G2M31 Abstract Algebra: Exam Paper, May 2000

SECTION 1 (Compulsory)

(1) (a) Find the multiplicative inverse of 169 modulo 391. **[3 marks]**

(b) Show that the matrix $\begin{pmatrix} 2 & 1 \\ 7 & 5 \end{pmatrix}$ is invertible modulo 26 and calculate its inverse. **[4 marks]**

(c) Let α = (147) (258) (369) (123456789) be an element of the permutation group S_9. Write α as a product of disjoint cycles, and also as a product of transpositions. **[4 marks]**

(d) Find the last two digits of 43^{4443}. **[4 marks]**

(e) Show that $n^{67} \equiv n$ modulo 469 for all $n \in \mathbf{Z}$. **[5 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Let * be a closed binary operation on a set S. Give the properties which need to be satisfied for S with * to form a group. **[3 marks]**

(b) Define x*y = 2xy+x+y where x, y ∈ ‘ and let S = ‘ \ {-½}. Show that S with * forms a group. **[12 marks]**

(3) (a) State Lagrange’s Theorem and Euler’s Theorem. **[2 marks]**

(b) Prove Euler’s Theorem using Lagrange’s Theorem. **[7 marks]**

(c) Find a generator for the multiplicative cyclic group $\boldsymbol{Z}_{11}^{*}$ = {1, 2, 3, ..., 9, 10}, and hence find all possible generators. **[6 marks]**

(4) (a) Describe the main features of the R.S.A. public key coding system. **[4 marks]**

(b) In an R.S.A. public key coding system the public key is n = 111 and e = 65. Encode the message 07, 15. **[4 marks]**

(c) By decomposing n into its prime factors decode the message 98, 29, 49, 18. **[7 marks]**

(5) A matrix code has the encoding matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Write down a parity check matrix for this code. **[1 mark]**

(b) Construct a table of message words and code words. **[3 marks]**

(c) Construct a table of coset leaders and syndromes and use it to decode the message 1110100, 0101111, 1000100. **[8 marks]**

(d) If p is the probability of correct reception of one digit, find in terms of p and q = 1-p the probability of correct decoding of one word. **[3 marks]**

(Questions done: 1, 2, 5)

G2M42 Linear Differential Equations: Exam Paper, May 2000

SECTION 1 (Compulsory)

(1) (a) Define the Wronskian W of two independent solutions $y_1(x)$ and $y_2(x)$ of the differential equation $D^2y+P(x)Dy+Q(x)y = 0$. Show that $y = v_1y_1+v_2y_2$ is a particular solution of $D^2y+P(x)Dy+Q(x)y=R(x)$ where $Dv_1 = -y_2R/W$ and $Dv_2 = y_1R/W$. **[10 marks]**

(b) Show that $y_1(x) = x$ and $y_2(x) = e^x$ are solutions of the differential equation $(x-1)D^2y-xDy+y=0$. By calculating the Wronskian $W = W(y_1, y_2)$ and by converting the non-homogeneous differential equation $(x-1)D^2y-xDy+y = (x-1)^2e^x$ into standard form, find its general solution. **[10 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Explain what is meant by the term "regular singular point" and show that the origin is a regular singular point of the differential equation $3xD^2y+4Dy+y=0$. Use the method of Frobenius to find the indical equation and a recurrence relation for the coefficients of the series solutions of this differential equation. Hence find the general solution giving the first 3 terms of each series. **[15 marks]**

(3) Let the Legendre polynomials be defined by the Rodrigués formula
$P_n(x) = \frac{1}{2^n n!} D^n((x^2-1)^n)$.
Show that P_n satisfies the differential equation $(1-x^2)D^2y-2xDy+n(n+1)y=0$. **[5 marks]**

Deduce that the functions $P_n(x)$ satisfy $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ for $m \neq n$. **[4 marks]**

Show that the polynomial $xP_n(x)$ is orthogonal to $P_k(x)$ for $k < n-1$. Deduce that there is a recurrence relation of the form $xP_n(x) = \alpha P_{n+1}(x) + \beta P_{n-1}(x)$. **[6 marks]**

(4) Define the Laplace transform Y(s) of y(t). Let D denote differentiation with respect to the dependent variable, $D = {}^{d}/_{dt}$.

(a) Use the method of Laplace transforms to solve the differential equation
$D^2x+4x = \sin t$, where $x(0) = Dx(0) = 0$. **[6 marks]**

(b) Use the method of Laplace transforms to solve the simultaneous differential equations $D^2y+z+y=0$ and $Dy+Dz = 0$, where $y(0) = Dy(0) = 0$ and where $z(0) = 1$. **[7 marks]**

(5) Find the Fourier half-range sine and half-range cosine series for the function f defined by: $f(x) = \pi-x$, $0 \leq x \leq \pi$. Show that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - ... = \frac{\pi}{4}$ and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$. **[15 marks]**

(Questions done: 1, 2, 4)

G2M55 Advanced Calculus: Exam Paper, May 2000

SECTION 1 (Compulsory)

(1) (a) Define the gradient of the scalar field f(x, y, z). Find the gradient and the directional derivative in the direction $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ of the scalar field $f = xy^2z^3$ at the point (2, 1, -1). **[5 marks]**

(b) Define the curl of the vector field **v**.
If $\mathbf{v} = (\mathbf{k}\times\mathbf{r})\times\mathbf{r}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, find curl **v**. **[7 marks]**

(c) Write down the Cauchy-Riemann equations satisfied by the real and imaginary parts u, v of the differentiable function f(z) = u(x, y) + iv(x, y). If the real and imaginary parts of a complex valued function satisfy the Cauchy-Riemann equations, is that function differentiable? Explain your answer.

Examine whether or not the real and imaginary parts of the function $f(z) = z^2+5iz+3-i$ satisfy the Cauchy-Riemann equations and state the domain of analyticity of the function. **[8 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Prove that $\text{curl}(\phi\mathbf{A}) = (\text{grad}\ \phi)\times\mathbf{A} + \phi\,\text{curl}\mathbf{A}$. **[4 marks]**

(b) Simplify curl(grad ϕ). **[2 marks]**

(c) Calculate $\text{curl}(r^2\mathbf{r})$ where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r = |\mathbf{r}|$. **[4 marks]**

(d) Is there a function ϕ such that $\text{grad}\ \phi = r^2\mathbf{r}$? If so, then find such a ϕ. If not, then explain why not. **[5 marks]**

(3) (a) Given a space curve $\mathbf{r} = r(t)$, define the unit tangent vector $\hat{\boldsymbol{T}}$, the unit principal normal vector $\hat{\boldsymbol{N}}$ and the curvature κ. **[3 marks]**

(b) Show that $\kappa = \frac{|\dot{\boldsymbol{r}}\ \ddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^3}$ **[4 marks]**

(c) If $\mathbf{r}(t) = t^4\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}$, find $\hat{\boldsymbol{T}}$, κ at the point P = (1, 1, 1). **[4 marks]**

(d) Calculate the work done in moving the force $\mathbf{F} = (3x-2y,\ y+2z,\ -x^2)$ from the origin to P along the curve given in part (c). **[4 marks]**

(4) (a) Give the definition of a harmonic function u(x,y). **[2 marks]**

(b) Show that $u = x^3-2xy^2+3x^2-3y^2+1$ is a harmonic function. **[3 marks]**

(c) Find the harmonic conjugate function v(x, y) such that w = u+iv is analytic. **[5 marks]**

(d) Express w in the form f(z) where z = x+iy. **[5 marks]**

(5) Determine the residues at the poles of f(z) where
$f(z) = \frac{z^4+1}{z^2(z-a)(z-a^{-1})}$, $0 < |a| < 1$. **[6 marks]**

Use the substitution $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, and the Residue Theorem to deduce that
$\int_0^{2\pi} \frac{\cos(2\theta)d\theta}{1-2a\cos(\theta)+a^2} = \frac{2\pi a^2}{1-a^2}$, $0 < |a| < 1$. **[9 marks]**

(Questions done: 1, 4, 5)

G2M81 Statistical Modelling: Exam Paper, May 2000

Answer 3 questions out of 4 (Questions Done: 1, 2, 3)

(1) You have taken a random sample of n observations $y_1, y_2, ..., y_n$ from a normal distribution with mean μ and variance σ^2.

(a) Find the maximum likelihood estimators of μ and σ^2. **[12 marks]**

(b) State the expected value of the m.l.e. of σ^2. **[3 marks]**

(c) State the effect this has in common statistical usage. **[5 marks]**

(2) The multiple linear regression model can be expressed as $y = X\beta+\varepsilon$ where y is a vector of response measurements y_i, X is a matrix of row vectors of explanatory variables l, $x_{i1}, x_{i2}, ..., x_{ip}$, $\beta_0....\beta_p$ is a vector of parameters and ε is a vector of random departures.

(a) Show that the least squares estimates of β are $\hat{\beta} = (X^tX)^{-1}X^ty$. **[12 marks]**

(b) Show that $\hat{\beta}$ is unbiased and that the covariance matrix of $\hat{\beta}$ is $\sigma^2(X^tX)^{-1}$. **[8 marks]**

(3) For a one factor Anova model where n observations have been randomly sampled from each of k groups, show that the total SS = $\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-y_{..})^2$ can be partitioned into two parts $\sum_{i=1}^{k} n(y_{i.}-y_{..})^2$ and $\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij}-y_{i.})^2$. **[8 marks]**

Describe in words the meaning of each of the two parts. **[4 marks]**

In an experiment at College farm 30 plots of land were used to study the effect of nitrogen fertiliser on barley yield. Six fertiliser levels were used (0, 40, 80, 120, 160, 200 kg/ha) and the yield of barley measured in tons/ha. The table below shows the mean and variance of the yield of barley for each fertiliser level. These were 5 replicates for each fertiliser level. The total SS is 43.50.

	0	40	80	120	160	200
Mean	4.02	5.23	5.81	6.99	7.27	6.99
Variance	0.088	0.264	0.096	0.147	0.074	0.041

Using the above information write out the analysis of variance table, state any further appropriate analyses you would perform and summarise your conclusions about the relationship between fertiliser level and barley yield. **[8 marks]**

(4) You have performed an experiment to examine the relationship between areal biomass and five chemical measurements of the soil. Show below is a table containing the regression SS for each model. Using the information that the total SS is 19 170 963 and that 45 observations were made,

(a) Write out the forward regression Anova table; **[8 marks]**

(b) Which model would you choose from the Anova table in (a) using:
(i) The residual mean square criterion. **[4 marks]**
(ii) Mallows C_p statistic. **[4 marks]**

(c) Which model would you recommend using from your answers to (a) and (b). **[4 marks]**

In the table below the following abbreviations have been used:
Salinity = S, pH = P, Potassium = K, Sodium = N and Zinc = Z.

1 Predictor	
S	204 048
P	11 490 388
K	806 574
N	1 419 069
Z	7 474 474
2 Predictors	
S P	11 567 715
S K	1 027 686
S N	1 487 655
S Z	10 594 197
P K	12 415 118
P N	12 622 789
P Z	11 661 321
K N	1 424 635
K Z	7 961 377
N Z	12 205 083
5 Predictors	
S P K N Z	12 984 915

3 Predictors	
S P K	12 503 299
S P N	12 634 567
S P Z	12 205 083
S K N	1 487 757
S K Z	11 052 633
S N Z	10 820 655
P K N	12 659 874
P K Z	12 501 663
P N Z	12 699 814
K N Z	8 244 127
4 Predictors	
S P K N	12 685 656
S P K Z	12 937 009
S P N Z	12 878 489
S K N Z	11 068 315
P K N Z	12 732 925

G2M85 Operations Reseach & Linear Programming: Exam Paper, May 2000

SECTION 1 (Compulsory)

(1) A toy manufacturer produces two types of toy, A and B. When running at full capacity, the factory has at most 1000 man-hours available in its machining department, 400 man-hours available in its assembly department and 250 man-hours in its painting department. Manufacturing time (in hours) required by each toy in the three departments is given by:

	Machining	Assembly	Painting
A	0.2	0.2	0.1
B	0.5	0.1	0.1

A toy of type A sells for £10 and one of type B for £8.

(a) Formulate a linear program model of this situation, explicitly defining the variables, the constraints they must satisfy and the objective function. **[8 marks]**

(b) Use the simplex algorithm to find the production levels that will maximise the income. **[12 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Consider the following minimisation problem:

Minimise $w = 4y_1 + 3y_2 + y_3$

subject to $y_1 + 2y_2 \geq 6$ (1)

$y_1 - y_2 + y_3 \geq 8$ (2)

$2y_2 + y_3 \geq 10$ (3)

where $y_1, y_2, y_3 \geq 0$.

Find the dual problem, solve the dual problem and form the dual optimal tableau, find the solution to the primal problem. State the amount of surplus in each primal constraint and identify and interpret the shadow prices. **[15 marks]**

(3) A small manufacturing group has 3 warehouses and 4 retail outlets. The relative costs of transporting one item from warehouse to outlet are given in the following table, as are the resources available (right hand column) and the supply requirements (bottom row):

Outlets:	1	2	3	4	Resources
Warehouse 1	20	15	5	6	5
Warehouse 2	5	12	10	2	10
Warehouse 3	15	7	4	1	15
Requirements:	4	4	10	12	

(a) Use the NW corner, the minimum cost **and** Vogel's method to find basic feasible solutions as starting positions for the transport algorithm solution of this problem. **[6 marks]**

(b) Solve to find the optimal assignment, starting with the minimum cost basic feasible solution. **[9 marks]**

(4) (a) The processing times for five jobs on each of two machines is shown in the table below:

Job	Machine 1	Machine 2
1	4	3
2	2	2
3	5	4
4	3	5
5	4	6

All jobs have to be processed first on Machine 1 before passing to Machine 2. Find the optimal order so as to minimise makespan and draw a Gantt diagram for your schedule giving the total time taken, and the total idle time. **[7 marks]**

(b) Seven jobs have to be processed on a single machine. Job, J_k, takes p_k days to process and has due date d_k. The various values of p_k and d_k are given in the following table:

J_k	J_1	J_2	J_3	J_4	J_5	J_6	J_7
p_k	1	2	3	4	1	3	3
d_k	10	8	16	6	9	17	13

Let C_k be the completion time for J_k and set $\bar{C} = (\Sigma C_k)/7$, the average completion time. Using Smith's algorithm find a schedule that minimises $\bar{C}$ subject to the constraint that no job can be late. Give the optimal value of $\bar{C}$ and the corresponding schedule. **[8 marks]**

(5) (a) An engineering firm has agreed to undertake the design, fabrication, and testing of a prototype transmission for a major car manufacturer. They have identified the following activities and their associated times and precedence relationships. Construct the network diagram that represents this project and find the critical path. **[7 marks]**

	ACTIVITY	TIME (WEEKS)	IMMEDIATE PREDECESSORS
A	Establish design specifications	1	-
B	Mechanical design	5	A
C	Electrical design	4	A
D	Final design review	1	B, C
E	Prepare test vehicle	2.5	B
F	Fabricate prototype	2	D
G	Conduct test	3.5	E, F
H	Prepare 'blueprints'	1	G
I	Prepare final report	1	H

With only one team of workers, the firm will take 21 weeks to finish the project. Show that by getting a second team in for some of the time to work on non-critical activities, the project can be finished within the total time of the critical path. Draw a Gantt diagram showing the feasibility of your plan. **[5 marks]**

(b) For small extra cost, the time for activity B could be cut to 4 weeks and that of G to 3 weeks. What should the firm do if it could gain prestige and a handsome bonus by finishing the project within 13 weeks? Justify your answer. **[3 marks]**

(Questions done: 1, 2, 3)