G2M30 Linear Algebra: Exam Paper, January 2000

SECTION 1 (Compulsory)

(1) (a) Find the matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-2x+y, x+3y)$ with respect to:

(i) the standard basis of the domain and image;
(ii) the basis {(2, -1), (2, 2)} in the domain and the standard basis in the image;
(iii) the basis {(2, -1), (3, 2)} in both the domain and the image.

[10 marks]

(b) The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}.$$

(i) Find the eigenvalues and eigenvectors of A.
(ii) Find a matrix P such that $P^{-1}AP$ is diagonal. **[10 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Let S denote the subset $\mathbb{R}^4$ of all vectors of the form (a, b, c, 2b-a-c).
(i) Show that S is a subspace of $\mathbb{R}^4$. **[4 marks]**
(ii) Show that {(1, 1, 0, 1), (1, -1, -3, 0), (0, 0, 1, -1)} forms a basis of S. **[5 marks]**
(iii) Apply the Gram Schmidt process to produce an orthonormal basis for S.
[6 marks]

(3) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by T(x, y, z) = (x+y+z, 2x-y+z).
(i) Write down the matrix of T with respect to the standard bases in the domain, $\mathbb{R}^3$ and image, $\mathbb{R}^2$. **[2 marks]**
(ii) Prove that X = {(2, 3), (1, 2)} is a basis for $\mathbb{R}^2$. **[2 marks]**
(iii) Find the matrix T with respect to the standard basis of $\mathbb{R}^3$ and the basis X of $\mathbb{R}^2$.
[4 marks]
(iv) Show that Y = {(1, 2, 3), (1, 0, 1), (5, 1, 3)} is a basis of $\mathbb{R}^3$. **[4 marks]**
(v) Find the matrix of T with respect to the bases Y and X. **[3 marks]**

(4) The linear transformation T is represented by the matrix

$$A = \begin{pmatrix} 4 & 8 & 4 \\ -3 & 1 & 11 \\ 2 & 7 & 8 \end{pmatrix}$$

(i) Convert A to row echelon form. **[4 marks]**

(ii) Find the kernel of T. **[3 marks]**

(iii) Find the dimension of the image of T. **[2 marks]**

(iv) Find a basis for the image of T. Is (1, 1, 1) in the image? **[6 marks]**

(5) The symmetric matrix A has right eigenvectors v_1, v_2 where

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}, \; v_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Find a third eigenvector v_3 and eigenvalue λ_3 to provide an orthogonal basis of eigenvectors.

Write the quadric ϕ: $x^2+2y^2+3z^2-4xy-4yz+6y+6z-{}^{11}/_{2} = 0$ in matrix form. Describe the changes of basis and transformation needed to convert to standard form and hence classify the quadric. **[15 marks]**

(Questions done: 1, 2, 4)

G2M56 Functions of Several Variables: Exam Paper, January 2000

SECTION 1 (Compulsory)

(1) (a) Calculate the Hessian of the function:
$f(x, y) = -xye^{-(x^2+y^2)/2}$.
Find and classify all the critical points of f. **[12 marks]**

(b) The volume V of a sphere of radius R is given in cartesian coordinates by
$V = 2\iint_\Omega \sqrt{(R^2 - (x^2 + y^2)}\,dxdy$,
where Ω is the disc of radius R centred at the origin. Use polar coordinates to show that $V = {}^4/_3\pi R^3$. **[8 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) By using the method of Lagrange multipliers, minimise xyz subject to the constraint $8x^3-8y^3+27z^3 = 1$, with $x \geq 0$, and $z \geq 0$, whilst $y \leq 0$. **[15 marks]**

(3) (a) Evaluate the triple integral $\int_0^2 \int_0^x \int_0^{4-x^2} xyz\,dz\,dy\,dx$. **[8 marks]**

(b) Use the transformation u = x+y, v = 2x-y to simplify the integral $\iint_\Omega (x+y)^2 dxdy$ where Ω is the parallelogram bounded by the lines x+y=0, x+y=1, 2x-y=0, 2x-y=3. **[7 marks]**

(4) (a) Determine whether or not the following function is continuous at the origin. If the function is continuous then prove it, otherwise provide reasons for the lack of continuity: $f(x, y) = \frac{2xy}{x^2+y^2}$, if $(x, y) \neq (0, 0)$; $f(0, 0) = 0$. **[6 marks]**

(b) A function f: S→‘ is defined on a convex subset S of ‘ ^{n}. You are told that at some point $\mathbf{x_0} \in S$, f has a local minimum (so there is some $\varepsilon > 0$ such that if $\|\mathbf{x}-\mathbf{x_0}\| < \varepsilon$ then $f(\mathbf{x}) \geq f(\mathbf{x_0})$.) Supposing that f is a convex function on S, prove that f has a global minimum at $\mathbf{x}_0$. **[9 marks]**

(5) Consider the mapping given by $u = e^x\cosh(y)$, $v = e^x\sinh(y)$.

(i) Find the Jacobian ${}^{\partial(u, v)}/_{\partial(x, y)}$ of the transformation.
Determine where local inverses exist. **[6 marks]**

(ii) Solve for x, y in terms of u, v to find the inverse mapping and calculate its Jacobian, giving your answer in terms of u, v and also in terms of x, y: ${}^{\partial(x, y)}/_{\partial(u, v)}$. **[6 marks]**

Finally verify ${}^{\partial(u, v)}/_{\partial(x, y)}\ {}^{\partial(x, y)}/_{\partial(u, v)} = I$, the two-by-two identity matrix. **[3 marks]**

(Questions done: 1, 3, 5)

G2M70 Mechanics: Exam Paper, January 2000

SECTION 1 (Compulsory)

(1) (a) (i) A particle moves under the influence of a central force. Write down the vector equation of motion. Hence write down the value of $\boldsymbol{r} \times \ddot{\boldsymbol{r}}$ and deduce that the motion lies in a plane and that the angular momentum per unit mass h is a constant. **[5 marks]**

(ii) Derive the radial and transverse components of velocity $\dot{\boldsymbol{r}}$ and acceleration $\ddot{\boldsymbol{r}}$ in plane polar coordinates (r, θ). **[4 marks]**

(b) A square lamina of side length a has uniform density σ per unit area and total mass M. Calculate the components of the inertia tensor relative to a set of axes Oxyz fixed in the lamina such that O is at the bottom left corner and Ox and Oy are along two sides of the square. Use the parallel axis theorem to obtain the moment of inertia about the z-axis relative to a set of axes with origin at the centre of the square. **[8 marks]**

SECTION 2 (Answer 2 out of 4 questions)

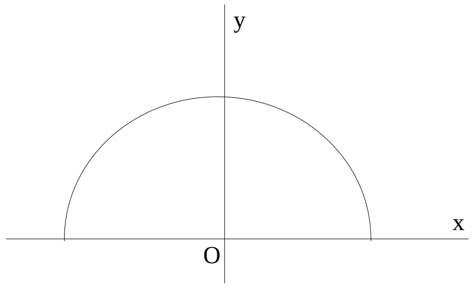
(2) (a) A particle moves under the influence of a central force. By referring to the equation of motion in plane polar coordinates (r, θ) as derived in Question 1 and introducing reciprocal place polar coordinates (u, θ), derive expressions for $\dot{r}$ and $\ddot{r}$ in terms of h, u, $^{du}/_{d\theta}$ and $^{d^2u}/_{d\theta^2}$. Hence derive the equation of orbit. **[6 marks]**

(b) Consider an inverse square law with force per unit mass given by $P = -^{\mu}/_{r^2}$, where μ is a positive constant. Write down the equation of orbit for this force. **[1 mark]**

(c) Find the angular momentum per unit mass and the constant speed v for which the particle can describe the circle r = a. **[3 marks]**

(d) By obtaining a general solution to the equation of orbit for the inverse square law of force, deduce that the orbit is a conic section with focus at the centre of force. **[5 marks]**

(3) A lamina is in the shape of a semi-circular disc of radius a and has constant surface density σ per unit area.



(a) Using place polar coordinates, calculate the components of the inertia tensor relative to a set of axes Oxyz aligned as shown in the above diagram. **[7 marks]**

(b) Calculate the angular momentum and kinetic energy of the lamina if it rotates with angular speed Ω about (i) the x-axis, (ii) the z-axis, (iii) the line in the xy-plane through the origin at an angle $^{\pi}/_{4}$ to Ox, (iv) the line perpendicular to the lamina through its centre of mass G. **[8 marks]**

(4) Two particles of equal mass m are attached at the points of trisection of an elastic spring of stiffness k and natural length $3l$ which is attached to two points 3a apart. The masses are constrained to move longitudally on a smooth surface. By a Lagrangian method, calculate the natural frequencies of the system and find the norm modes. Describe the amplitude and phase of the particles' displacements in each mode. **[15 marks]**

(5) (a) State the two basic postulates for Einstein's Special Theory of Relativity. **[1 mark]**

(b) Consider two inertial frames S and S' which are in standard configuration with relative velocity V along the x-axis. State the Galilean and Lorentz transformations relating the description of an event at x, y, z, t in frame S with its description x', y', z', t' in frame S'. Discuss and compare the two transformations. **[4 marks]**

(c) Derive an expression for the Lorentz-Fitzgerald contraction and an expression for time dilation. **[6 marks]**

(d) A car moves with a speed of 108 km per hour. If the length of the car is 3.4m, calculate the relative decrease in its length as noted by a stationary observer. (Take the speed of light to be $3\times10^{8}ms^{-1}$). **[2 marks]**

(e) Calculate the speed with which the car must move in order that its length be shortened to half its proper length. **[2 marks]**

<u>(Questions done: 1, 3, 4)</u>

G2M84 Fuzzy Sets, Uncertainty and Information: Exam Paper, January 2000

SECTION 1 (Compulsory)

(1) (a) Define t-norm between two fuzzy sets giving a set of axioms. Give the name and a mathematical expression for each axiom. **[5 marks]**

(b) Let A be a fuzzy set on some universal set U. Let A' be the closest crisp set of A, $\bar{A}$ be the complement of A, A* be a sharpened version of A, and A_α be the α-cut of A. There are exactly four correct statements in the below table. Find and rewrite them (one mark each). (Make sure you write down *only four*.) **[4 marks]**

$A \subseteq A^*$	$core(A) \subseteq A_{0.5}$	$A \subseteq A_{0.0}$
$A \subseteq A_{1.0}$	$A \subseteq \bar{A}$	$A \subseteq A'$
$A_{0.5} \subseteq A_{0.2}$	$A^* \subseteq A_{0.0}$	$core(A) \subset A_{1.0}$

(c) The table below shows three fuzzy sets, A, B and C on the same universal set U.

$U \rightarrow$	u_1	u_2	u_3	u_4	u_5
μ_A	0.1	0.4	0.9	0.3	0.2
μ_B	0.1	0.6	0.1	0.0	0.7
μ_C	0.1	0.0	0.0	0.2	0.4

(i) Aggregate the three fuzzy sets using OWA (ordered weighted averaging) coefficients [0, ½, ½]. **[2 marks]**

(ii) Find $|core(A \cup B)|$ and $height(A \cup \bar{A})$. **[1 mark]**

(iii) Calculate the fuzziness $\nu(A)$ of A. **[1 mark]**

(iv) Calculate the similarity between A and B using a measure of similarity of your choice. **[2 marks]**

(d) (i) Draw membership functions for the linguistic labels "small" and "large" for $x_1 \in [5, 50]$. Draw membership functions for the linguistic labels "early" and "approximately on time" on a time scale $x_2 \in [-10, 10]$. **[2 marks]**

(ii) Formulate all possible if-then rules for the two variables and calculate their firing strengths for input $[20, -4]^T$ (you can estimate approximately the degree of membership from your graph). **[3 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Prove that the Hamacher product $T(a,b) = \frac{ab}{a+b-ab}$ is a t-norm. **[15 marks]**

(3) Let R(X, X) be a relation on the (crisp) set $X = \{x_1, ..., x_4\}$ expressed as the matrix

$$R = \begin{bmatrix} 0.1 & 0.6 & 0.4 & 0.1 \\ 0.9 & 0.6 & 0.1 & 0.3 \\ 1.0 & 0.0 & 0.0 & 0.4 \\ 0.0 & 0.1 & 0.5 & 0.8 \end{bmatrix}$$

(a) Find an α such that the α-cut of R has the maximal possible cardinality and is a *mapping* (could be incompletely specified). Characterise the mapping obtained. **[3 marks]**

(b) Characterise the relation $R_{0.6}$ (the 0.6-cut of R). **[3 marks]**

(c) Prove that antitransitivity involves antireflexivity (for any relation). **[5 marks]**

(d) Calculate the max-min composition $R_{0.3} \circ R^{-1}$, where R^{-1} is the inverse of relation R. **[4 marks]**

(4) Let X and Y be sets of messages.

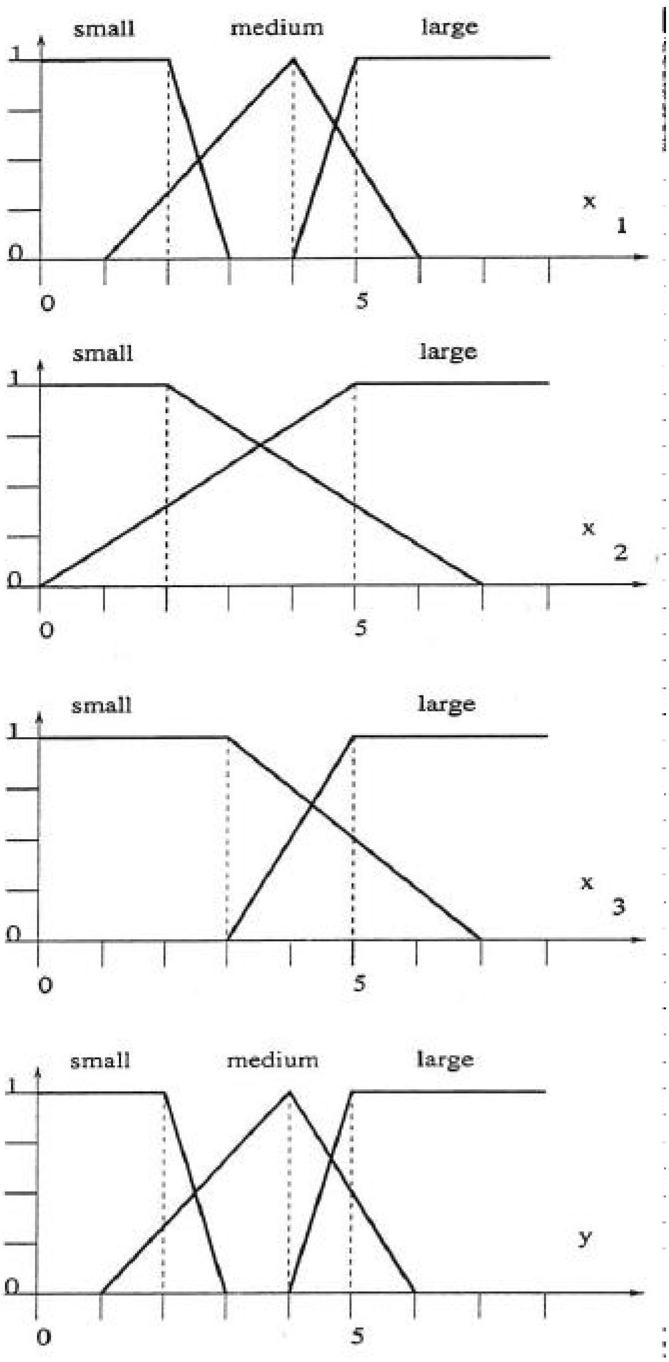
(a) Let R(X, Y) be a crisp binary relation which defines a completely specified bijective mapping $f: X \rightarrow Y$. Knowing that the cardinality of X is N, find the information transmission T(X, Y). **[3 marks]**

(b) Give an example of a crisp relation R(X, Y) that defines a completely specified injective mapping $f: X \rightarrow Y$ with joint information value I(X, Y) = 2. **[4 marks]**

(c) Let $|X| = N$ and $|Y| = M$. What are the minimal and the maximal possible values of the joint information I(X, Y) if R defines a completely specified surjective mapping $f: X \rightarrow Y$? **[4 marks]**

(d) Let x be a random variable taking values in the set X = {-1, 0, 1, 2, 3, 4}. Knowing that the expected value (the mean) of x is 3, suggest a probability mass function with minimal value of the Shannon entropy (no proof is required). **[4 marks]**

(5) (a) Assume you have a MISO MA-type fuzzy system. The membership functions partitioning the input feature axes as given in the figure below.



(i) *Calculate* the firing strength of the rule:
IF x_1 is *medium* AND x_2 is *large* AND x_3 is *small* THEN y is *medium* for input $[4, 2, 1]^T$ and product as the AND operation. **[4 marks]**

(ii) Draw the output fuzzy set for this rule using minimum as the implication. **[3 marks]**

(b) The firing strength of the 4 rules of a SISO TSK fuzzy system for input x = 6 are respectively 0.4, 0.7, 0.1 and 0.2. The consequent parts are as follows:

R_1: ... THEN y = 3x
R_2: ... THEN y = 2x-1
R_3: ... THEN y = x+1
R_4: ... THEN y = 3x

(i) Find the output of the system. **[3 marks]**

(ii) Assume that the consequents are normal singletons at the respective values of y, and the system is an MA model. Find the output fuzzy set of the system and calculate the output by the Mean Of Maximums (MOM) defuzzification method. **[5 marks]**

(Questions done: 1, 2, 5)