6$ to win -£2000. Choose *which* lottery to play!

<u>Step 1:</u> Identify the *most and least* favourable outcomes: +£1,000,000 and -£2000. **<u>Step 2:</u>** For each possible *reward* r_i, ask the DM to determine the probability p_i such that he/she is *indifferent between* choosing r_i (with probability 1) and choosing between £1,000,000, with *probability* p_i, and -£2000, with probability $1-p_i$. The **outcomes** are {+£1,000,000, £5000, £4000, £2000, £0, £-1000, £-2000}. Our "*decisions*" are shown below.

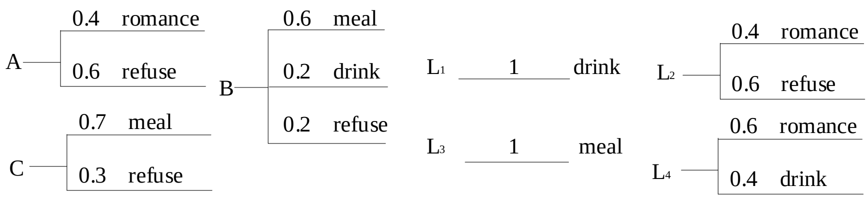




<u>Step 3:</u> Construct *new* lotteries L_i' such that the DM is **indifferent** between L_i and L_i', and that L_i' **only** involves the *best/worst* outcomes. An <u>example</u> for L_1' is shown boxed above. Similarly, for L_2', we **have** 0.0004 for £1M, *and* 0.9996 for -£2K. For L_3', we **have** 0.01099 for £1M, *and* 0.989901 for -£2K. And for L_4', we **have** 0.00058 for £1M, *and* 0.99942 for -£2K. **<u>Conclusion</u>**: L_3 is *better* than L_1, which is *better* than L_4, which is *better* than L_2.

22nd March 2001

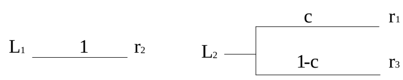
Q: John likes *3 girls from his class:* Alice, Belinda and Claire. John's chances of success for a <u>drink</u>/<u>romance</u>/<u>meal</u>/<u>refusal</u> are as shown for each girl. John is also *indifferent* between L_1 and L_2; and L_3 and L_4. (i) Which girl should John *ask out*? (ii) What is John's *certainty equivalent of lottery* L_2? (iii) If a **romance** counts 10 points, and the embarrassment of a **refuse** counts -30 points, what are the *expected values* of lotteries L_2 and L_4?



A: (i) The **best** and **worst** outcomes are romance/refused. With L_3 i L_4, we can *replace* the **drink** branch in L_4 with L_2, to give L_4' = 0.76 romance; 0.24 refusal. So we have *4 simple lotteries:* 1 romance *indifferent* to 1 romance / 0 refusal; 1 meal *indifference* to 0.76 romance / 0.24 refusal; 1 drink *indifferent* to 0.4 romance / 0.6 refusal, and 1 refusal *indifferent* to 0 romance / 1 refusal. To get **which** girl John should ask out, we *plug in the simple lotteries* into A, B and C. A is *already* in the required form; B **becomes** 0.536 Ro / 0.464 Re; and C **becomes** 0.532 Ro / 0.468 Re. Conclusion: John should ask <u>Belinda</u> out (0.536 > 0.532 > 0.4). For (ii), we need *more theory* first.

<u>Lottery Axioms</u>

(1) **<u>Complete ordering</u>** (*total* order). For any *two* outcomes r_1 and r_2, either the DM prefers r_1 to r_2, or r_2 to r_1, or is *indifferent* between r_1 and r_2. If r_1 is preferred to r_2, and if r_2 is preferred to r_3, then r_1 is preferred to r_3. (2) **<u>Continuity</u>**: If the DM prefers r_1 to r_2, and r_2 to r_3, then, for *some* $0 < c < 1$, the DM is *indifferent* between L_1 and L_2 as shown in the *diagram*.



(3) **Independence:** If the DM is *indifferent* between rewards r_1 and r_2, then the DM is *indifferent* between lotteries L_1 and L_2, as shown in diagram 1. (4) **Unequal Probability:** If r_1 and r_2 are the *only two rewards*, if we have what is shown in diagram 2, if the DM **prefers** r_1 to r_2, and if $c > p$, *then* clearly the DM prefers L_1 to L_2. (5) **Compound Lottery:** The DM is *indifferent* between (3) and (4), etc.

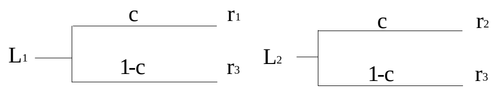


Diagram 1

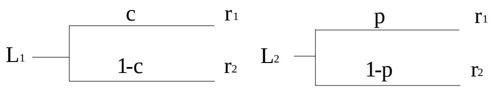
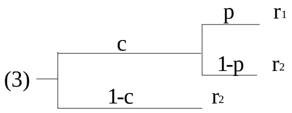
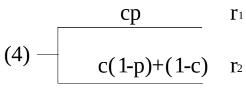


Diagram 2





Definition: The certainty equivalent (CE(L)) of a lottery L is the *number* such that the DM is *indifferent* between L and ——1——CE(L). In the **example** we have on the *previous* page, we have L_2: 0.4 romance / 0.6 refusal, and we are *told* that L_2 i L_1 ——1——drink. Therefore, $CE(L_2)$ = "*drink*". (c) If **Romance** = 10pts, and if **Refusal** = -30pts, then $Ev(L_2) = {}^2/_5(10)+{}^3/_5(-30) = 4-18 = -14$, and $Ev(L_4) = {}^3/_5(10)+{}^2/_5(-14) = 6-5.6 = 0.4$.

Tutorial

Q: Two restaurants determine the *price of a pizza*. Pizza King believes that Noble Greek's price is a **random** variable D having the following pmf: P(D=£6) = 0.25, P(D=£8) = 0.5, and P(D=£10) = 0.25. If Pizza King *charges* a price p_1, and if Noble Greek *charges* a price p_2, then Pizza King will **sell** $100+25(p_2-p_1)$ pizzas. It **costs** Pizza King £4 to make a pizza, and it is considering **charging** £5, £6, £7, £8 or £9 for a pizza. Use each of the *four design criteria* to determine the price that Pizza King should charge.

A: What Noble Greek charges is the *state of the world*. We can charge between £5 and £9 for a pizza. In the table, an entry represents a profit **multiplied** by a probability. For example, cell (1,1) represents us selling 100+25(6-5) = 125 pizzas, with a *profit* of 125×(5-4) = £125. But the probability of us **obtaining** this profit is 0.25, so that the entry is 125×0.25 = £31.25. The second table is the table of regrets. (CORRECTION: **DO NOT** MULTIPLY THE PROBABILITIES IN — ONLY USE THE PROBABILITIES WITH THE EXPECTED VALUE CRITERION!).

	6	8	10	maximin	maximax	$\bar{r}_i$
5	31.25	87.5	56.25	31.25	87.5	87.5
6	50	150	100	50	150	100
7	56.25	187.5	131.25	56.25	187.5	125
8	50	200	150	50	200	133.33
9	31.25	187.5	156.25	31.25	187.5	125

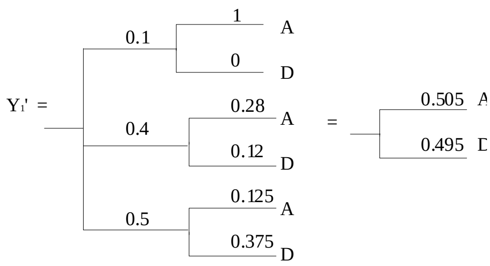
	6	8	10	max
5	25	112.5	100	112.5
6	6.25	50	56.25	56.25
7	0	12.5	25	25
8	6.25	0	6.25	6.25
9	25	12.5	0	25

Q: X is planning which *two of the courses* Y_1, Y_2 and Y_3 to take next semester. X estimates his chance to get **grade** A in course Y_1 to be 10%, B 40%, and C 50%. For course Y_2, X's prediction is 70% for B, 25% for C, and 5% for D. For Y_3, we have 6% for A, 62% for B, 20% for C, and 12% for D.

X is *indifferent* between L_1: With *probability* 1, X gets grade C; and L_2: With *probability* 0.25, X gets grade A, and with *probability* 0.7, X gets grade D. X is **also** indifferent between L_3: With *probability* 1, X gets grade B; and L_4: 0.70: X gets grade A, and 0.30: X gets grade D.

(a) **Rank** X's preferences, and find out *which two courses* X should take. (b) If grades are <u>quantified</u> as A = 10, B = 8, C = 6, and D = 4, find out the expected **values** and the expected **utilities** for the four lotteries.

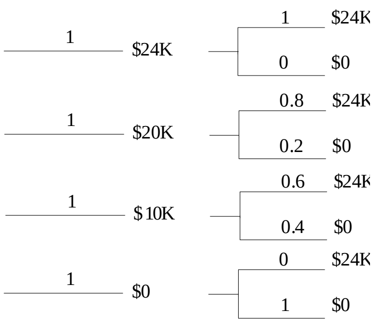
A: We also have *indifference* between L_5: 1 gets A; and L_6: 1 gets A and 0 gets D; and *indifference* between L_7: 1 gets D; and L_8: 0 gets A and 1 gets D. Using these *four sets of indifferences,* we can construct the Y_i', e.g. Y_1' as shown. For Y_2', we **have** 0.5525 for A, and 0.4475 for D. And for Y_3', we **have** 0.544 for A, and 0.456 for D. <u>**Conclusion**</u>: X should take courses Y_2 and Y_3 as there is <u>more chance</u> to get an A in these courses.



(b) We have *utilities* 1, 0.7, 0.25 and 0 for A, B, C and D. <u>**Expected Values**</u>: For L_1, we *have* 6. For L_2, we *have* (0.25×10)+(0.75×4) = 5.5. For L_3, we *have* 8. And for L_4, we *have* (0.7×10)+(0.3×4) = 8.2. <u>**Expected Utilities:**</u> for L_1, we *have* 0.25. For L_2, we *have* (0.25×1)+(0.75×0) = 0.25. For L_3, we *have* 0.7. And for L_4, we *have* (0.7×1)+(0.3×0) = 0.7.

Q: We are willing to **pay** \$50,000 for a painting. We can buy *today* for \$40,000, or wait until *tomorrow* and buy for \$30,000 (if it has not been sold). On the *third* day, it is \$26,000, and *no longer available* after this. Each day, there is a 0.60 **probability** that the painting will be sold. What strategy **maximises** the dealer's expected profit?

A: For **Today**, L_1, there is *probability 1* that we buy, with a profit of \$10,000. For **tomorrow**, L_2, there is *probability* 0.6 that we buy, with a profit of \$20,000; and probability 0.4 that we get \$0 profit. For **day 3**, there is probability 0.36 (= 0.6×0.6) that we get a profit of \$24K, and probability 0.64 that we get \$0 profit. <u>**Step 1:**</u> \$24K is the *most* favourable, and \$0 is the *least* favourable.



<u>**Step 2:**</u> The *indifferences* are as shown on the left. <u>**Step 3**</u>: Construct the L_i' as normal. L_1': 0.6 to get \$24K, 0.4 to get \$0. L_2': 0.48 to get \$24K, 0.52 to get \$0. L_3': 0.36 to get \$24K, 0.64 to get \$0. <u>**Conclusion**</u>: buy *immediately*. <u>**Note**</u>: it is better to use *decision trees* in this question — there is no need to **make up** probabilities for the <u>indifferences</u> in the diagrams.

(Table for the *next* question):

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Opening Session	3	4	2		4	2
3	8	5	6	7	5	7
Coffee Break						
1	6	8	1	2	1	4
Lunch Break						
5		6	3	4	7	**Closing Session**
	City Tour	2	**Banquet**	8	8	

Q: An exhibition of *8 companies* is held 1 hour away. We have to attend at least **one** presentation from each company, and at least **2** for company 2. The budget does not allow you to attend *everything*, or even to *stay overnight*. You cannot go to the **Friday afternoon** session. On **Wednesday** and **Sunday,** there is a prize draw, one of which you are *determined* to attend. You would also like to attend at least **2** of the major events: *opening, closing, city tour and banquet.* Formulate an **IP problem** to minimise the number of your trips to the conference site.

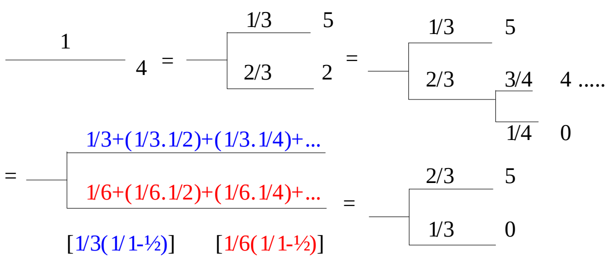
A: Let x_{ij} denote the *time slots* (i = **day** of week, j = **slot** 1-5). Let x_{ic} denote the attendance of a *coffee break*. We want x_{ij} = 0 or 1 *for all i and j*. **Constraints**: $x_{13}+x_{43}+x_{63} \geq 1$ *(company 1)*, $x_{35}+x_{41}+x_{53}+x_{71} \geq 2$ *(company 2)*, $x_{12}+x_{21}+x_{44} \geq 1$ *(company 3)*, and so on, **down** to *(company 8)*, noting that we do **not** include the x_{ij} for **Friday** afternoon. $x_{11}+x_{25}+x_{45}+x_{74} \geq 2$ *(events)*, $x_{31}+x_{32}+x_{33}+x_{34}+x_{35}+x_{71}+x_{72}+x_{73}+x_{74} \geq 1$ *(prize draw)*, $x_{1c}+x_{2c}+x_{3c}+x_{4c}+x_{5c}+x_{6c}+x_{7c} \leq 6$ *(not full attendance).* The **objective** function is to **minimise** $x_{1c}+x_{2c}+x_{3c}+x_{4c}+x_{5c}+x_{6c}+x_{7c}$ (to minimise the number of *coffee breaks* attended).

26th March 2001

Definition: The *expected value* EV(L) of a lottery L = (p_1, r_1; ..., p_n, r_n) is given by EV(L) = $\Sigma_{i=1}^{n} p_i r_i$. The *expected utility* for L is E(U for L) = $\Sigma_{i=1}^{n} p_i U(r_i)$. **Lottery Question:** A CD is *rated from 0 to 5*. Sid is **indifferent** between L_1: with *probability* 1. the grade is 4; and L_2: with *probability* $^1/_3$. the grade is 5, and with *probability* $^2/_3$. the grade is 2.

We are **also** indifferent between L_3: 1 for 3; and L_4: ¼ for 5, ¾ for 1; **also** between L_5: 1 for 2; and L_6: ¾ for 4, ¼ for 0; and **also** between L_7: 1 for 1; and L_8: $^3/_5$ for 3, $^2/_5$ for 0. Sid *estimates* the band's chance of obtaining **rating** {5, 4, 3, 2, 1, 0} as {10%, 35%, 15%, 25%, 10%, 5%}. Calculate the **expected utility** for this CD.

A: We start by obtaining *simple lotteries*: indifferent **between** 1 for i, **and** u for 5/1-u for 0. Bu if we use the *diagrammatic* approach here, for example as **shown** for rating 4, we have to use *geometric series*. But there is an alternative method: let U_5 = the utility of *grade 5*, etc. We are given a set of simultaneous equations: $U_4 = {}^1/_3U_5+{}^2/_3U_2$ (---(4)), $U_3 = {}^1/_4U_5+{}^3/_4U_1$ (---(3)), $U_2 = {}^3/_4U_4+{}^1/_4U_0$ (---(2)), and $U_1 = {}^3/_5U_3+{}^2/_5U_0$ (---(1)) (with U_5 = 1, and U_0 = 0).

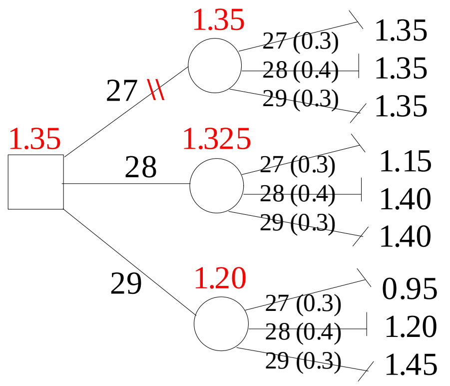


Now (4) ⇒ $U_4 = {}^1/_3+{}^2/_3U_2$, and (2) ⇒ U_2 = ¾U_4+0. We can *solve these simultaneously* to give $U_4 = {}^1/_3+{}^2/_3.{}^3/_4U_4$; ½$U_4 = {}^1/_3$; $U_4 = {}^2/_3$, and *so* U_2 = ½. *Similarly*, (3) and (1) ⇒ $U_3 = {}^5/_{11}$ and $U_1 = {}^3/_{11}$. So now we can *write out the six simple lotteries*: e.g. —1—3 is **indifferent** to $^5/_{11}$ for 5, and $^6/_{11}$ for 0. **Note** that U_i gives the "(...) for 5" part in the *lottery*. So we can construct the *table* shown, and so EU = 1(0.1) + $^2/_3$(0.35) + ... + 0(0.05) ≈ 0.55.

Reward	Utility	Prob.
5	1	10%
4	2/3	35%
3	1/2	15%
2	5/11	25%
1	3/11	10%
0	0	5%

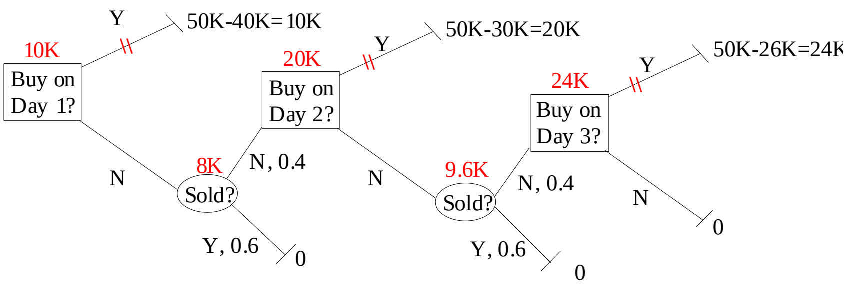
29th March 2001

Decision Trees

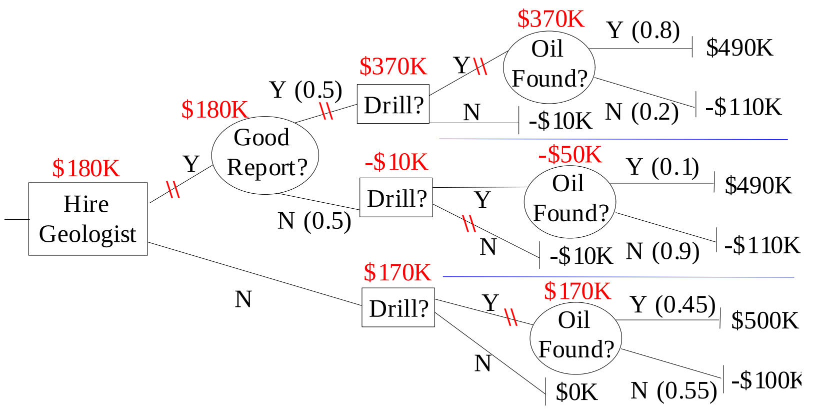


These trees have *3 types of node:* (1) **Decision** nodes (*square* boxes); (2) **Event** Nodes (*round* boxes) — where we have probabilities on the edges; where the **sum** of the probabilities is 1; and where the events are *beyond* the DM's control; and (3) **Terminal** nodes. **Example 1**: We return to the *newspaper vendor* where, in this question, A = {27, 28, 29} = the number purchased, and S = {27, 28, 29} = the demand. The demands have **probabilities** P(27) = 0.3, P(28) = 0.4, and P(29) = 0.3. We *backtrack* to find the expected profit at each event node. For example, for the *second* branch in the diagram, $^3/_{10}(1.15) + {}^4/_{10}(1.40) + {}^3/_{10}(1.40) = 1.325$. At a *decision node*, we mark the **best** branch with a (\\).

Q: We are willing to **pay** $50,000 for a painting. We can buy *today* for $40,000, or wait until *tomorrow* and buy it for $30,000, if it has not been sold. On the *third* day, it is $26,000, and *no longer available* after this. Each day, there is a 0.60 **probability** that the painting will be sold. What strategy **maximises** the dealer's expected profit? A: This question was answered earlier using *lottery methods*, and we will now attack it using **decision trees**. As you can see, the *expected* profit is $10K.



Q: It costs $100,000 to **drill**, and if oil is **found**, its value is $600,000. There is a 45% chance that the field contains oil. We can, before drilling, hire a **geologist** to obtain more information about the *likelihood* that the field will contain oil. There is a 50% chance that the geologist will issue a favourable report, and a 50% chance of an unfavourable report. Given a **favourable** report, there is an 80% chance that the field **contains** oil (10% chance with an **unfavourable** report). Determine the *optimal* course of action. A: as shown in the *diagram* above.



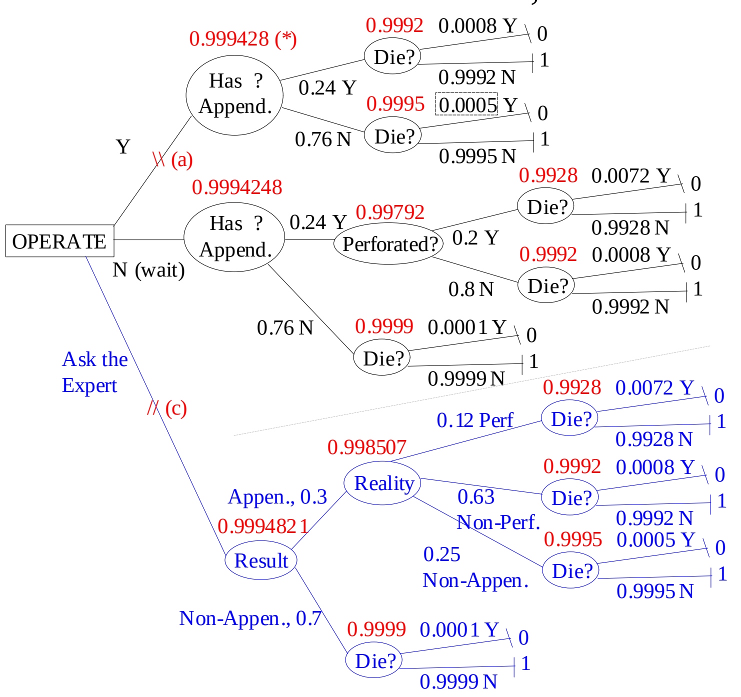
Q: Statistics show that *24% of patients* entering casualty with acute abdominal pain have **appendicitis**. The doctor can operate immediately, or wait until the morning. In the meantime, if the patient has appendicitis, there is a 20% chance that the appendix may **perforate**. The risk for a patient with *acute abdominal pain, but without appendicitis*, is 0.0001 if left with no operation. The mortality rates of the operations are as shown in the table.

Operation	P(patient dies)
Patients with appendicitis	0.0008
Patients with perforated appendix	0.0072
Patients without appendicitis	0.0005

(a) Using a **decision tree,** find the strategy that the doctor should take for *maximising* the chance for the patient to survive. (b) Assume that the statistical figure about the mortality rate of the operation on patients without appendicitis was **not** available. What is the range of *possible values of the mortality rate* that will not **change** the doctor's strategy?

(c) Assume that the doctor has a *3rd alternative — consulting an expert.* The expert takes 6 hours to make his decision, and the probability for a perforated appendix is **reduced**. The patients that the expert classifies as *appendicitis* (30% of all cases) will be 12% with perforated appendix, 63% with non-perforated appendicitis, and 25% without appendicitis. The patients that the expert classifies as *non-appendicitis* (70%) will **all** be without appendicitis. Find the **optimal** strategy of the doctor, and calculate the *total chance that a patient has to survive.*

A: In the *diagram*, I assume that the "*reward*" for the **best** outcome is 1, and that the "*reward*" for the **worst** outcome is 0. (a) Ignoring the blue branch for the time being, we see that the **optimal** course of action is to operate *immediately*. (b) In order for the doctor to change his/her mind, value * has to be less than 0.9994248, **where** (0.24)0.9992 + 0.76p = * (p is 1-the *mortality* we now do not know). If we want * < 0.9994248, then (0.24)0.9992 + 0.76p < 0.9994248; 0.76p < 0.7596168; p < 0.99949578... So the *mortality rate* will have to be in the range 0 ≤ mortality ≤ 1-p for the doctor **not** to change his/her mind. (c) We add in the blue branch to the diagram. When we do this, as you can see, the *optimal* course of action is now to **ask the expert** for his opinion first.



2nd April 2001

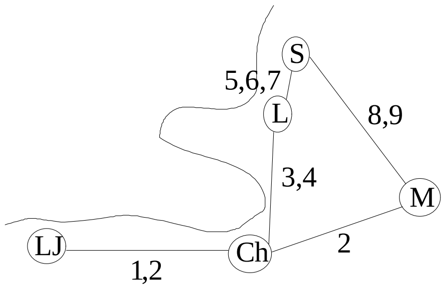
Tutorial

A rail traveller is planning a journey *from Llandudno Junction to Southport*, with either connections at **Chester** and **Liverpool,** or a connection at **Manchester**. The traveller is considering the trains shown in the table *above*. In each case, when a connection is <u>missed</u>, there is an *alternative* connection later. Trains are assumed to arrive either on time, 10 minutes late, or 20 minutes late, as detailed in the table on the left of **probabilities**.

Train No.	1	2	3	4	5	6	7	8	9
Lland. Jun.	11:45	11:55							
Chester	12:30	12:50	12:45	13:45					
Liverpool		↓	13:50	14:50	14:00	14:30	15:00		
Manchester		13:50			↓	↓	↓	13:55	14:55
Southport					15:15	15:45	16:15	15:20	16:20

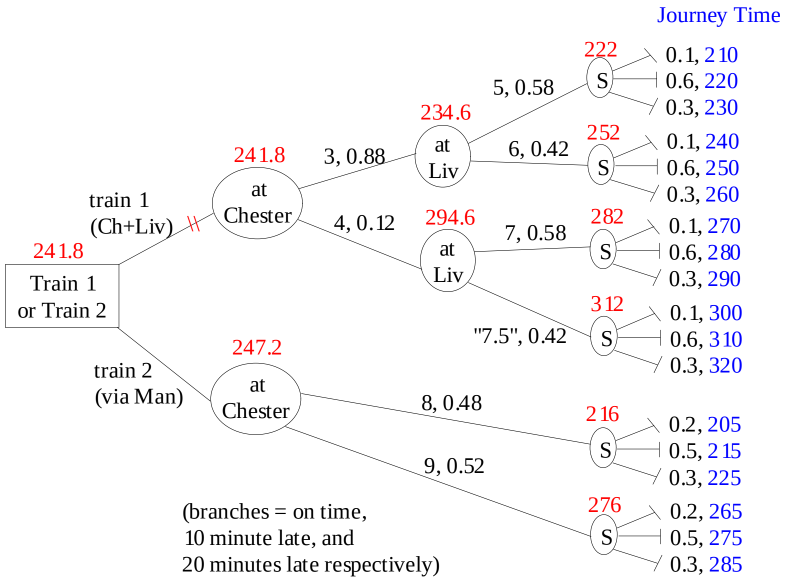
train	arrival at	on time	10mins	20mins
1,2	Llandudno Junct.	1.0	0.0	0.0
1,2	Chester	0.5	0.3	0.2
2	Manchester	0.2	0.5	0.3
3,4	Chester	0.6	0.3	0.1
3,4	Liverpool	0.2	0.6	0.2
5,6,7	Liverpool	0.4	0.5	0.1
5,6,7	Southport	0.1	0.6	0.3
8,9	Manchester	0.5	0.4	0.1
8,9	Southport	0.2	0.5	0.3

(1) What is the probability that, *when train 1 is taken*, the connection is **made** at Chester? (2) What is the probability that, *when train 2 is taken*, the connection is **made** at Manchester? (3) What is the probability that, *when train 3 is taken*, the connection is **made** at Liverpool? (4) Decide **which** train the traveller should catch from Llandudno Junction so as to **minimise** the expected journey time.



A: (1) Consider **connection** Train 1 → Train 3. The train can *arrive* at Chester on time at 12:30 (with probability 0.5), or 10 minutes **late** at 12:40 (0.3), or 20 minutes **late** at 12:50 (0.2). The *departing* train 3 can be **on** time at 12:45 (0.6); 10 minutes **late** at 12:55 (0.3); or 20 minutes **late** at 13:05 (0.1). There is therefore *only one way* to miss the train, with probability $0.2\times0.6 = 0.12$. The connection is thus made with *probability* 1-0.12 = 0.88 (this value is also given by summing up over all combinations).

(2) **Arrival**: on time at 13:50 (0.2); 10 minutes late at 14:00 (0.5), or 20 minutes late at 14:10 (0.3). **Departing**: at 13:55, 14:05, and 14:15, with respective *probabilities* 0.5, 0.4, and 0.1. Here, there are 6 ways to **catch** the train, and 3 ways to **miss** is. We could set up a *3×3 grid* to see the possibilities, and then work out the *probability*. Similarly for (3). For (4), we construct the decision tree shown on the right, where the **final** decision is to catch train 1 at Llandudno Junction to *minimise* the journey time.

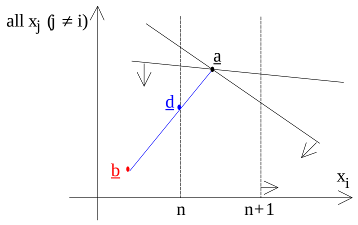


30th April 2001

An Alternative Proof on an Earlier Theorem

Proposition: Let S be a *subproblem* of an IP problem P, with solution at a in which the basic variable is $x_i = a_i \in (n, n+1)$. Let S_1 be the **subproblem** of S obtained by *adding* the constraint $x_i \le n$. Then S_1 has an **optimal** solution to its LP-relaxation, with $x_i = n$.

Proof by Contradiction: Let b be a point where the *LP-max* of S_1 is obtained, with $x_i = b_i < n$. Let $k = (n-b_i/a_i-b_i)$, where $0 < k < 1$, and let $d = ka+(1-k)b$, so that $d_i = ka_i+(1-k)b_i = b_i+(n-b_i/a_i-b_i)(a_i-b_i) = n$. By **convexity**, d is **in** the feasible region for S, and *so* for S_1. The **values** of the objective function satisfy $M_b \le M_d \le M_a$. ($c.b \le c.d \le c.a$). Since M_b is *optimal* for S_1, it **follows** that $M_b = M_d$. **So** (either b = d or) d is an *alternative optimal point* for S_1. **End of Proof**.



Exam Paper: May 2001

Answer 3 questions out of 5 (Questions Done: 2, 3, 4)

(1) The LP-relaxation of the pure **IP** problem

P: $\max(M = 5x_1 + 6x_2 + 7x_3 + 8x_4)$

such that $3x_1 + 2x_2 + 4x_3 + 4x_4 \le 52$

$4x_1 + 3x_2 + 2x_3 + 3x_4 \le 47$

$6x_1 + 5x_2 + 4x_3 + 3x_4 \le 59$

$x_1, x_2, x_3, x_4 \in \mathbf{Z}^{\ge 0}$.

has solution tableau

M	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs	basic
10	33				11	11	1	1,148	M = 114.8
	3			10	1	6	-4	98	$x_4 = 9.8$
	11	10			-3	2	2	56	$x_2 = 5.6$
	-1		10		3	-7	3	4	$x_3 = 0.4$

(a) Apply the cutting plane algorithm to the second constraint (x_2-row) in the solution tableau and solve the resulting LP-relaxation using the dual simplex method.

[8 marks]

(b) Let $P_{1,3}$ be the subproblem of P obtained by setting $x_1 = x_3 = 0$. Solve the LP-relaxation of $P_{1,3}$ graphically. Solve $P_{1,3}$ graphically. **[10 marks]**

(c) State the best upper and lower bounds for the solution of P currently available.

[2 marks]

(2) (a) Let F1 be the linear fractional program

$\mathbf{max}\left(\frac{\boldsymbol{p}.\boldsymbol{x}+\alpha}{\boldsymbol{q}.\boldsymbol{x}+\beta}\right)$ s.t. $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$,

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\alpha, \beta \in \mathbb{R}$, A is an m×n matrix, and $\mathbf{q}.\mathbf{x} + \beta \neq 0$ for all $\mathbf{x}$ in the feasible region S.

(i) Explain why we may assume that $\mathbf{q}.\mathbf{x} + \beta > 0$ for all $\mathbf{x} \in S$.

(ii) Show that the transformation $\mathbf{y} = z\mathbf{x}$, $z^{-1} = \mathbf{q}.\mathbf{x} + \beta$ converts F1 into a formally equivalent LP, F2: $\mathbf{max}(\mathbf{p}.\mathbf{y} + \alpha z)$ s.t. $A\mathbf{y} - z\mathbf{b} \le \mathbf{0}$, $\mathbf{q}.\mathbf{y} + \beta z = 1$, $(\mathbf{y}, z) \ge \mathbf{0}$. **[10 marks]**

(b) Solve the linear fractional problem

$$F : \mathbf{opt}\left(\frac{10 - x - 2y}{25 - 2x + y}\right) \text{such that} \begin{cases} 3 \le x + y \le 9, \\ -5 \le x - y \le 7, \\ x, y \ge 0, \end{cases}$$

in the cases (i) **opt** = **min**; (ii) **opt** = **max**. **[10 marks]**

(3) (a) Let P: $\max(\mathbf{c}.\mathbf{x})$, $A\mathbf{x}+\mathbf{s} = \mathbf{b}$, $\mathbf{x},\mathbf{s} \geq \mathbf{0}$ be a standard max problem, and let D(P): $\min(\mathbf{b}.\mathbf{u})$, $A^t\mathbf{u}-\mathbf{e} = \mathbf{c}$, $\mathbf{u},\mathbf{e} \geq \mathbf{0}$ be the dual of P. Prove the following:

(i) If $\mathbf{x}_0$, $\mathbf{u}_0$ are feasible for P, D(P) respectively, then $\mathbf{c}.\mathbf{x}_0 \leq \mathbf{b}.\mathbf{u}_0$. **[3 marks]**

(ii) If $\mathbf{c}.\mathbf{x}_0 = \mathbf{b}.\mathbf{u}_0$, then $(\mathbf{x}_0, \mathbf{s}_0)$ and $(\mathbf{u}_0, \mathbf{e}_0)$ are complementary slack. **[3 marks]**

(iii) If P and D(P) are both feasible, then both have optimal solutions.

[4 marks]

(b) Let P be the standard max **LP** problem

$$\text{P: } \max\ (M = 4x_1 + 3x_2 + 5x_3 + 6x_4)$$

such that

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + s_1 &= 9,\\ 2x_1 + 3x_2 + 3x_3 + 2x_4 + s_2 &= 10,\\ 2x_1 + x_2 + 5x_3 + 3x_4 + s_3 &= 8,\\ 3x_1 + 2x_2 + 5x_3 + 4x_4 + s_4 &= 11,\\ \mathbf{x}, \mathbf{s} &\geq \mathbf{0}. \end{aligned}$$

Write down the dual problem D(P).
Test the following proposed vectors for optimality:

(i) $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, (ii) $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}$, (iii) $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. **[10 marks]**

(4) (a) A traveller wishes to buy some books to take on a journey. The estimated time taken to read each of the 6 books, and the purchase price of each book, are shown in the following table:

book	A	B	C	D	E	F
reading time (hours)	15	13	9	8	7	6
purchase price (£)	12	10	7	6	6	5

Which of these books should the traveller buy so as to give the maximum amount of reading time without spending more than £21? Use a branch and bound method to solve this problem, calculating the LP-relaxation solution at each stage. Show that your answer is unique. **[12 marks]**

(b) Jane estimates her chances of obtaining grades {A, B, C, D} as {20%, 50%, 20%, 10%} in module M.

Jane is indifferent between: L_1: with probability 1.0 she gains grade B;
and: L_2: with probability 0.4 she gains grade A,
with probability 0.6 she gains grade C;
and also between: L_3: with probability 1.0 she gains grade C;
and: L_4: with probability 0.5 she gains grade B,
with probability 0.5 she gains grade D.

Calculate the expected utility for module M. **[8 marks]**

(5) A rail traveller is planning a journey from Bangor to Cardiff with *either* a connection at Birmingham *or* a connection at Crewe (for the Shrewsbury and Hereford line). The traveller is considering the following trains:

Train No.	1	2	3	4	5	6
Bangor	09:20			09:50		
Crewe	↓			11:25	11:30	12:30
Birmingham	11:50	12:05	13:05		↓	↓
Cardiff		13:50	14:50		14:00	15:00

In each case, when a connection is missed, there is an alternative connection one hour later. Trains are assumes to arrive either *on time*, or *10 minutes late*, or *20 minutes late*, as detailed in the following table of probabilities:

train	arrival at	on time	10 min late	20 min late
1, 4	Bangor	1.0	0.0	0.0
1	Birmingham	0.2	0.4	0.4
2, 3	Birmingham	0.3	0.6	0.1
2, 3	Cardiff	0.2	0.5	0.3
4	Crewe	0.2	0.6	0.2
5, 6	Crewe	0.4	0.5	0.1
5, 6	Cardiff	0.1	0.6	0.3

Show that, when train 1 is taken, the traveller makes the connection at Birmingham with probability 0.88. What is the probability that, when train 4 is taken, the connection is made at Crewe? **[6 marks]**

Decide which train the traveller should catch from Bangor so as to minimise the expected journey time. **[10 marks]**

Train 1 is also scheduled to stop at Crewe at 10:50, allowing a connection with train 5. Discuss whether this extra possibility changes your decision. **[4 marks]**