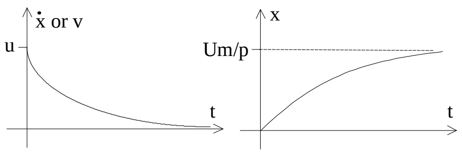
> $\gt$ 28th September 1999

# Introduction, Recall

A **Basic** problem in particle mechanics is: Given a force $\underline{F} = F(\underline{r}, \dot{\underline{r}}, t)$, find the path of the *particle* $\underline{r} = \underline{r}(t)$. To do so, we must **solve** $\underline{F} = \frac{d}{dt}(m\dot{\underline{r}})$. If m is *constant*, $\underline{F} = m\frac{d^2\mathbf{r}}{dt^2}$, or $\underline{F} = m\ddot{\underline{r}}$. (*A Differential* equation of motion). In **one-dimensional** motion, the equation of motion is $m\ddot{x} = F(x, \dot{x}, t)$.

**Special** cases. (1) $F = F(t)$, with *equation* of motion $m\ddot{x} = F(t)$, or $m\frac{d^2x}{dt^2} = F(t)$. In this case, integrate the equation *twice* w.r.t. time. Example: $F = kt$ ($k$ = constant; $x = 0$ and $\dot{x} = U$ when $t = 0$). So $m\frac{d^2x}{dt^2} = kt$; $\frac{d^2x}{dt^2} = \frac{k}{m}t$. Integrating, $\frac{dx}{dt} = \frac{1}{2}\frac{k}{m}t^2 + c_1$ ...(1). And *then* $x = \frac{1}{6}\frac{k}{m}t^3 + c_1t + c_2$ ...(2). From (1), $\dot{x} = U$ at $t = 0$ gives $U = c_1$; and from (2), $x = 0$ at $t = 0$ gives $0 = c_2$. So the **solution** is $x = \frac{1}{6}\frac{k}{m}t^3+Ut$.

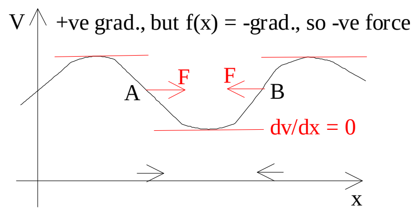
(2). $F = F(\dot{x})$ or $F = F(v)$. The *differential* equation to be solved is $m\ddot{x} = F(v)$. Use $\ddot{x} = v\frac{dv}{dx}$, and integrate w.r.t. *x*; or use $\ddot{x} = \frac{dv}{dt}$, and integrate w.r.t. *time*. Example: $F = -pv$, where p is a constant; and $v = U$ and $x = 0$ at $t = 0$. **Using** $\ddot{x} = \frac{dv}{dt}$, we obtain the equation $m\frac{dv}{dt} = -pv$. So $\int\frac{dv}{v} = \int-\frac{p}{m}dt$; $\log v = -\frac{p}{m}t + c_1$; $v = e^{-(p/m)t+c_1}$; $v = e^{-(p/m)t}e^{c_1}$; $v = Ae^{-(p/m)t}$. At $t = 0$, $v = U$, so $U = A$ and $v = Ue^{-(p/m)t}$. To obtain x, *integrate* w.r.t. time, i.e. $\frac{dx}{dt} = Ue^{-(p/m)t}$; $x = U(-\frac{m}{p})e^{-(p/m)t} + c_2$. At $t = 0$, x is zero, so $c_2 = \frac{Um}{p}$; and therefore $x = U\frac{m}{p}(1-e^{-(p/m)t})$. This gives the *graphs* as shown.

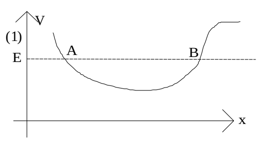


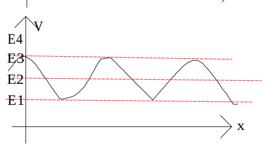

> $\gt$ 30th September 1999

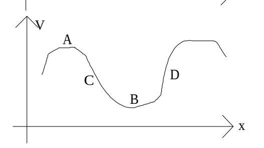
(3) $F = F(x)$. The **equation** to be solved is $m\ddot{x} = F(x)$. Using $m\frac{d^2x}{dt^2} = F(x)$, we *cannot* go further, so we use $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ ($= \frac{dv}{dx}\frac{dx}{dt}$) to get $mv\frac{dv}{dx} = F(x)$. Integrating gives $m\int v\,dv = \int F(x)dx$, to give $\frac{1}{2}mv^2 = W(x)+E$, where $W(x) = \int F(x)dx$ is the *work* function — the work done by the force during the **displacement**, and E is a constant of integration.

This may be **written** as the principle of conservation for *mechanical* energy: $T+V(x) = E$, where $T = \frac{1}{2}mv^2$ is the kinetic energy; $V(x) = -W(x) = -\int f(x)dx$ is the *potential* energy; and E is the total energy (a constant). We need to <u>integrate</u> again to obtain an expression for x. Consider $\frac{1}{2}mv^2+V(x) = E$; $\frac{1}{2}mv^2 = E-V(x)$; $V^2 = \frac{2}{m}(E-V(x))$; $V = \sqrt{(\frac{2}{m})(E-V(x))}$. So $\frac{dx}{dt} = \sqrt{(\frac{2}{m})(E-V(x))}$. In principle, this may be *integrated* as $\int dt = \int\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}}$+constant. Therefore, $t = \int\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}}$ + constant. The **constant** E is determined by the initial *conditions*, e.g. values for x and $\dot{x}$ at $t = 0$.




**Potential** Energy diagrams. Note that $V(x) = -\int f(x)dx$ so *that* $f(x) = -\frac{dV}{dx}$. Also note that $T = \frac{1}{2}mv^2 \geq 0$; so $E-V(x) \geq 0$; $V(x) \leq E$. *Examples*: (1) If the particle is <u>released</u> at rest at position $x_A$ (with energy E), it will *oscillate* back and forth between $x_A$ and $x_B$. The particle cannot <u>escape</u> from the potential well.

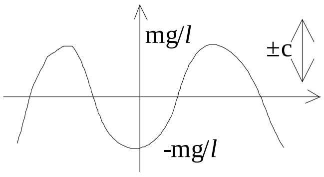
(2) If the particle has **total** energy $E_1$, it is in equilibrium and *cannot* move. If the particle has total energy $E_2$, it will oscillate within a potential well. If the particle has energy $\geq E_3$, it will *continue* its motion. At C, $\frac{dv}{dx} < 0 \Rightarrow F > 0$. At D, $\frac{dv}{dx} > 0 \Rightarrow F < 0$. At A, $\frac{dv}{dx} = 0$ and $\frac{d^2v}{dx^2} < 0 \Rightarrow F = 0$ in *unstable* equilibrium. At B, $\frac{dv}{dx} = 0$ and $\frac{d^2v}{dx^2} > 0 \Rightarrow F = 0$ in *stable* equilibrium.

➢ 1st October 1999

# Tutorial

Q: Calculate the **potential** V(x) for the force F = -kx, where k is a *positive* constant. Find the position(s) of equilibrium, and show whether they are *stable* or *unstable*. Sketch the graph of V(x). A: $V(x) = -W(x) = -\int -kx\,dx$; $V(X) = \frac{1}{2}kx^2+\text{constant}$. Positions of *equilibrium* occur at maxima or minima of V(x), when $\frac{dV}{dx} = 0$. An equilibrium point is *stable* if it is a minimum, that is $\frac{d^2V}{dx^2} > 0$. In **this** case, $V = \frac{1}{2}kx^2+c$; $\frac{dV}{dx} = kx$. And so $\frac{dV}{dx} = 0 \Rightarrow kx = 0$, so $x = 0$. Now $\frac{d^2V}{dx^2} = k > 0$, so we have a point of *stable* equilibrium. The sketch is any quadratic graph (positive).

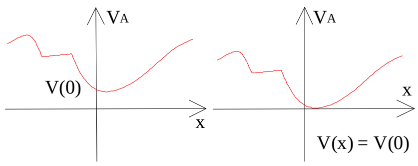
Q: The equation of motion for a **pendulum** may be written as $m\ddot{\theta} = -\frac{mg}{l}\sin\theta$. Calculate the *potential* V(θ), and sketch its graph. Determine the points of **equilibrium**. A: $F = -\frac{mg}{l}\sin\theta$. $V(x) = -\int -\frac{mg}{l}\sin\theta\,d\theta = \frac{mg}{l}\int\sin\theta\,d\theta = \frac{mg}{l}(-\cos\theta)+\text{constant}$. A *Sketch* is shown on the right. Now $\frac{dV}{d\theta} = \frac{mg}{l}\sin\theta$, so $\frac{dV}{d\theta} = 0$ at $\theta = 0, \pi, 2\pi, \ldots$, with $\frac{d^2V}{d\theta^2} = \frac{mg}{l}\cos\theta$. At *the specified points*, this is 1, -1, 1, -1, ..., so **stable** at $\theta = 0, 2\pi, 4\pi, \ldots$; and **unstable** at $\theta = \pi, 3\pi, 5\pi, \ldots$




➢ 5th October 1999

# Small Oscillations about Stable Equilibrium

Here, we *linearise* the equation of motion about a point of **stable** equilibrium. For convenience, take the point of equilibrium to be at the origin, so x = 0. Therefore, f(0) = 0 or V'(0) = 0. We now expand V(x) in the form of a **Taylor** series about the origin (A Maclaurin series): $V(x) = V(0)+V'(0)x+V''(0)\frac{x^2}{2!}+V'''(0)\frac{x^3}{3!}+\ldots$ Because x = 0 is a point of *equilibrium*, V'(0) = 0. Also, V(0) is a constant term, and constitutes an arbitrary additive of integration, which we take to be **zero** for convenience.



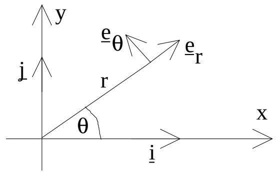

So $V(x) = V''(0)\frac{x^2}{2!}+O(x^3)$. If the distance from the origin is *small*, we ignore terms of $O(x^3)$, so $V(x) \approx V''(0)\frac{x^2}{2}$. To obtain the linearised equation of motion, differentiate V(x) to give $F(x) = -\frac{dV}{dx} = -V''(0)x$. For small displacements, $V(x) = \frac{kx^2}{2}$, and $F(x) = -kx$, where $k = V''(0)$. Remember that the point of *equilibrium* is **stable** if $k = V''(0) > 0$, and **unstable** if $k = V''(0) < 0$.

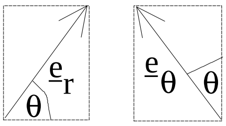
The equation of motion for *small* displacements is $m\ddot{x} = -kx$, or $m\ddot{x}+kx = 0$. For $k > 0$, this is the linear *oscillator* equation, $\ddot{x}+\omega^2x = 0$, where $\frac{k}{m} = \omega^2 > 0$. This has *solutions* of the form $x = A\sin\omega t+B\cos\omega t$, where A & B are constants of *integration*. An alternative representation is $x = A\sin(\omega t+\delta)$, where A is the *amplitude*; δ is the phase angle; ω is the *angular* frequency (in radians per second); and $f = \frac{\omega}{2\pi}$ is the frequency (in *revs* per second). $T = \frac{1}{f}$ is the period (the time taken for a revolution) $= \frac{2\pi}{\omega}$. In terms of *potential* energy, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{V''(0)}{m}}$, hence $T = 2\pi\sqrt{\frac{m}{V''(0)}}$.

➢ 7th October 1999

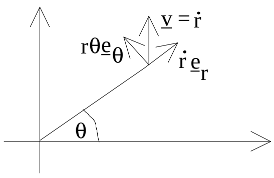
# Plane Polar Co-ordinates

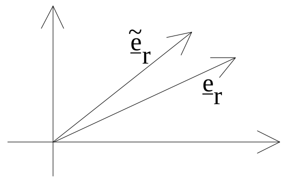
(Note: in the following, where I cannot underline a term in the notes, I will **embolden** it instead, i.e. take $\underline{e}_r$ = $\mathbf{e}_r$). An **alternative** to Cartesian co-ordinates (x,y) is to use *polar* co-ordinates (r,θ), where r is the radial co-ordinate and θ is the transverse co-ordinate. Take **unit** vectors in the r and θ directions, namely $\underline{e}_r$ and $\underline{e}_\theta$ (sometimes *denoted* by $\hat{r}$ and $\hat{\theta}$). Note that the position **vector** of a particle may be represented as $\underline{r} = r\underline{e}_r$. The **velocity** vector is given by $\dot{\mathbf{r}} = d\underline{r}/dt = dr/dt(\underline{e}_r)+\underline{r}(d\underline{e}_r/dt)$. So $\dot{\mathbf{r}} = \dot{r}\underline{e}_r + r\dot{\mathbf{e}}_r$.

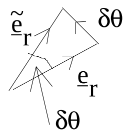



From the diagram, we note that $\underline{e}_r = \cos\theta\underline{i}+\sin\theta\underline{j}$, and $e_\theta = -\sin\theta\underline{i}+\cos\theta\underline{j}$. *Differentiating* the expression for $\underline{e}_r$ gives $d\underline{e}_r/dt = -\sin\theta(d\theta/dt)\underline{i}+\cos\theta(d\theta/dt)\underline{j}$ (by the chain rule) $= d\theta/dt(-\sin\theta\underline{i}+\cos\theta\underline{j})$. *Therefore, we have* $d\underline{e}_r/dt = (d\theta/dt)\underline{e}_\theta$, or $\dot{\mathbf{e}}_r = \dot{\theta}\underline{e}_\theta$. **Similarly**, for the time derivation of $\underline{e}_\theta$, $d\underline{e}_\theta/dt = -\cos\theta(d\theta/dt)\underline{i} - \sin\theta(d\theta/dt)\underline{j} = -d\theta/dt(\cos\theta\underline{i}+\sin\theta\underline{j})$. Therefore, $d\underline{e}_\theta/dt = (-d\theta/dt)\underline{e}_r$, or $\dot{\mathbf{e}}_\theta = -\dot{\theta}\underline{e}_r$. Looking at the **diagram** on the right, $\frac{d}{dt}(\underline{e}_r) = \lim_{\delta t\to 0}\{(\tilde{\mathbf{e}}_r-\underline{e}_r)/\delta t\} = \lim_{\delta t\to 0}\{\frac{\delta\theta}{\delta t}(\underline{e}_\theta)\}$. So $\dot{\mathbf{r}} = \dot{r}\underline{e}_r + r\dot{\mathbf{e}}_r$. The *expression* for the **velocity** vector becomes $\dot{\mathbf{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$.




To obtain the **acceleration** vector in plane polar components, *differentiate* the velocity vector to give $\ddot{\mathbf{r}} = d/dt(\dot{\mathbf{r}}) = \ddot{r}\underline{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\underline{e}_r + r\ddot{\theta}\underline{e}_r + r\dot{\theta}\dot{\mathbf{e}}_r$. Using the **expressions** for $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ *gives* the following: $\ddot{\mathbf{r}} = \ddot{r}\underline{e}_r + r\dot{\theta}\underline{e}_r + r\dot{\theta}\underline{e}_r + r\ddot{\theta}\underline{e}_\theta + r\dot{\theta}(-\dot{\theta}\underline{e}_r)$. Now *gather terms*: $\ddot{\mathbf{r}} = (\ddot{r}-r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta}+2\dot{r}\dot{\theta})e\theta$. This is *equivalent* to $\ddot{\mathbf{r}} = (\ddot{r}-r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}(d/dt)(r^2\dot{\theta})\underline{e}_\theta$.

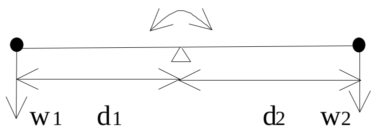
**Example**: Motion in a circle that has radius a. The equation for the *particle's path* is r = a (constant). The **velocity** vector is $\dot{\mathbf{r}} = a\dot{\theta}\underline{e}_\theta$. The **acceleration** vector is $\ddot{\mathbf{r}} = -a\theta^2\underline{e}_r + a\ddot{\theta}\underline{e}_\theta$. This is the *radial* component of acceleration (= $a\omega^2$, where $\omega = \dot{\theta}$).

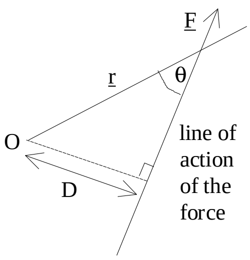
➢ 8th October 1999

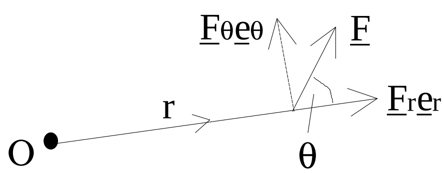
# Newton's Second Law of Motion in Plane Polar Co-ordinates

**Consider** $m\ddot{\mathbf{r}} = F(r,\dot{r},t)$, and take *components* in the r and θ directions, so that $\ddot{\mathbf{r}} = (\ddot{r}-r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}\frac{dr}{dt}(r^2\dot{\theta})\underline{e}_\theta$. Let $\underline{F} = F_r\underline{e}_r + F\theta\underline{e}_\theta$. *r-component*: $m(\ddot{r}-r\dot{\theta}^2) = F_r(r,\dot{r},\theta,\dot{\theta},t)$. *q-component*: $m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = F_\theta(r,\dot{r},\theta,\dot{\theta},t)$.

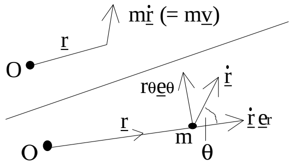
**<u>Moments of Vectors in Plane Polar co-ordinates</u>**. The moment of a force about a point O is *defined* as $\underline{M}_0 = \underline{r}\times\underline{F}$. ($\underline{r}$ is the position vector of the point of *application* of the force F). If θ is the angle between $\underline{r}$ and $\underline{F}$, the magnitude of $\underline{M}_0$ is given by $|\underline{M}_0| = |\underline{r}||\underline{F}|\sin\theta$. From the *diagram* (on the left), $D = |\underline{r}|\sin\theta$, so $|\underline{M}_0| = |\underline{F}|D$. Now consider the components of F in plane polar co-ordinates ($\underline{F} = F_r\underline{e}_r + F\theta\underline{e}_\theta$). $F_r = |\underline{F}|\cos_\theta$ and $F_\theta = |\underline{F}|\sin\theta$. So $|\underline{M}_0| = |\underline{r}||\underline{F}|\sin\theta = rF_\theta$.



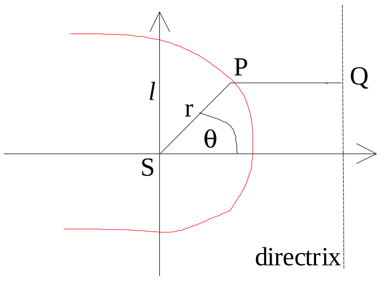



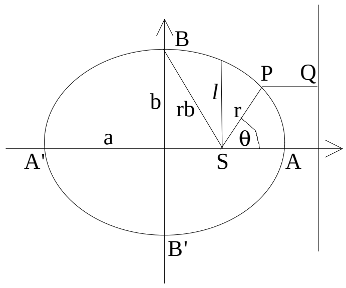
We may take the moment of *any* vector about a point. In particular, the **angular** momentum about a point O is defined as the moment about O of the *linear* momentum: $\underline{L} = \underline{r}\times m\dot{\underline{r}}$ ; $\underline{L} = \underline{r}\times m\underline{v}$. In *plane polar co-ordinates*, $\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$. So $L = |\underline{L}| = mr^2\dot{\theta}$ (= $r(mr\dot{\theta})$ which follows from $|M_0| = rF\theta$) We denote the *angular momentum* per unit mass as $h = {}^L/_m$, so $h = r^2\dot{\theta}$.

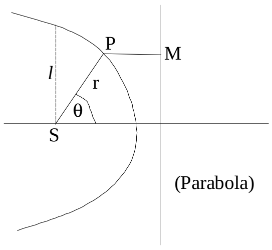



# Geometry of Conic Sections

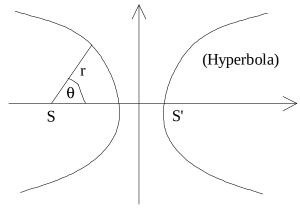



*Cartesian/Polar* Co-ordinates. Circle: $x^2+y^2 = a^2$. Ellipse: ${}^{x^2}/_{a^2}+{}^{y^2}/_{b^2} = 1$. Parabola: $y^2 = 4ax$. Hyperbola: ${}^{x^2}/_{a^2}-{}^{y^2}/_{b^2} = 1$. In polar co-ordinates, we have ${}^l/_r = 1+e\cos\theta$, where $l$ and e are *constants* $\geq 0$. $l$ is called the semi-latus rectum, and e is the eccentricity. In the diagram, e is the *ratio* ${}^{SP}/_{PQ}$.




**Different** values of e give different curves. e = 0: circle. ${}^1/_r = 1 \Rightarrow r = l$. For $0\leq e\leq 1$, we have an *ellipse*. It may be proved that $l = a(1-e^2)$, and that $r_b = a$. **e = 1: parabola**. ${}^l/_r = 1+\cos\theta = 1+2\cos^2{}^\theta/_2-1$. **Therefore,** ${}^l/_r = 2\cos^2{}^\theta/_2$; $r = {}^l/_2\sec^2{}^\theta/_2$. At $\theta = 0$, $r = {}^l/_2$. At $\theta = {}^\pi/_2$, ${}^1/_{\cos^2(\pi/4)} = 2$, so $r = l$. As $\theta \to \pi$ (and $-\pi$ too), ${}^1/_{\cos^2(\pi/2)} \to {}^1/_0$, so $r \to \infty$.

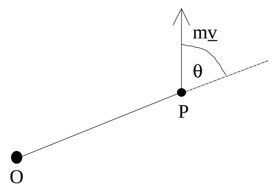



**e > 1: hyperbola**. $\theta = 0 \Rightarrow r = {}^l/_{1+e}$. $\theta = {}^\pi/_2, {}^{3\pi}/_2,\ldots \Rightarrow r = l$. $r = {}^l/_{1+e\cos\theta} \to \infty$ when $e\cos\theta \to -1$. As $e>1$, $\cos\theta \Rightarrow -{}^1/_e$; $\theta \Rightarrow \cos^{-1}(-{}^1/_e)$. As $\theta\to\cos^{-1}(-{}^1/_e)$, $r\to\infty$.

➢ 14th October 1999

# Orbital Motion / Central Orbits

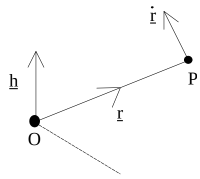
$m\ddot{\underline{r}} = \underline{F}(\underline{r}, \dot{\underline{r}}, t)$ or $m(\ddot{r}-r\dot{\theta}) = F_r(r, \dot{r}, \theta, \dot{\theta}, t)$; and $m{}^1/_r{}^d/_{dt}(r^2\theta) = F_\theta(r, \dot{r}, \theta, \dot{\theta}, t)$. We assume that the **force** is directed *towards* or *away* from a fixed point, called the centre of force. In addition, we assume that the **magnitude** of the force is a function of radial distance r only: $\underline{F}(\underline{r}) = F(r)\underline{e}_r$, where $r = |\underline{r}|$.




**Example**: $F(r) = {}^\mu/_{r^2}\underline{e}_r$ (the *inverse square law*). $\mu > 0$: **repulsive** force; $\mu < 0$: **attraction** force. Equation of *motion:* $m\ddot{\underline{r}} = F(r)\underline{e}_r$ ---(1). Reminder From the *picture*, $\underline{L}_0 = \underline{r}\times m\dot{\underline{r}}$; $|\underline{L}_0| = |\underline{r}||m\dot{\underline{r}}|\sin\theta = mr^2\dot{\theta}$ .

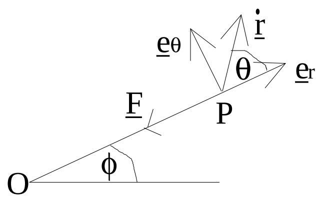
**Theorem**: under a *central force*, the angular momentum about the centre of force O is conserved. **Proof**: Take the vector product of (1) with $\underline{r}$, which gives $\underline{r}\times m\ddot{\underline{r}} = \underline{r}\times F(r)\underline{e}_r$. **Therefore,** $m\underline{r}\times\ddot{\underline{r}} = \underline{0}$ because $\underline{r}$ and $\underline{e}_r$ are *collinear*. Also, ${}^d/_{dt}(\underline{r}\times\dot{\underline{r}}) = \underline{r}\times\ddot{\underline{r}} + \dot{\underline{r}}\times\dot{\underline{r}} = \underline{r}\times\ddot{\underline{r}} + \underline{0}$. (Since $\dot{\underline{r}}\times\dot{\underline{r}} = \underline{0}$ (*collinear* vectors)). So the equation of **motion** becomes $m{}^d/_{dt}(\underline{r}\times\dot{\underline{r}}) = \underline{0}$ ---**(2)**. Hence $m\underline{r}\times\dot{\underline{r}} = \underline{L}_0 = m\underline{h}$ is a *constant vector*. Hence the **angular momentum** is conserved. The constant vector $\underline{h}$ is the angular momentum *per unit mass*, which is a constant vector for motion governed by a **central** force. Its magnitude is defined by $h = |\underline{h}|$.

**Corollary**: Motion lies in a *plane*. The above theorem shows that motion is confined to the plane having $\underline{h}$ as normal vector, $\underline{h} = r^2\dot{\theta}$.




➢ 15th October 1999

# Central Forces; Polar Components of the Equation of Motion




The **vector** equation of motion is $m\ddot{\underline{r}} = F(r)\underline{e}_r$. Resolving *radially* and *tangentially*, the r-component is $m(\ddot{r}-r\dot{\theta}^2) = F(r)$, while the θ-component is $m\,{}^1/_r\,{}^d/_{dt}(r^2\dot{\theta}) = 0$. [Aside: ${}^d/_{dt}(\underline{r}\times\dot{\underline{r}}) = 0$; $\underline{r}\times\dot{\underline{r}} = h$; $|\underline{r}||\dot{\underline{r}}| = \sin\theta$] The θ-component *shows* that $r^2\dot{\theta}$ is a constant **throughout** the motion, so that $r^2\dot{\theta} = h$, the *angular moment per unit mass*, is constant.

The r component may be **written** as $\ddot{r}-r\dot{\theta}^2 = P(r)$, where $P = {}^F/_m$ is the force *per unit mass*. The equations may then be **written as**: r-component: $\ddot{r}-r\dot{\theta}^2 = P(r)$ ---(3); θ-component: $r^2\dot{\theta} = h$ (constant) ---(4). The constant h is *determined* from the initial conditions.

# Energy Equation

**Substitute** for $\dot{\theta}$ from (4) into (3) to give $\ddot{r}-r({}^h/_{r^2})^2 = P$; $\ddot{r}-{}^{h^2}/_{r^3} = P$ **(5)**. [**Aside**: $m\ddot{x} = F(x)$; $mv\,{}^{dv}/_{dx} = F(x)$; and integrate to get the *energy equation*. Now $m\dot{x}\ddot{x} = F(x)\dot{x}$; ${}^d/_{dt}(\dot{x}^2)$ is a **trick** we use]. So multiply by $\dot{r}$ to give $\dot{r}\ddot{r}-(\dot{r}){}^{h^2}/_{r^3} = P(r)\dot{r}$; ${}^d/_{dt}({}^1/_2\dot{r}^2) + {}^d/_{dt}({}^1/_2{}^{h^2}/_{r^2}) = P(r){}^{dr}/_{dt}$. (Question: work out **where** that came from).

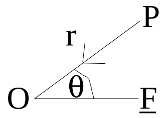
Integrate w.r.t. *time* to give ${}^1/_2\dot{r}^2+{}^1/_2{}^{h^2}/_{r^2} = \int P(r)dt$ + constant. This may be *written as follows:* ${}^1/_2(\dot{r}^2+{}^{h^2}/_{r^2}) + \tilde{V}(r) = \tilde{E}$, which is the *energy equation* per unit **mass**. Note that $\tilde{V}$ is the *potential* energy per unit mass, while $\tilde{E}$ is the *total* energy per unit mass. We may **substitute** back from (4) for $h = r^2\dot{\theta}$ to give [**Aside**: ${}^{h^2}/_{r^2} = (r^2\dot{\theta})^2/r^2 = r^2\dot{\theta}^2 = (r\dot{\theta})^2$] ${}^1/_2(\dot{r}^2+(r\dot{\theta})^2) + \tilde{V}(r) = \tilde{E}$.

**Note** that $\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$, so *that* $|\dot{\underline{r}}|^2 = v^2 = \dot{r}^2+(r\dot{\theta})^2$, and the *energy equation* may be **written** as $½v^2+\tilde{V}(r) = \tilde{E}$. **Multiplying** by m *gives* $½mv^2+V(r) = E$.

➢ 19th October 1999

# Differential Equation of Orbit

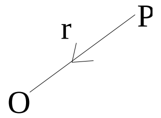
To obtain the **geometrical** equation of the particle path, we need to *express* r in terms of θ, i.e. $r = r(\theta)$. We have (5) and (4): $\ddot{r}-{}^{h^2}/_{r^3} = P(r)$ ---**(5)** [where $\ddot{r}$ is ${}^d/_{dt}({}^{dr}/_{dt})$]; $r^2\dot{\theta} = h$ ---**(4)** [where h is a *constant* and $r^2\dot{\theta}$ is ${}^{d\theta}/_{dt}$]. **Note** that ${}^d/_{dt} = {}^{d\theta}/_{dt}\,{}^d/_{d\theta}$, and that ${}^{d\theta}/_{dt} = \dot{\theta} = {}^h/_{r^2}$ (from (4)). *So* ${}^d/_{dt} = {}^h/_{r^2}\,{}^d/_{d\theta}$, and (5) *becomes* ${}^h/_{r^2}\,{}^d/_{d\theta}({}^h/_{r^2}\,{}^{dr}/_{d\theta}) = {}^{h^2}/_{r^3} = P(r)$ ----**(6)**.



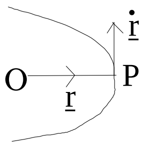

r(θ)

*Expanding* gives ${}^{d^2r}/_{d\theta^2} - {}^2/_r({}^{dr}/_{d\theta})^2 - r = (r^4/h^2)P(r)$. This is a **nasty** non-linear ordinary differential equation for r(θ). Its solution requires *numerical procedures on a computer*. *But*, the equation becomes linear if it is written in terms of the reciprocal of r, i.e. $u = {}^1/_r$.

The **co-ordinate** pair (u,θ) are known as reciprocal polar co-ordinates. If $u = \frac{1}{r}$, then $r = \frac{1}{u}$. Hence $\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$. **Substitute** for r and $\frac{dr}{d\theta}$ in (6) to give $hu^2\frac{d}{d\theta}(hu^2(-\frac{1}{u^2}\frac{du}{d\theta})) - h^2u^3 = P(u)$. Thus $-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = P(u)$, so that $\frac{d^2u}{d\theta^2} + u = -\frac{P(u)}{h^2u^2}$ ---**(7)**. This is a differential **equation** of orbit. It is a linear *second order differential equation* for u(θ). **Example**: Inverse Square. Here, $P(r) = -\frac{\mu}{r^2}$ (an *attraction* force). The equation of orbit is $\frac{d^2u}{d\theta^2} + u = --\frac{\mu u^2}{h^2u^2}$. So $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$; $\frac{d^2x}{dt^2} + x = 0$.

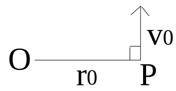



## Definition




An **apse** is a point where the *radius vector* and the *tangent to the orbit* are perpendicular. The corresponding radius vector is termed an **apse line**. The point of *nearest* approach to the centre of force is called the perihelion; the *furthest approach* is called the aphelion. Notes: At an apse, $\underline{r}$ and $\dot{\underline{r}}$ are *perpendicular* (see the **diagram** on the left).

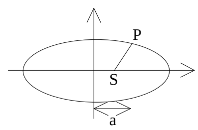
At an **apse**, r is stationary ($r = r_{max}$ or $r = r_{min}$). So $\dot{r} = 0$, and $\frac{dr}{d\theta} = 0$, and $\frac{du}{d\theta} = 0$. Given the speed $v_0$ at an apse *when* $r = r_0$, then $h = r_0v_0$. (Note: $v_0 = r_0\dot{\theta}$). See the diagrams on the *right*.




➢ 21st October 1999

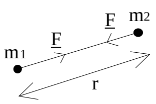
## Planetary Motion

**Kepler's** laws of planetary motion. Kepler produced *three laws* as the result of observation (this predates Newton). Take S to be the **sun**, and P to be a **planet**. Then (i) P describes an ellipse with S at the **focus**; (ii) SP sweeps out equal areas in *equal* times; (iii) The period of revolution of P is proportional to the (semi-major axis)$^{3/2}$ Newton later unified these laws through his laws of motion together with his laws of *universal gravitation*.

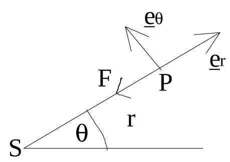



## Newton's Law of Gravitation

Two particles **attract** each other with a force directed along the *line* joining the particles, with **magnitude** proportional to their masses, and **inversely proportional** to the square of their distance apart. (An *inverse square law*). Thus for 2 particles having mass $m_1$ and $m_2$, being a distance of r apart, the force of attraction between them is *given* by $Gm_1m_2/r^2$, where G (or χ) is the *gravitational* constant. (G is approximately $6.67\times10^{-11}$ $Nm^2kg^{-2}$).

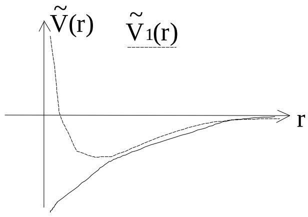



Newton's law of **gravitation**, together with Newton's laws of **motion,** enable us to obtain Kepler's laws from *first principles.* Assumptions: (1) S and P are particles of *masses* $m_S$ and $m_P$ respectively. (2) S is fixed in an *inertial frame of reference.* (3) Neglect **attraction** between planets and effects of moons, etc. So the *force* is given by $F(r) = -Gm_Sm_P/r^2$, where we have the -ve sign because the force is in the negative direction *relative* to polar co-ordinates centred at S.




It follows that the force on P per unit mass is $P(r) = F(r)/m_P = -Gm_S/r^2$. This may be written as **$P(r) = -\mu/r^2$**, where $\mu = Gm_S$ is a constant. The potential energy *per unit mass* is given by $\tilde{V}(r) = -\int P(r)dr = -\int -\frac{\mu}{r^2}dr$, i.e. $\tilde{V}(r) = -\frac{\mu}{r}$.

The **energy equation** is $\frac{1}{2}mv^2 + m\tilde{V}(r) = m\tilde{E}$ (KE + PE = Total energy). So $\frac{1}{2}(\dot{r}^2+(r\dot{\theta})^2) - {}^{\mu}/_{r} = \tilde{E}$; $\frac{1}{2}(\dot{r}^2+{}^{h^2}/_{r^2}) - {}^{\mu}/_{r} = \tilde{E}$ (Notes: $\dot{r}^2$ is the **radial** KE; ${}^{h^2}/_{r^2}$ is the **centrifugal** KE; $-{}^{\mu}/_{r}$ is the PE; and $\tilde{E}$ is *total* energy). This can be represented by a graph as shown. We can write the **above** as $\frac{1}{2}\dot{r}^2 + \tilde{V}_1(r) = \tilde{E}$, where $\tilde{V}_1(r) = \frac{1}{2}{}^{h^2}/_{r^2} - {}^{\mu}/_{r^2}$. Consider a *pseudo potential energy* diagram for $\tilde{V}_1(r)$, also shown on the **graph** as a dotted line.




Consider **three** cases of the total energy for the *graph with the dotted line.* (i) $\tilde{E}_1 \geq 0$: (a) Higher $\tilde{E}_1$ gives smaller *minimum r;* (b) the particle escapes to **infinity with an unbounded** orbit. (ii) $\tilde{E}_2 < 0$. (a) *Potential* well; *bounded* orbit. (iii) $\tilde{E}_3 = \frac{1}{2}{}^{\mu^2}/_{h^2}$. (a) r = constant: motion in a *circle*. (b) $\tilde{E}_3$ is the minimum energy *for a given h.*

➢ 22nd October 1999

# Equation of Orbit

Using reciprocal **polar co-ordinates**: $P = -{}^{\mu}/_{r^2} = -\mu u^2$, the equation of orbit is ${}^{d^2u}/_{d\theta^2} + u = -{}^{P}/_{h^2u^2} = {}^{\mu u^2}/_{h^2u^2}$. Therefore, ${}^{d^2u}/_{d\theta^2} + u = {}^{\mu}/_{h^2}$. *Note*: For ${}^{d^2y}/_{dx^2}+y = 0$, we get $y = A\cos(x+\alpha)$. For $m\ddot{x} = -k$, we get $m\ddot{x}+kx = 0$; $\ddot{x}+{}^{k}/_{m}x = 0$. This is a *2nd order linear ordinary differential equation, whose* solution comprises of **two** parts:

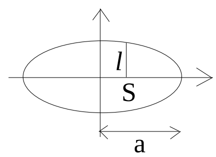
(i) The *complimentary function*, CF, which is the solution to the **homogenous** equation ${}^{d^2u}/_{d\theta^2}+u = 0$. (ii) The *particular integral*, PI. In this situation, the CF is $A\cos(\theta+\alpha)$, while the PI is ${}^{\mu}/_{h^2}$, so the **general** solution is $u = A\cos(\theta+\alpha) + {}^{\mu}/_{h^2}$. There are *two* constants of integration: the **amplitude** A, and the **phase angle** $\alpha$. We measure $\theta$ from an apse, i.e. when ${}^{du}/_{d\theta} = 0$, $-A\sin(\theta+\alpha) = 0$, so $\alpha = 0$.

Now the **solution** is $u = {}^{\mu}/_{h^2} + A\cos\theta$. Now substitute in $u = {}^{1}/_{r}$, and *rearrange*, to give ${}^{(h^2/\mu)}/_{r} = 1+A{}^{h^2}/_{\mu}\cos\theta$. The *general equation* of a conic is ${}^{l}/_{r} = 1+e\cos\theta$. So the solution for an *inverse square orbit* is a conic section with $l = {}^{h^2}/_{\mu}$, and $e = A{}^{h^2}/_{\mu} = Al$ (the *eccentricity*). Therefore, we get **elliptic**, **parabolic** or **hyperbolic** paths, depending on the value of $e = A{}^{h^2}/_{\mu}$, where A is a *constant of integration* obtained from the initial conditions.

To investigate the **physical** meaning of A, substitute the solution into the *energy equation.* After manipulation, we obtain $E = \frac{1}{2}{}^{\mu^2}/_{h^2}(e^2-1)$. So the amount of **total** energy governs the *type* of orbit (linked to the $\tilde{E}_k$ seen before). *e = 0 means* $E = -{}^{1}/_{2}{}^{\mu^2}/_{h^2}$ (**A Circle**). $0<e<1$ means $-{}^{1}/_{2}{}^{\mu^2}/_{h^2} < E < 0$ (An **Ellipse**). e = 1 means E = 0 (A **Parabola**). And e > 1 means E > 0 (A **Hyperbola**).

# Time in Orbit

We start with the *angular momentum per unit mass*, $h = r^2\dot{\theta}$. Here, $r^2\,{}^{d\theta}/_{dt} = h$; so $h\int dt = \int r^2 d\theta$. *Therefore*, $\int dt = {}^{1}/_{h}\int r^2 d\theta$. Substitute ${}^{l}/_{r} = 1+e\cos\theta$ for r in the $\theta$ **integral,** so that $r^2 = ({}^{l}/_{1+e\cos\theta})^2$ and $t = {}^{l^2}/_{h}\int {}^{d\theta}/_{(1+e\cos\theta)^2}$. Take $\theta = 0$ at $t = 0$, *so the time* to $\theta = \theta_1$ is $t = {}^{l^2}/_{h}\int_0^{\theta_1} {}^{d\theta}/_{(1+e\cos\theta)^2}$. For the *ellipse*, (e < 0), the time for a **complete** revolution is the periodic time given by $T = {}^{l^2}/_{h}\int_0^{2\pi} {}^{d\theta}/_{(1+e\cos\theta)^2}$. To *integrate*, substitute $z = \tan{}^{\theta}/_{2}$ to give $T = 2\pi(a^{3/2}/\mu^{1/2})$, where a is the *semi-major axis* of the ellipse, which is related to $l = {}^{h^2}/_{\mu}$ by $l = a(1-e^2)$.



**Kepler's Laws**: Verification. (1) An *elliptic* path follows if $E < 0$. (2) The equal area *sweep* is true for **any** central force. (3) $T \propto a^{3/2}$ has *just been proved.*

➢ 26th October 1999

# Tutorial

Q: Write down the **equation of orbit, the potential energy function, and the energy equation** for $P(u) = -\mu u^5$. A: Let $P(r) = -\mu/r^5$. The equation of *orbit* is $\frac{d^2u}{d\theta^2} + u = -\frac{P(u)}{h^2u^2}$. So $\frac{d^2u}{d\theta^2} + u = -(-\mu u^5)(\frac{1}{h^2u^2})$; $\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2}$. Now the *potential energy function is* $V(r) = \int -\mu/r^5 \, dr = \frac{1}{4}\mu r^{-4}$. Therefore, the *energy equation* is $\frac{1}{2}mv^2+V(r) = E$; $\frac{1}{2}mv^2+(\mu/4r^4) = E$.

Q: If a particle moves in an *ellipse of eccentricity e,* under a central force directed towards a fixed point O, with O being one of the **foci** of the ellipse, prove that the force is *inverse square.* A: The *general* equation is $\frac{l}{r} = 1+e\cos\theta$. Does the shape *being an ellipse* ⇒ P(r) = inverse square?

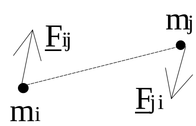
Substitute for **u** into $\frac{d^2u}{d\theta^2}+u = -\frac{P}{h^2u^2}$. From the *general equation,* we see that $u = \frac{1+e\cos\theta}{l}$. $\frac{du}{d\theta} = -\frac{e}{l}\sin\theta$; $\frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$. *Substituting* for u and $\frac{d^2u}{d\theta^2}$ in the LHS gives $-\frac{e}{l}\cos\theta + \frac{1+e\cos\theta}{l} = -\frac{P}{h^2u^2}$; $\frac{-e\cos\theta+1+e\cos\theta}{l} = -\frac{P}{h^2u^2}$; $\frac{P}{h^2u^2} = -\frac{1}{l}$; $P = -\frac{h^2u^2}{l}$. Therefore, *because* $u = \frac{1}{r}$, the force **is** inverse square.

➢ 28th October 1999

# Systems of Particles

We have been considering the *motion of single particles*. We now use Newton's laws of motion to investigate **many** particle systems. It is assumed that all vectors are measured in an inertial frame of reference, so that Newton's laws of motion hold. We label **each** particle with an integer, i, say. Let the i^th particle have mass $m_i$, and the position vector $\underline{r}_i$. The force acting on particle i is made up of *two* contributions: (1) **externally** applied force $\underline{F}_i$; (2) **internal** force due to other particles in the system, $\underline{f}_i$.

**Consider** any two particles in the system. They interact by *exerting* forces on each other, e.g. gravitational forces, electrostatic forces, etc. Let us denote the force exerted on particle i by particle j as $\underline{f}_{ij}$. Then, by *Newton's* third law, $\underline{f}_{ji} = -\underline{f}_{ij}$ The total *internal* force on particle i exerted by all the **other** particles in the system is $\underline{f}_i = \Sigma_j \underline{f}_{ij}$.

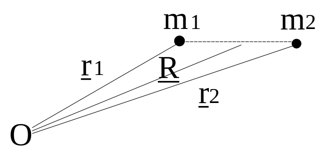



The **summation** is taken over *all particles of the system*. (Take $\underline{f}_{ii} = 0$). The total force on particle i is *given* by $\underline{f}_i+\underline{F}_i = \Sigma_j\underline{f}_{ij}+\underline{F}_i$. So Newton's *second law of motion* for particle i is $m\ddot{\underline{r}}_i = \Sigma_j\underline{f}_{ij}+\underline{F}_i$, or *equivalently,* $\frac{d}{dt}(m_i\dot{\underline{r}}_i) = \Sigma_j\underline{f}_{ij}+\underline{F}_i$. We may sum this equation over **all** particles: $\Sigma_i m\ddot{\underline{r}}_i = \Sigma_i\Sigma_j \underline{f}_{ij} + \Sigma_i\underline{F}_i$. The *term* $\Sigma_i\Sigma_j\underline{f}_{ij}$ is analogous to a *matrix* with $\underline{f}_{ij}$ as the entries. In this matrix, all the *diagonal entries* are zero, and, for example, $\underline{f}_{13} = -\underline{f}_{31}$.

By Newton's **third** law, $\underline{f}_{ji} = -\underline{f}_{ij}$, so the **double** sum gives a *zero* vector. So $\Sigma_i m\ddot{\underline{r}} = \Sigma_i\underline{F}_i$, or *equivalently*, $\Sigma_i \frac{d}{dt}(m_i\dot{\underline{r}}) = \Sigma_i\underline{F}_i$. Definitions: $\underline{F} = \Sigma_i \underline{F}_i$, the total *external* force. $M = \Sigma_i m_i$, the total *mass*. And $\underline{P} = \Sigma_i \underline{p}_i = \Sigma_i m_i\dot{\underline{r}}_i$, the total *linear momentum*.

# Principle of Total Linear Momentum

The **centre** of mass of a system of particles moves *as if it were a single particle of mass* equal to the total mass of the system and subject to a force **equal** to the total external force. The centre of mass (C or G) of a system of particles is *defined* by $\underline{R} = (\Sigma_i m_i\dot{\mathbf{r}}_i)/(\Sigma_i m_i)$ [$= (\Sigma_i m_i\dot{\mathbf{r}}_i)/M$]. This is a **position** vector.

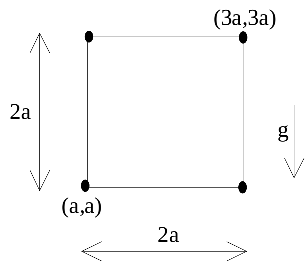



**Proof**: From the *definition* of the centre of mass, $\dot{\mathbf{R}} = (\Sigma_i m_i\dot{\mathbf{r}}_i)/M$, i.e. $M\dot{\mathbf{R}} = \Sigma_i m_i\dot{\mathbf{r}}_i$ (the total *linear* momentum $\underline{P}$). Differentiating **again** gives $M\ddot{\mathbf{R}} = \Sigma_i m_i\ddot{\mathbf{r}}_i$. From the *above*, $\Sigma_i m_i\ddot{\mathbf{r}}_i = \Sigma\underline{f}_i = \underline{F}$. Therefore, $\mathbf{M\ddot{R} = \underline{F}}$, or *equivalently*, $\mathbf{{}^{d}/_{dt}(\underline{P}) = \underline{F}}$.

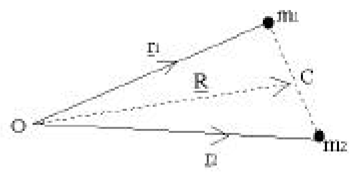
➢ 29th October 1999

# Exercises

Consider 4 particles of **equal** mass m, attached by *straight massless rods* positioned at the corners of a vertical square of side length 2a, with the lower left particle **positioned** at (a,a), and the upper right at (3a,3a). The only external force is that due to **gravitational** acceleration g, acting vertically downwards. Find (a) the total *mass* M; (b) the total external *force* $\underline{F}$; (c) the position $\underline{R}$ of the **centre** of mass. (C or G). (a) $M = \Sigma_i m_i = m+m+m+m = 4m$. (b) $\underline{F} = \Sigma_{i=1}^{4} \underline{F}_i = 4mg$. (c) $\underline{R} = (\Sigma_{i=1}^{4} m_i\underline{r}_i)/(\Sigma_{i=1}^{4} m_i) = {}^{m(a,a)+m(3a,a)+m(a,3a)+m(3a,3a)}/_{4m} = {}^{(8a,8a)}/_{4} = (2a,2a)$. You can *introduce* $\underline{i}$ and $\underline{j}$ unit *vectors* into the above.




Q: Consider a system consisting of **two particles** of mass $m_1$ and $m_2$. Show that the position of the *centre of mass* lies on the straight line joining the **two** particles. A: $\underline{R} = \Sigma_{i=1}^{2} m_i\underline{r}_i/m_i = (m_1\underline{r}_1+m_2\underline{r}_2)/(m_1+m_2)$. A point on the line *joining the two particles* is given by the relation $\underline{r} = \underline{r}_1+\lambda(\underline{r}_2-\underline{r}_1)$. *Transforming*, $\underline{R} = (m_1\underline{r}_1+m_2\underline{r}_2)/(m_1+m_2) = (m_1\underline{r}_1+m_2\underline{r}_1-m_2\underline{r}_1+m_2\underline{r}_2)/(m_1+m_2) = [(m_1+m_2)\underline{r}_1+m_2(\underline{r}_2-\underline{r}_1)]/[m_1+m_2] = \underline{r}_1+{}^{m_2}/_{m_1+m_2}(\underline{r}_2-\underline{r}_1)$, i.e. $\underline{R} = \underline{r}_1+\lambda(\underline{r}_2-\underline{r}_1)$, where $\lambda = {}^{m_2}/_{m_1+m_2}$.

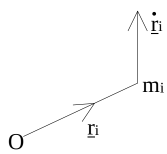



➢ 2nd November 1999

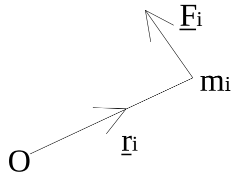
# Principle of Total Linear Momentum (Continued)

**Corollary**: If a system is isolated so that there are *no external forces*, ($\underline{F} = \underline{0}$), then the centre of mass moves with *uniform* velocity. It can therefore be used as the origin of a new inertial frame of reference. The previous sections show that it is **always** possible to predict the motion of the centre of mass. But what about the *motion of the system relative to this point?* Progress is possible by using moments of force and momentum.

**Definition** (Total Angular Momentum): The total angular momentum of a system of particles about a point O is *defined as the sum* of the angular moments of the individual particles. So let $\underline{L}_0 = \Sigma_i \underline{r}_i\times m_i\dot{\mathbf{r}}_i = \Sigma_i \underline{r}_i\times m_i\underline{v}_i$, where $\underline{r}_i$ is the *position* vector of particle i *relative* to O, and $\dot{\mathbf{r}}_i$ is its velocity *relative* to O.



**Definition** (Total moment of external forces): This is the *sum of the moments* of the external forces about O. So let $\underline{N}_0 = \Sigma_i \underline{r}_i \times \underline{F}_i$, where $\underline{r}_i$ is the *position vector* of the particle i relative to O, and $\underline{F}_i$ is the *external* force acting on particle i.




# The Principle of Total Angular Momentum

Consider a *system of particles* where the internal forces between any two particles acts along the line joining them. The rate of change of angular momentum about a point O (fixed in an **inertial** frame) is equal to the total moment of *external* forces about O, i.e. $\mathbf{d\underline{L}_0/dt = \underline{N}_0}$. Or, equivalently, (in the *linear* case), $d\underline{P}/dt = \underline{F}$, where $\underline{P} = M\underline{V} = M\dot{\mathbf{R}}$. The proof will be done *later*.

**Corollary**: *The conservation of total angular momentum:* If $\underline{N}_0 = 0$, then $\underline{L}_0$ is a *constant* vector. **Corollary**: *The principle of total angular momentum about an axis:* The rate of change of angular momentum about any **axis** through a point O (fixed in an *inertial* frame) is equal to the total moment of external forces about the axis.

**Proof**: Let â be the *unit vector* directed along the axis. Take the scalar product of the principal of angular momentum **with** â, to give $\underline{\hat{a}}.{}^{d}/_{dt}\underline{L}_0 = \underline{\hat{a}}.\underline{N}_0$.

➢ 4th November 1999

**Definition**: The total *angular momentum* about the centre of mass is $\underline{L}_G = \Sigma_i \underline{r}'_i \times m\dot{\mathbf{r}}_i$, where $\underline{r}'_i = \underline{r}_i - \underline{R}$ is the position vector of *particle i* relative to the centre of mass, G. **Proposition**: Velocities may be taken relative to G when *calculating* $\underline{L}_G$, i.e. $\underline{L}_G = \Sigma_i \underline{r}'_i \times m_i \dot{\mathbf{r}}'_i$.

**Proof**: From the definition of $\underline{L}_G$, $\underline{L}_G = \Sigma_i \underline{r}'_i \times m\dot{\mathbf{r}}_i = \Sigma_i \underline{r}'_i \times m_i(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_i) = \Sigma_i \underline{r}'_i \times m\dot{\mathbf{R}} + \Sigma_i \underline{r}'_i \times m_i \dot{\mathbf{r}}'_i = (\Sigma_i m_i \underline{r}'_i) \times \dot{\mathbf{R}} + \Sigma_i \underline{r}'_i \times m_i \dot{\mathbf{r}}'_i$. **Hence** $\Sigma_i m_i \rho'_i = 0$. By the definition of the *centre of mass*, $\Sigma_i m_i \underline{r}'_i = \underline{0}$. **Hence** $\underline{L}_G = \Sigma_i \underline{r}'_i \times m_i \dot{\mathbf{r}}'_i$.

(*Asides*: $\underline{r}'_i = \underline{r}_i - \underline{R}$, so $\underline{r}_i = \underline{R} + \underline{r}'_i$; $\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}'_i$. Also, $\Sigma_i m_i \underline{r}'_i = \Sigma_i m_i \underline{r}_i - \Sigma_i m_i \underline{R}$. From the **definition** of $\underline{R}$, $\underline{R} = \Sigma m_i \underline{r}_i / \Sigma m_i$, so $\Sigma m_i \underline{R} = \Sigma m_i \underline{r}_i$).

**Definition**: The total moment of external forces about the *centre* of mass: $\underline{N}_G = \Sigma_i \underline{r}'_i \times \underline{F}_i$. ($\underline{r}'_i$ is the position *vector* relative to G). **Proposition**: $\underline{L}_0 = \underline{L}_G + \underline{R} \times M\dot{\mathbf{R}}$. This states that $\underline{L}_0$ is equal to $\underline{L}_G$ plus the *angular momentum* about O of a mass M moving with **velocity** $\dot{\mathbf{R}}$. (**Proof** Omitted).

**Proposition**: The principle of *total angular momentum* about the centre of mass: ${}^{d}/_{dt}\underline{L}_G = \underline{N}_G$. (*Proof* omitted).

**Corollary**: if $\underline{N}_G = \underline{0}$, then $\underline{L}_G$ is a *constant* vector. **Corollary**: Principle of total angular momentum about an axis through G: ***(Energy):*** The total *kinetic* energy of a system of particles is given by $T = \Sigma_i \tfrac{1}{2}\dot{\mathbf{r}}_i^2$ ($= \Sigma_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i . \dot{\mathbf{r}}_i$) ($= \Sigma_i \tfrac{1}{2} m_i \underline{v}_i^2$). *Proposition*: $T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \Sigma_i \tfrac{1}{2} m_i \underline{r}_i'^2$ (T is the K.E. of the *total mass concentrated at G* + the K.E. due to the motion *relative* to G).

# Rigid Bodies

We now consider the motion of *rigid* bodies. We assume that the distance between any two points on the body remains a **constant**. We ignore any consideration of the body's elasticity. We also assume that the body is a *continuum*, so that we may ignore the true molecular structure of matter, and work with entities like density at a point, i.e. we may take a mathematical limit at **individual** points.

➢ 5th November 1999

Consider a point **P** within a rigid body, surrounded by a *small volume* $\delta V$ of mass $\delta m$. The density is given by $\rho = \lim_{\delta V \to 0} {}^{\delta m}/_{\delta V}$. So we have $dm = \rho dV$ (Aside: $\int dm = \int \rho dV$). All the concepts in the *section* on systems of particles may be adapted for rigid bodies by replacing the summation signs $\Sigma_i$ by integrals $\int_V$.

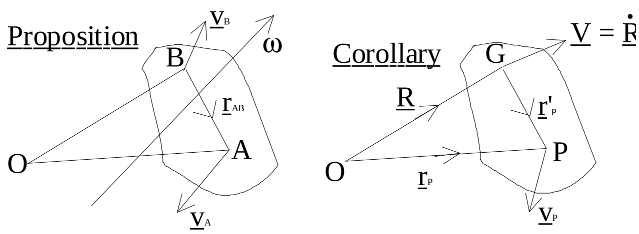
**Examples**: (1) *Total mass of body*: $M = \int dm = \int_V \rho dV$. (2) *Centre of mass*: $\underline{R} = \int \underline{r} dm / \int dm = \int_V \underline{r} \rho dV / \int_V \rho dV = \int_V \underline{r} dV/M$. (3) *Total linear momentum:* $P = \int \dot{\mathbf{r}} dm = \int_V \dot{\mathbf{r}} \rho dV$. (4) *Angular momentum about an origin* O: $\underline{L}_0 = \int \underline{r} \times \dot{\mathbf{r}} dm = \int_V \underline{r} \times \dot{\mathbf{r}} \rho dV$.

# Kinematics (Motion) of Rigid Bodies

*Definition of Degrees of freedom:* This is the number of independent scalar co-ordinates required to describe the state of a *mechanical* system precisely at a given instant. Examples: A single particle in **general** motion has 3 d.o.f. A single particle constrained to move on a **plane** of a **surface** has 2 d.o.f. A single particle constrained to move on a **line** or a **curve** has 1 d.o.f. A system of particles *containing N particles*, all in unconstrained motion, has 3N d.o.f.

**Proposition**: In general, unconstrained motion a solid body has 6 d.o.f. **Proof** (outline): 3 d.o.f. are required to fix *one* point, e.g. the centre of mass G. And a further 3 d.o.f. are required to specify the orientation of the body about G.

**Proposition**: In any motion of a rigid body, there exists an *angular velocity* $\omega$ such that if A & B are any two points fixed in the body, then their **velocities** $\underline{v}_A$ and $\underline{v}_B$ are linked by $\underline{v}_B = \underline{v}_A + \omega \times \underline{r}_{AB}$, where $\underline{r}_{AB} = \underline{r}_A - \underline{r}_B$. The angular velocity $\omega$ is independent of the *points* chosen. (Proof omitted). **Corollary**: In particular, the velocity of any point P of the body can be **determined** if we know (i) the velocity of the *centre* of mass: $\underline{V} = \dot{\mathbf{R}}$; and (ii) the *angular* velocity $\omega$ (Calculate by using $\underline{v}_P = \underline{V} + \omega \times \underline{r}'_P$).



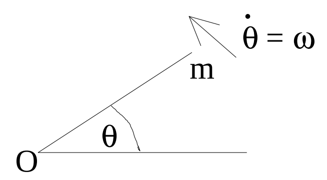

➢ 9th November 1999

# Rigid Bodies (Continued)

**Summary**: Linear: ${}^{d}/_{dt}\underline{P} = \underline{F}$ (motion *of* G). Angular: ${}^{d}/_{dt}\, \underline{L}_0 = \underline{N}_0$ (motion *about* G).

**<u>Linear momentum</u>**: $\underline{P} = M\underline{V} = M\dot{\mathbf{R}}$, and $M\ddot{\mathbf{R}} = \underline{F}$. The principle of linear momentum gives a (vector) *differential* equation for the centre of mass G. In theory, we can solve for the motion of G. What remains is to determine the motion of the body **about** G. We need to determine the *angular velocity* ω of the body. But the principle of *angular* momentum relates $\underline{L}_0$ to $\underline{N}_0$ with no *mention* of ω.

We now develop the **relationship** between $\underline{L}_0$ and ω. This may then be used to obtain a *differential equation* for ω. Before looking at the general case, consider the angular momentum of a **single** particle under a central force $l = mh = mr^2\dot{\theta} = mr^2\omega$. This is the relationship between $l$ and ω in this *simple* case. The result is generalised in the *following* proposition:




**<u>Proposition</u>**: If a set of Cartesian axes is fixed in a **rigid** body, and $\underline{L}_G$ and ω are written as column vectors with respect to *these* axes, then there exists a **symmetric** matrix $I_{\sim G}$ such that $\underline{L}_G = I_{\sim G}\omega$ (Notes: $\underline{L}_G$ is the angular *momentum;* ω is the angular *velocity;* and in $I_{\sim G}$, the tilde (~) is supposed to be **UNDER** the I, i.e. read $I_\sim$ as $\underset{\sim}{I}$). Compare this with the case for *linear* momentum and *linear* velocity: $\underline{P} = M\underline{V}$.

**<u>Proof</u>**: Select the origin of the axes at G, giving the *co-ordinate* system $Gx_1,x_2,x_3$, and measure *position* and *velocity* relative to G. So $\underline{L}_G = \Sigma_i\, \underline{r}_i \times m_i\dot{\mathbf{r}}_i = \Sigma_i\, \underline{r}_i \times m_i(\omega\times\underline{r}_i) = -\Sigma_i\, m_i\underline{r}_i\times(\underline{r}_i\times\omega)$. To **clarify** this expression, note that we may write the vector *product* $\underline{a}\times\underline{b}$ (where $\underline{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\underline{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$) as $\underline{a}\times\underline{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$.

So $\underline{a}\times(\underline{a}\times\underline{b}) = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}^2\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$. Applying this formula to the *expression* for $\underline{L}_G$ gives $\underline{L}_G = -\Sigma_i\, m_i \begin{pmatrix}0 & -x_3 & x_2\\ x_3 & 0 & -x_1\\ -x_2 & x_1 & 0\end{pmatrix}^2_i\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$, where $\underline{r}_i$ and ω have been written as *column* vectors: $r_i = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_i$ and $\omega = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$. So $\underline{L}_G =$

$$\Sigma_i\ m_i\begin{bmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3\\ -x_2x_1 & x_3^2+x_1^2 & -x_2x_3\\ -x_3x_1 & -x_3x_2 & x_1^2+x_2^2\end{bmatrix}_i\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}\Sigma_i m_i(x_2^2+x_3^2)_i & -\Sigma_i m_i(-x_1x_2)_i & -\Sigma_i m_i(-x_1x_3)_i\\ -\Sigma_i m_i(-x_2x_1)_i & \Sigma_i m_i(x_3^2+x_1^2)_i & -\Sigma_i m_i(-x_2x_3)_i\\ \Sigma_i m_i(-x_3x_1)_i & -\Sigma_i m_i(-x_3x_2)_i & \Sigma_i m_i(x_1^2+x_2^2)_i\end{bmatrix}_i\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

Therefore, $\underline{L}_G = I_{\sim G}\omega$, as *required*.

**<u>Definition</u>**: *Inertia tensor — moments and products of inertia*. The inertia tensor at G with respect to the *axes* $Gx_1x_2x_3$, is $I_{\sim G} = \begin{pmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}$, where the components are *given* by: **Moments of inertia**: $I_{11}$, $I_{22}$, $I_{33}$; (e.g. $I_{11} = \Sigma_i\, m_i(x_2^2+x_3^2)_i$); **Products of inertia:** $I_{21}$, $I_{21}$, $I_{23}$, $I_{32}$, $I_{31}$, $I_{13}$. (e.g. $I_{21} = I_{12} = -\Sigma_i\, m_i\,(x_1x_2)_i$).

Note that for *continuous* solid bodies, the components of $I_{\sim G}$ are found by **multiple** integrals, e.g. $I_{11} = \int_v \rho(x_2^2+x_3^2)dV$, or $I_{xx} = \int_v \rho(y^2+z^2)dV$. Sometimes, we *write* $I_{\sim G} = \begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}$, i.e. $x_1\to x$, $x_2\to y$, and $x_3\to z$.

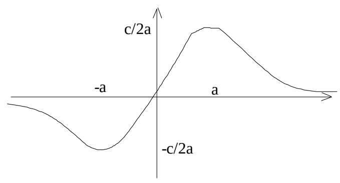
# Assignment 1

Q: A particle of *mass* m moves on Ox under the action of a force whose potential is given by $V(x) = {}^{cx}/_{a^2+x^2}$, where $a > 0$ and $c > 0$. Find the position of *stable* equilibrium and the period of the small oscillations about it. The particle is **projected** from this point with velocity $v_0$. Find the range of $v_0$ for which the particle *(i) oscillates, (ii) escapes to -infinity, (iii) escapes to infinity.*

A: For *stable* equilibrium at $x = x_0$, we require ${}^{dV}/_{dx}(x_0) = 0$ and ${}^{d^2V}/_{dx^2}(x_0) > 0$. If *such* a point exists, then the period of the **small** oscillations about the point is given by $T = 2\pi\sqrt{({}^{m}/_{V''(x_0)})}$. Now ${}^{dV}/_{dx} = {}^{(a^2+x^2)c-cx(2x)}/_{(a^2+x^2)^2} = {}^{c(a^2-x^2)}/_{(a^2+x^2)^2}$. For *equilibrium*, $V''(x) = 0$, hence $a^2-x^2 = 0$. Therefore, $x = \pm a$ are points of equilibrium.

${}^{d^2V}/_{dx^2} = {}^{(a^2+x^2)c(-2x)-c(a^2-x^2)2(a^2+x^2)(2x)}/_{(a^2+x^2)\wedge 4} = {}^{2cx}/_{(a^2+x^2)^3}(x^2-3a^2)$. At $x = a$, $V''(a) = -{}^{c}/_{2a^3} < 0$. At $x = -a$, $V''(-a) = {}^{c}/_{2a^3} > 0$. Hence $x = a$ is a point of *unstable* equilibrium, and $x = -a$ is a point of *stable* equilibrium. The period of the **small** amplitude oscillations about $x = -a$ is given by $T = 2\pi\sqrt{({}^{m}/_{V''(-a)})}$. Therefore, $T = 2\pi\sqrt{(m^{2a^3}/_{c})}$.

A **sketch** of the potential energy diagram is shown. Note that $V(a) = {}^{ca}/_{2a^2} = {}^{c}/_{2a}$; and $V(-a) = {}^{-ca}/_{2a^2} = {}^{-c}/_{2a}$. A particle is *projected* from $x = -a$ with velocity $v_0$ (which could be +ve or -ve). Note that $V(-a) = -{}^{c}/_{2a}$. (i) For **oscillations**, $E < 0$, therefore $\frac{1}{2}mv_0^2-{}^{c}/_{2a} < 0$. It follows that $v_0^2 < {}^{c}/_{ma}$, or $-\sqrt{({}^{c}/_{ma})} < v_0 < \sqrt{({}^{c}/_{ma})}$.



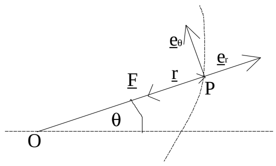

(ii) To escape to *-infinity*, we have two possibilities: (a) $v_0 < 0$ and $E \geq 0$, so $\frac{1}{2}mv_0^2-{}^{c}/_{2a} \geq 0$. Therefore, $v_0^2 \geq {}^{c}/_{ma}$; $v_0 \geq -\sqrt{({}^{c}/_{ma})}$. (b) $v_0 > 0$ and $0 \leq E < {}^{c}/_{2a}$. (It must not have *enough* energy to escape to +infinity). Therefore, $0 \leq {}^{1}/_{2}mv_0^2-{}^{c}/_{2a} < {}^{c}/_{2a}$, so ${}^{c}/_{ma} \leq v_0^2 < {}^{2c}/_{ma}$; $+\sqrt{({}^{c}/_{ma})} \leq v_0 < \sqrt{({}^{2c}/_{ma})}$. (iii) To escape to *infinity*, $E > {}^{c}/_{2a}$ and $v_0 > 0$. (Otherwise it escapes to *-infinity*). Therefore, $\frac{1}{2}mv_0^2-c/2a > {}^{c}/_{2a}$, so $v_0^2 > {}^{2c}/_{ma}$; $v_0 > +\sqrt{({}^{2c}/_{ma})}$.

Q: Write down the *equation of orbit, the potential energy function, and the energy equation* for the following central force: $P = -\mu/r^5$. A: The *differential* equation of orbit is given by ${}^{d^2u}/_{d\theta^2}+u = -{}^{P(u)}/_{h^2u^2}$. The potential energy function is given by $V(r) = -\int P(r)dr$. And the energy equation is given by $\frac{1}{2}mv^2+V(r) = E$.

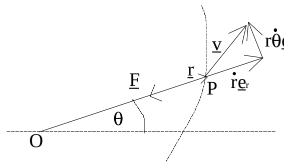
So *using* $P = -\mu/r^5$; $P(u) = -\mu u^5$, so the equation of *orbit* is ${}^{d^2u}/_{d\theta^2}+u = -(-\mu u^5)({}^{1}/_{h^2u^2})$; ${}^{d^2u}/_{d\theta^2}+u = \mu u^3/h^2$. The potential energy function is *given* by $V(r) = -\int-(\mu/r^5)dr = -\int-\mu r^{-5}dr = -{}^{1}/_{4}\mu r^{-4} = -\mu/4r^4$. Therefore, the *energy* equation is $\frac{1}{2}mv^2-\mu/4r^4 = E$.

Q: Show that the **velocity** components in polar co-ordinates for a particle under a central force are given by $\dot{r} = -h^{du}/_{d\theta}$ and $r\dot{\theta} = {}^{h}/_{r}$; and that the *energy* equation may be written as $({}^{du}/_{d\theta})^2+u^2 = {}^{2(E-V)}/_{mh^2}$. A: (see over).

Let F be the *central force* acting on a particle situated at a point P. The particle moves in a central orbit at a *distance* $\underline{r}$ from an origin O. Note that the vector $\underline{r}$ is the magnitude of r multiplied by the **unit** vector in the direction of r, $\underline{e}_r$. So $\underline{r} = r\underline{e}_r$. To get the *velocity*, we differentiate $\underline{r}$ with *respect* to time.




So $\underline{v} = \dot{r} = {}^{d}/_{dt}(\underline{r}) = {}^{d}/_{dt}(r\underline{e}_r) = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r$ (---(1)). It can be shown by **considering** Cartesian components that $d\underline{e}_r/d\theta = \underline{e}_\theta$. Hence, using the *chain* rule, $\dot{\underline{e}}_r = (d\underline{e}_r/d\theta)\times(d\theta/dt) = \dot{\theta}\underline{e}_\theta$. So (1) becomes ${}^{d}/_{dt}(r\underline{e}_r) = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$, i.e. $\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$ (---(2)). We now must *show* that the velocity components may be written as $\dot{r} = -h{}^{du}/_{d\theta}$, and $r\dot{\theta} = {}^{h}/_{r}$. Note that h is a *constant*, so that $h = r^2\dot{\theta}$ (---(3)); $r^2\dot{\theta} = h$; $r\dot{\theta} = {}^{h}/_{r}$.

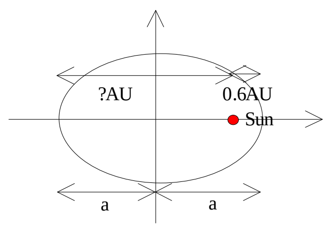



We now must **show** that $\dot{r} = -h{}^{du}/_{d\theta}$. Consider *reciprocal polar co-ordinates*. If $u = {}^{1}/_{r}$, it follows that ${}^{du}/_{d\theta} = -{}^{1}/_{r^2}{}^{dr}/_{d\theta}$ (--(4)). So the RHS of (4) *becomes* $-h{}^{du}/_{d\theta} = -h(-{}^{1}/_{r^2}{}^{dr}/_{d\theta}) = \ldots = {}^{dr}/_{dt}$. So $-h{}^{du}/_{d\theta} = \dot{r}$. Now consider the *energy* equation $\frac{1}{2}mv^2+V = E$, giving $v^2 = {}^{2(E-V)}/_{m}$. So equation (2) may be rewritten as $\underline{v} = -h{}^{du}/_{d\theta}\underline{e}_r + {}^{h}/_{r}\underline{e}_\theta$.

*Therefore*, the magnitude of $\underline{v}$, $|\underline{v}| = v$, is given by $v = |-h{}^{du}/_{d\theta}\underline{e}_r + {}^{h}/_{r}\underline{e}_\theta|$. Using *pythagoras'* rule, $v^2 = (-h{}^{du}/_{d\theta})^2 + ({}^{h}/_{r})^2 = h^2({}^{du}/_{d\theta})^2+{}^{h^2}/_{r^2}$. Therefore, $h^2({}^{du}/_{d\theta})^2+{}^{h^2}/_{r^2} = {}^{2(E-V)}/_{m}$; $({}^{du}/_{d\theta})^2+{}^{1}/_{r^2} = {}^{2(E-V)}/_{mh^2}$; $({}^{du}/_{d\theta})^2+u^2 = {}^{2(E-V)}/mh^2$. QED.

Q: In 1986, **Halley's** comet returned for the 7th time since 1456. At its previous perihelion in 1910, its distance from the Sun was 0.6AU. Show that the comet's *greatest* distance from the Sun is approximately 35AU. A: It has a period of ${}^{530}/_{7}$ years. *Perihelion* = 0.6AU, Aphelion = ? To help us find the aphelion, we use the equation $T = 2\pi(a^{3/2}/\mu^{1/2})$, or $T/a^{3/2} = 2\pi/\mu^{1/2}$. The RHS is a constant, which we can *find*.

We find the constant by applying the equation to the revolution of the **Earth** around the Sun. Here, we have a period of 1 year, and mean distance 1AU. So $T/a^{3/2} = 2\pi/\mu^{1/2}$ becomes $1/1^{3/2} = 2\pi/\mu^{1/2}$, i.e. $2\pi/\mu^{1/2} = 1$. Now we know the *constant*, so we have $T = a^{3/2}$. For Halley's comet, $T = {}^{530}/_{7}$. Therefore, $a = T^{2/3} = (530/7)^{2/3} \approx 17.89$ AU. We can find the aphelion from the *diagram*: aphelion = $(2\times a) - 0.6 \approx 35.19 \approx 35$AU. QED.

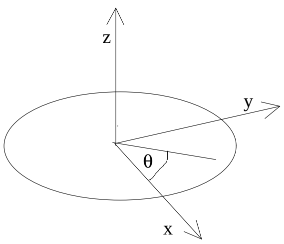



➢ 11th November 1999

# Calculation of the Inertia Tensor

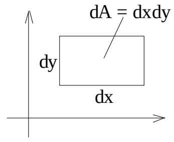
Calculating the *moments* and products of inertia for a specific set of axes involves the geometry, density and mass distribution of the body, together with multiple integrals. Example: A lamina (a surface with infinitesimal thickness) is in the **shape** of a thin circular disc of radius a, with constant surface density of σ per unit area. Calculate the *inertia* tensor relative to co-ordinates fixed at the disc **centre**.

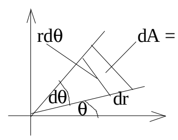
(Note: 3-D **mass** per unit volume: (ρ, $kgm^{-3}$). 2-D **mass** per unit area: (σ, $kgm^{-2}$)).




By *definition*, $I_{zz} = \Sigma m(x^2+y^2) = \iint_A (x^2+y^2)dm$. But $dm = \sigma dA = \sigma dxdy$. Because of the *geometry* of the disk, it is much easier to use **polar** co-ordinates, $(r,\theta)$. But we must **remember** that $dA = (dr)(rd\theta) = rdrd\theta$. So $I_{zz} = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \sigma(x^2+y^2)rdrd\theta$. But $x^2+y^2 = r^2$, so $I_{zz} = \int_0^a\int_0^{2\pi} \sigma r^3drd\theta = \sigma\int_0^a r^3dr\int_0^{2\pi}d\theta = \sigma[r^4/4]_0^a[\theta]_0^{2\pi} = \sigma(a^4/4)2\pi = \frac{1}{2}(\sigma\pi a^2)a^2 = \frac{1}{2}Ma^2$, where M is the total mass of the lamina, $\sigma\pi a^2$.

$$\begin{pmatrix} I_{xx} & I_{xy} & i_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ i_{zx} & I_{zy} & i_{zz} \end{pmatrix}$$




$I_{xx} = \Sigma m(y^2+z^2) = \Sigma my^2$ ($z = 0$ for lamina) $= \iint_A \sigma y^2dA = \sigma\int_0^a\int_0^{2\pi}(r\sin\theta)^2rdrd\theta = \sigma\int_0^a r^3dr \int_0^{2\pi} \sin^2\theta d\theta$. By *symmetry*, $I_{xx} = I_{yy}$. But $I_{zz} = I_{xx}+I_{yy} = \frac{1}{2}Ma^2$. So $I_{xx} = \frac{1}{4}Ma^2$; $I_{yy} = \frac{1}{4}Ma^2$; $I_{yz} = I_{zy} = -\Sigma myz = 0$ ($z = 0$); $I_{zx} = I_{xz} = -\Sigma mxz = 0$ ($z = 0$).

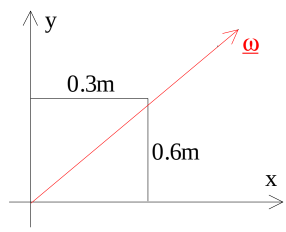
Now $I_{xy} = I_{yx} = -\Sigma mxy = -\iint_A \sigma xydA = -\int_0^a\int_0^{2\pi} \sigma r\cos\theta r\sin\theta rdrd\theta = -\sigma\int_0^a r^3dr\int_0^{2\pi}\sin\theta\cos\theta d\theta = -\sigma[r^4/4]_0^a\int_0^{2\pi} \frac{1}{2}\sin 2\theta d\theta = -\sigma(a^4/4)^1/_4[-\cos 2\theta]_0^{2\pi} = -\sigma(a^4/4)(-{}^1/_4)(\cos 4\pi-\cos\theta) = -\sigma(a^4/4)(-{}^1/_4)[1-1] = 0$. By **symmetry**, if $(x,y)$ lies in the disc, then so do $(\pm x, \pm y)$, hence the integrals *cancel* over four quadrants. Therefore, $I_{\sim G}$ is the matrix *shown on the right*.

$$\begin{bmatrix} \frac{1}{4}Ma^2 & 0 & 0 \\ 0 & \frac{1}{4}Ma^2 & 0 \\ 0 & 0 & \frac{1}{2}Ma^2 \end{bmatrix}$$

➢ 12th November 1999

# Tutorial

Q: A **rectangular** plate lies in the x-y plane of the body-fixed *co-ordinate* system. Its mass is 4kg and it has constant **density** $\sigma$. (a) Determine the plate's *moments* and *products* of inertia relative to the Oxy co-ordinate system. (b) If the plate is rotating about the **fixed** point O, with angular velocity $\underline{\omega} = 4\underline{i}-2\underline{j}$ rads$^{-1}$, what is the plate's *angular momentum* about O? (Use $\underline{L}_{\sim 0} = \underline{I}_{\sim 0}\underline{w}$).




A: We calculate the *matrix* $\underline{I}_{\sim 0}$ first. $I_{xx} = \int_A \sigma(y^2+z^2)dA = \iint\sigma(y^2+z^2)dxdy = \iint\sigma y^2dxdy$ (because $z = 0$) $= \int_0^{0.3}dx\times\int_0^{0.6}\sigma y^2dy = \sigma[0.3][^{0.6^3}/_3] = 0.48$. (Where $\sigma$ = density $= {}^{mass}/_{area} = {}^4/_{0.18}$ is used). *Similarly*, $I_{yy} = \sigma.(^{0.3^3}/_3)(0.6) = 0.12$. Now we can *use* $I_{zz} = I_{xx}+I_{yy} = 0.48+0.12 = 0.6$, or use $\iint\sigma(x^2+y^2)dxdy$, which is more *complicated* to compute.

Now $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$, *because* $z = 0$. But $I_{xy} = I_{yx} = -\int_A \sigma xydA = -\sigma\int_0^{0.3}xdx\int_0^{0.6}ydy = -\sigma[^{0.3^2}/_2][^{0.6^2}/_2] = -0.18$. So we have the *matrix* $\underline{I}_{\sim 0} = \begin{pmatrix} 0.48 & -0.18 & 0 \\ -0.18 & 0.12 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}$. Finally, use $\underline{L}_{\sim 0} = \underline{I}_{\sim 0}\begin{pmatrix}4\\-2\\0\end{pmatrix}$ to get $\begin{pmatrix} 0.48 & -0.18 & 0 \\ -0.18 & 0.12 & 0 \\ 0 & 0 & 0.6 \end{pmatrix}\begin{pmatrix}4\\-2\\0\end{pmatrix} = \begin{pmatrix}2.28\\-0.96\\0\end{pmatrix}$.

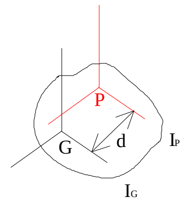
➢ 16th November 1999

**<u>Proposition</u>**: If the body is rotating about *one fixed point* Q, which is fixed in an inertial frame, then $\underline{L}_Q = \underline{I}_{\sim Q}\underline{\omega}$. Moment of inertia about a *general* axis: $I = \Sigma_i m_ir_i^2$. ($I_{zz} = \Sigma m_i(y_i^2+z_i^2)$, where $y_i^2+z_i^2 = r_i^2$).

**<u>Radius of Gyration.</u>** Let I be the *moment* of inertia about an axis. The radius of gyration, K, about the axis, is defined by the <u>formula</u> $I = MK^2$. So $I = \Sigma_i m_ir_i^2 = MK^2$ ($= \iiint_V r^2dm$).

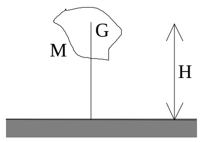
**Lamina Theorem**. Consider a lamina *lying* in the x-y plane. Then $I_{zz} = I_{xx}+I_{yy}$. *Proof*: $I_{xx} = \Sigma_i m_i(y_i^2+z_i^2) = \Sigma_i m_i y_i^2$. $(z = 0)$. $I_{yy} = \Sigma_i m_i(z_i^2+x_i^2) = \Sigma_i m_i x_i^2$. $(z = 0)$. Therefore, $I_{zz} = \Sigma_i m_i(x_i^2+y_i^2) = \Sigma_i m_i x_i^2 + \Sigma_i m_i y_i^2$. It follows that $I_{zz} = I_{xx}+I_{yy}$.

**Parallel axis theorem**. The moment of *inertia* about an axis through a point P, $I_P$, is related to the *moment* of inertia about a parallel axis through the *centre* of mass G, $I_G$, by $I_P = I_G + Md^2$, where M is the *total mass*, and d is the *distance* between the two axes. Proof omitted.




## Energy

**Potential Energy**: The treatment of potential energy depends only on the *external* forces, and follows the same argument as for a particle. The gravitational potential energy of a body is the same as that of the **total** mass concentrated at the centre of mass, so that P.E. = MgH.




**Kinetic Energy**: If a body is in general motion, its kinetic energy can be *expressed* as $T = \frac{1}{2}M\underline{V}^2 + \frac{1}{2}\underline{\omega}.\underline{L}_G$, where $\underline{V}$ is the *velocity* of the centre of mass; $\underline{\omega}$ is the angular velocity of the body; and $\underline{L}_G$ is the *angular* momentum about the centre of mass. Note that T consists of **2** parts: one is the *linear kinetic energy* of a single particle of mass M positioned at G; and the second is the *rotational kinetic energy* about G. *Proof omitted*.

**Proposition**: If a body is rotating *about a fixed point Q*, then $T = \frac{1}{2}\underline{\omega}.\underline{L}_Q$. Note that both *linear* and *rotational* kinetic energies are **quadratic** forms: $T^{linear} = \frac{1}{2}\underline{V}^t M \underline{V}$, and $T^{rotational} = \frac{1}{2}\underline{\omega}^t \underline{I}_G \omega$. (Note: $T^{linear} = \frac{1}{2}(v_1\ v_2\ v_3)M\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$; $T^{rorational} = \frac{1}{2}(w_1\ w_2\ w_3)\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, where $\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is $\underline{L}_G = \underline{I}_{\sim G}\underline{\omega}$).

➢ 18th November 1999

## Transformation of Axes

It may be necessary to calculate the inertia tensor with respect to *different* axes fixed within the body. Consider an arbitrary set of rectangular Cartesian axes at a fixed point P of a rigid body. Now choose a **new** set of axes at P by rotating the original axes about some axes.

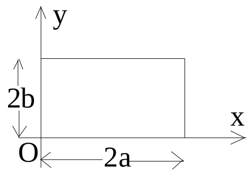
If $\underline{b}$ is any column vector as specified in the *original* co-ordinates, it becomes $\underline{b}' = U_\sim \underline{b}$ in the *new* co-ordinates, where $U_\sim$ is the orthogonal **rotation** matrix so that $U_\sim U_\sim^t = 1_\sim$ (the *identity matrix*). So $U_\sim = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $U_\sim t = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore, $U_\sim U_\sim^t$ = the *identity* matrix = $1_\sim$.

Proposition: Suppose that the *inertia tensor* at P is $I_{\sim P}$ relative to the *original* axes, and $I_\sim'_P$ relative to the *new* axes. Then $I_\sim'_P = U_\sim I_{\sim P} U_\sim^t$. **Proof**: from matrix algebra. **Proposition**: With respect to any point P of a body, *there exists a set of axes* with origin at P, such that the inertia tensor is diagonal. Proof: from matrix algebra. *(The orthogonal diagonalisation of a real symmetric matrix).*

**Definition**: The *principal axes of inertia* at P are the axes at P with respect to which the inertia tensor is diagonal, i.e. the products of inertia are **zero**. Definition: The principal moments of inertia are those *associated* with the principal axes of inertia. Note: For **principal** axes, $\underline{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_2 \end{pmatrix}$.

In *general*, $\underline{L}$ will not be in the same **direction** as $\underline{\omega}$. However, $\underline{L}$ *will* be in the same direction as $\underline{\omega}$ if (a) the **principal** moments of inertia are *equal:* (total symmetry): $I_{11} = I_{22} = I_{33}$; (b) $\underline{\omega}$ is along a *principal* axis, e.g. if $\underline{\omega} = \begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix}$, then $\underline{L} = \begin{pmatrix} I_{11}\Omega \\ 0 \\ 0 \end{pmatrix}$.

*Example*: A rectangular *lamina* of sides (2a, 2b); uniform density ρ; and mass M rotates about **one** corner. (a) Calculate the components of the inertia tensor *referred* to axes Oxyz fixed in the lamina, with Ox and Oy along the **sides** of the rectangle. (b) Find the product of inertia $I'_{yy}$, referred to new axes obtained from the *original* axes by rotation through an angle θ about Oz. Show that $I'_{yy} = 0$, when $\tan 2\theta = \frac{3ab}{2(a^2-b^2)}$.




A: (a) $I_{xx} = \Sigma m(y^2+z^2) = \Sigma my^2 = \int_0^{2a}\int_0^{2b} y^2\rho\,dydx = \rho\int_0^{2b} y^2dy \int_0^{2a} dx = \rho[y^3/3]_0^{2b}[x]_0^{2a} = \rho\,{}^{8b^3}/_3 2a = {}^{16\rho ab^3}/_3 = {}^{4b^2M}/_3$. $I_{yy} = \Sigma m(x^2+z^2) = \Sigma mx^2 = \int_0^{2a}\int_0^{2b} \rho x^2dxdy = \ldots = \rho\,{}^{8a^3}/_3 2b = {}^{4Ma^2}/_3$. And $I_{zz} = I_{xx}+I_{yy}$ (by the *lamina* theorem) $= {}^{4M}/_3(a^2+b^2)$. Now $I_{xz} = I_{zx} = I_{zy} = I_{yz} = 0$. And $I_{xy} = -\Sigma mxy = -\int_0^{2a}\int_0^{2b} \rho xydxdy = \ldots = -Mab$. So $I_{\sim 0} = \begin{pmatrix} 4Mb^2/3 & -Mab & 0 \\ -Mab & 4Ma^2/3 & 0 \\ 0 & 0 & (4M/3)(a^2+b^2) \end{pmatrix}$.

Therefore, $I_{\sim 0} = U_\sim I_{0\sim} U_\sim^t = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4Mb^2/3 & -Mab & 0 \\ -Mab & 4Ma^2/3 & 0 \\ 0 & 0 & (4M/3)(a^2+b^2) \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now concentrate on $I'_{xy}$ only. *Multiplying* the above gives $I'_{xy} = (I_{yy}-I_{xx})^1/_2\sin 2\theta + I_{xy}\cos 2\theta$. So for $I'_{xy} = 0$, $\tan 2\theta = -2I_{xy}/I_{yy}-I_{xx}$, i.e. $\tan 2\theta = \frac{2Mab}{(4/3)M(a^2-b^2)} = \frac{3ab}{2(a^2-b^2)}$. If a = b, then $\tan 2\theta \to \infty$, so that $2\theta = {}^\pi/_2$, i.e. $\theta = {}^\pi/_4$.

An alternative (*better*?) method, which is used to find the **eigenvalues**, is to solve the characteristic equation, etc.: find *eigenvalues* $\lambda_1, \lambda_2, \lambda_3$; and eigenvectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$, such that $I_\sim\underline{v} = \lambda\underline{v}$, i.e. $(I_\sim - \lambda 1_\sim)v = 0$. This is true *only* if $|I_\sim - \lambda 1_\sim| = 0$, which gives the *characteristic* equation.

If $I_\sim$ is a 3×3 tensor, then the **characteristic equation** is a *cubic* in λ. This gives **three** eigenvalues (the principal moments of inertia), and the eigenvectors are the orientation of the new axes. In detail, we **solve** $\begin{vmatrix} (4/3)Mb^2-\lambda & -Mab & 0 \\ -Mab & (4/3)Ma^2-\lambda & 0 \\ 0 & 0 & (4/3)M(a^2+b^2)-\lambda \end{vmatrix}$, or $({}^4/_3M(a^2+b^2)-\lambda)[({}^4/_3Mb^2-\lambda)({}^4/_3Ma^2-\lambda)-M^2a^2b^2]$.

➢ 19th November 1999

**Example**: A skater of mass 70kg is rotating about a *vertical axis* with arms outstretched horizontally at 100 revolutions per minute on **smooth** ice. He then folds his arms, which are each of mass 2.5kg. If the radius of *gyration* k of his arms is 0.5m when extended, and 0.1m when folded, and if the remainder of his body has a radius of gyration of 0.1m, calculate his *new rate of rotation.*

A: Since there are **no** external moments of forces about the *vertical* axis, the angular momentum is conserved. The skater's original moment of inertia about the vertical axis ($I = Mk^2$) is (mass of body)×$(0.1)^2$+(mass of arms)×$(0.5)^2$ = 65×0.01 + 5×0.25 = 1.9kgm² = $I_{OLD}$.

Also, $\omega_{OLD} = {}^{100}/_{60}\times 2\pi$ rads$^{-1}$ = 10.5 rads$^{-1}$. So the *original* angular momentum is $L_{OLD} = I_{OLD}\times\omega_{OLD}$ = 1.9×10.5 =19.95. The moment of *inertia* is $I_{NEW} = 65\times 0.1^2 + 5\times 0.1^2 = 0.7$ kgm². Also, $L_{NEW} = 0.7\omega_{NEW}$. But $L_{NEW} = L_{OLD}$, so $0.7\omega_{NEW} = 19.95$; $\omega_{NEW} = 28.5$ rads$^{-1}$. So the rate of **rotation** is ${}^{28.5}/_{2\pi}\times 60 = 272$ revolutions per *minute*.

➢ 23rd November 1999

# Analytical Mechanics (Lagrangian)

**Analytical** mechanics extends the ideas of Newtonian mechanics by placing the theory into a more *abstract* setting. Newton's laws (in vector form) can be tedious to apply. Analytical mechanics gives a method to formulate the equations of motion in a scalar manner. The concepts introduced are **prerequisites** for *quantum mechanics and relativistic mechanics.*

**Generalised** co-ordinates. Definition: *Degrees of freedom*: The number of d.o.f. of a mechanical system is the number of independent scalar co-ordinates required to describe its position precisely at a given instant in an inertial frame of reference. **Denote** this number by f.

Examples: (i) A single particle moving *freely* in 3-D has f = 3. [x,y,z; or r,θ,φ, etc.]. (ii) A single particle *restricted* to move on a surface z = z(x,y) has f = 2. The constraint reduces the d.o.f. by one. (iii) A single particle constrained to move on a **curve** has f = 1. Now consider a system with **N** particles having position vectors $\underline{r}_i$ for i = 1,2,...,N. Note that f ≤ 3N.

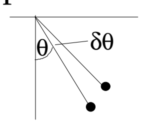
**Definition: Generalised co-ordinates**. Consider a system of N particles with d.o.f. ≤ 3N. The *configuration* of the system may be specified by f independent co-ordinates $q_1,q_2,\ldots,q_f$. The set $q_\alpha$, for α = 1,...,f, are called *generalised co-ordinates.* The choice of generalised co-ordinates is **not** unique (indeed, there is an infinite choice).

It is always possible to **express** the particle position $\underline{r}_i$ uniquely in terms of the $q_\alpha$, where α = 1,...,f. Therefore, $\underline{r}_i = \underline{r}_i(q_1,q_2,\ldots,q_f)$ for i = 1,...,N. (Here, if $\underline{r}_i = (x_i,y_i,z_i)$, then $x_i = x_i(q_1,\ldots,q_f)$, $y_i = y_i(q_1,\ldots,q_f)$, and $z_i = z_i(q_1,\ldots,q_f)$).

**Definition: Constraints:** If a system is described by a set of *scalar* co-ordinates $q_1,q_2,\ldots,q_k$, then a fixed *holonomic* constraint is $\phi(q_1,q_2,\ldots,q_k) = 0$ for some function φ.

**Definition: Generalised velocities:** The *time* derivatives of the generalised co-ordinates are called generalised velocities. These are denoted by $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_f$. Note that the **real** velocities are functions of *both* the $q_\alpha$ and the $\dot{q}_\alpha$, i.e. $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1,\ldots,q_f, \dot{q}_1,\ldots,\dot{q}_f)$. To see this, *consider* $\dot{\boldsymbol{r}}_i = \frac{d\boldsymbol{r}_i}{dt} = \sum_{\alpha=1}^{f} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\frac{dq_\alpha}{dt} = \sum_{\alpha=1}^{f} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha}\dot{q}_\alpha$, where the **chain** rule has been used.

**Definition: Virtual Displacement:** A virtual displacement is a *small* displacement compatible with the constraints. It is denoted by $\delta q_1, \delta q_2, \ldots, \delta q_f$. **Example**: Simple pendulum. Note that a virtual displacement is arbitrary, in that it need not correspond to any "real" displacement of the system under the applied forces. It is a mathematical trick.



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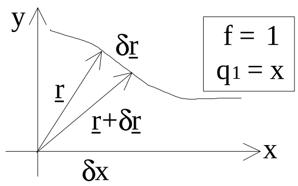



**Definition: Virtual Work:** Let $\delta r_1, \delta r_2, \ldots, \delta r_N$ be the displacements of the *particles* that correspond to the virtual displacements $\delta q_1, \delta q_2, \ldots, \delta q_f$. Let $F_i$ be the total force *acting* on the $i^{th}$ particle. $(i = 1,\ldots,N)$. The *virtual work done* by the forces during the virtual displacements is given by $\delta W = \Sigma^N_{i=1} \underline{F}_i.\delta\underline{r}_i$ (---(1)).

**Definition: Generalised forces**: The generalised forces $Q_\alpha$ $(\alpha = 1,\ldots,f)$ are defined under virtual *displacements* $\delta q_1,\delta q_2,\ldots,\delta q_f$ by $\delta W = \Sigma^f_{\alpha=1} Q_\alpha \delta q_\alpha$ (---(2)). So the *generalised* force $Q_\alpha$ is found by calculating the **virtual** work done in a virtual *displacement* $\delta q_\alpha$. By equating (1) and (2), we get $\Sigma^f_{\alpha=1} Q_\alpha q_\alpha = \Sigma^N_{i=1} \underline{F}_i.\delta\underline{r}_i$ (---(3)).

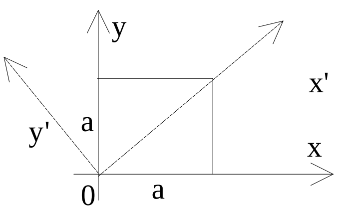
Now $\underline{r}_i = \underline{r}_i(q_1,q_2,\ldots,q_f)$, so by the *chain rule* for the total derivative, $\delta r_i = \Sigma^f_{\alpha=1} (\partial\underline{r}_i/\partial q_\alpha)\delta q_\alpha$ (---(4)). *Substituting* (4) into (3) gives $\Sigma^f_{\alpha=1} Q_\alpha \delta q_\alpha = \Sigma^N_{i=1} \underline{F}_i$. And $\Sigma^F_{\alpha=1} (\partial\underline{r}_i/\partial q_\alpha)\delta q_\alpha = \Sigma^f_{\alpha=1}(\Sigma^N_{i=1} \underline{F}_i.(\partial\underline{r}_i/\partial q_\alpha)\delta q_\alpha$. Since the $q_\alpha$ are *independent*, then $Q_\alpha = \Sigma^N_{i=1} \underline{F}_i .(\partial r_i/\partial q_\alpha)$ (---(5)).

# Assignment 2

Q: Consider a **cube** of side a; uniform density ρ; and total mass M. Calculate the *components* of the inertia tensor at a vertex O w.r.t axes fixed in the cube such that Oxyz are coincident with the edges of the cube. Calculate the angular momentum and kinetic energy of the cube when rotating with angular speed Ω about (a) The Ox axis; (b) the **diagonal** line joining O to the vertex (a,a,a).

A: $I_{xx} = \Sigma m(y^2+z^2) = \int^a_0\int^a_0\int^a_0 \rho(y^2+z^2)dxdydz = \iiint\rho y^2dxdydz + \iiint\rho z^2dxdydz = \rho a^{a^3}/_3 a+\rho a^{a^3}/_3 a = {}^2/_3\rho a^5$. Therefore, $I_{xx} = {}^2/_3Ma^2$. By *symmetry*, $I_{yy} = I_{zz} = {}^2/_3Ma^2$. Now $I_{xy} = -\Sigma mxy = -\int^a_0\int^a_0\int^a_0 \rho xydxdydz = -\rho^{a^2}/_2{}^{a^2}/_2 a = -\rho a^5/4$. By *symmetry*, $I_{yz} = I_{zx} = I_{xy} = -{}^1/_4Ma^2$. So $\underline{I}_{\sim 0} = Ma^2\begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$. (a) $\omega = \begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = \Omega\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. $\underline{L} = \underline{I}_{\sim 0}.\omega = Ma^2\Omega\begin{pmatrix} 2/3 \\ -1/4 \\ -1/4 \end{pmatrix}$. And $T = ½\underline{\omega}.\underline{L} = ½Ma^2\Omega^2(1\ 0\ 0)\begin{pmatrix} 2/3 \\ -1/4 \\ -1/4 \end{pmatrix} = {}^1/_3Ma^2\Omega^2$. (b) $\omega = {}^\Omega/_{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\underline{L} = \underline{I}_{\sim 0}.\omega = Ma^{2\Omega}/_{\sqrt{3}}\begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \end{pmatrix} = Ma^{2\Omega}/_{6\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. And $T = ½\underline{\omega}.\underline{L} = ½{}^\Omega/_{\sqrt{3}}.Ma^{2\Omega}/_{6\sqrt{3}}(1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = {}^1/_{12}Ma^2\Omega^2$.

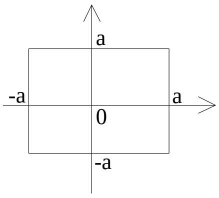
Q: Consider a square lamina of side a; *uniform* density ρ; and total mass M. Let O be the bottom left hand corner. Calculate the components of the *inertia tensor* at O with respect to axes Oxyz fixed in the lamina, with Ox and Oy along the sides of the square. By using the transformation law for tensors, obtain the components of the inertia tensor referred to axes Ox'y'z' **obtained** from Oxyz by a rotation about Oz so that Oz' lies along the diagonal of the square.




A: Lamina: $z = 0$; $M = \rho a^2$ for a *constant* ρ. $I_{xx} = \Sigma m(y^2+z^2) = \Sigma my^2 = \int^a_0\int^a_0 \rho y^2dxdy = r[x]^a_0[{}^{y^3}/_3]^a_0 = \rho a^4/3 = {}^1/_3Ma^2$. By symmetry, $I_{yy} = \Sigma mx^2 = I_{xx} = {}^1/_3Ma^2$. Now $I_{zz} = \Sigma m(x^2+y^2) = I_{xx}+I_{yy} = {}^2/_3Ma^2$. And $I_{xy} = -\Sigma mxy = -\int^a_0\int^a_0 \rho dxdy = -\rho[{}^{x^2}/_2]^a_0[{}^{y^2}/_2]^a_0 = -\rho a^4/4 = -{}^1/_4Ma^2$ *Therefore*, $\underline{I}_{\sim 0} = Ma^2\begin{pmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$.

**Transformation** law for tensors: $I' = UIU^t$. For a rotation $\theta$ about Oz, $U = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$. For $\theta = \pi/4$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}$. Therefore, $I' = \frac{Ma^2}{2}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}1/3 & -1/4 & 0\\ -1/4 & 1/3 & 0\\ 0 & 0 & 2/3\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix} = \frac{1}{2}Ma^2\begin{pmatrix}1/12 & -7/12 & 0\\ 1/12 & 7/12 & 0\\ 0 & 0 & 2\sqrt{2}/3\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}$. Therefore, $I' = \frac{1}{12}Ma^2\begin{pmatrix}1 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 8\end{pmatrix}$.

Q: A *square* lamina of side 2a has variable density given by $\rho = \rho_0+\rho_1(xy/a^2)$, where $\rho_0$ and $\rho_1$ are constants, and x and y are measured from the *centre* of the square along axes parallel to the edge. Calculate the inertia tensor of the lamina at its centre, w.r.t. the given axes. Find the principal axes of inertia, and specify the principle moments.




A: $\rho = \rho_0+\rho_1 xy/a^2$. $M = \Sigma m = \int_{-a}^{a}\int_{-a}^{a}dm = \int_{-a}^{a}\int_{-a}^{a}\rho\,dxdy$. Therefore, $M = \int_{-a}^{a}\int_{-a}^{a}\rho_0 dxdy + \int_{-a}^{a}\int_{-a}^{a}\rho_1 \frac{xy}{a^2}dxdy = \rho_0 2a2a+\rho_1 0 = 4a^2\rho_0$. Now $I_{xx} = \Sigma my^2 = \iint\rho y^2dxdy = \iint(\rho_0+\rho_1(xy/a^2))y^2dxdy = \iint\rho_0 y^2dxdy + \iint\rho_1 \frac{xy^3}{a^2}dxdy = 2a\frac{2a^3}{3}\rho_0 + 0 = (4a^4/3)\rho_0 = \frac{1}{3}Ma^2$. By *symmetry*, $I_{yy} = \Sigma mx^2 = I_{xx}$. Now $I_{zz} = I_{xx}+I_{yy} = \frac{2}{3}Ma^2$. And $I_{xy} = -\Sigma mxy = -\iint\rho xydxdy = -\iint\rho_0 xydxdy - \iint\rho_1 x^2y^2dxdy = -\frac{4}{9}a^4\rho_1$. Finally, $I_{yz} = I_{zx} = 0$ because z = 0. *Therefore*, $\underset{\sim}{I}_0 = \begin{bmatrix}\frac{1}{3}Ma^2 & -\frac{4}{9}a^4\rho_1 & 0\\ -\frac{4}{9}a^4\rho_1 & \frac{1}{3}Ma^2 & 0\\ 0 & 0 & \frac{2}{3}Ma^2\end{bmatrix}$.

To find the **principal** axes and principal moments, we could use $I' = AIA^t$, and calculate $\theta$ such that $I'_{xy} = 0$. Alternatively, find the *eigenvalues* & *eigenvectors* by solving the characteristic equation given by $|\underset{\sim}{I}_0-\lambda\underset{\sim}{1}| = 0$, i.e. the *yellow* bit on the left. Expanding the **determinant** by minors of the *third* row, we get $(\frac{2}{3}Ma^2-\lambda)[(\frac{1}{3}Ma^2-\lambda)^2-(\frac{4}{9}a^4\rho_1)^2] = 0$.

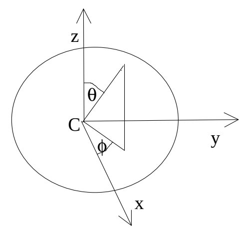
$$\begin{vmatrix}\frac{1}{3}Ma^2 & -\frac{4}{9}a^4\rho_1 & 0\\ -\frac{4}{9}a^4\rho_1 & \frac{1}{3}Ma^2 & 0\\ 0 & 0 & \frac{2}{3}Ma^2\end{vmatrix} = 0$$

So $(\frac{2}{3}Ma^2-\lambda)[(\frac{1}{3}Ma^2-\lambda)+\frac{4}{9}a^4\rho_1][(\frac{1}{3}Ma^2-\lambda)-\frac{4}{9}a^4\rho_1] = 0$. Therefore, the *three* eigenvalues are: $\lambda_1 = \frac{2}{3}Ma^2$, $\lambda_2 = \frac{1}{3}Ma^2+\frac{4}{9}a^4\rho_1$, and $\lambda_3 = \frac{1}{3}Ma^2-\frac{4}{9}a^4\rho_1$. *The eigenvector* $\underline{x}_{\lambda_1}$ for $\lambda_1$ is given by $(I_0-\lambda_1)\underline{x} = 0$, i.e. what is shown in **blue** on the left. Therefore, $\underline{x}_{\lambda_1} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Similarly, *eigenvector* $\underline{x}_{\lambda_2}$ is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, and eigenvector $\underline{x}_{\lambda_3}$ is $\begin{pmatrix}1\\1\\0\end{pmatrix}$. So $\underset{\sim}{I}'_0 = \begin{pmatrix}\lambda_3 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_1\end{pmatrix}$ = what is shown in *green* of the right. It has **principal** axes such that Ox' is along the diagonal of the *lamina*, i.e. rotated from the original axes by $\pi/4$ about Oz.

$$\begin{bmatrix}\frac{1}{3}Ma^2 & -\frac{4}{9}a^4\rho_1 & 0\\ -\frac{4}{9}a^4\rho_1 & \frac{1}{3}Ma^2 & 0\\ 0 & 0 & \frac{2}{3}Ma^2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = 0$$

$$\begin{pmatrix}\frac{1}{3}Ma^2-\frac{4}{9}a^4\rho_1 & 0 & 0\\ 0 & \frac{1}{3}Ma^2+\frac{4}{9}a^4\rho_1 & 0\\ 0 & 0 & \frac{2}{3}Ma^2\end{pmatrix}$$

Q: Use **multiple** integrals to calculate the inertia tensor at the centre of mass of a solid sphere of *radius* a and mass M. A: $M = \frac{4}{3}\pi a^3\rho$. By symmetry, the *centre* of mass is at the centre of the sphere. Using spherical polar co-ordinates, $(r,\theta,\phi)$, where $dV = r^2\sin\theta drd\theta d\phi$; $x^2+y^2 = r^2\sin^2\theta$; $x = r\sin\theta\cos\phi$; and $y = r\sin\theta\sin\phi$, we get:




$I_{zz} = \Sigma m(x^2+y^2) = \int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho(r^2\sin^2\theta)r^2\sin\theta drd\theta d\phi = \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4\sin^3\theta drd\theta d\phi = \rho[\phi]_0^{2\pi}[r^5/5]_0^a\int_0^{\pi}\sin^3\theta d\theta = \rho 2\pi(a^5/5)\int_0^{\pi}\sin^3\theta d\theta$. Now $\int_0^{\pi}\sin^3\theta d\theta = \int_0^{\pi}\sin^2\theta\sin\theta d\theta = \int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta$. Put $x = \cos\theta$, so that $dx = -\sin\theta d\theta$. $\theta = 0$ implies $x = 1$, and $\theta = \pi$ implies $x = -1$. Therefore, $\int_0^{\pi}\sin^3\theta d\theta = -\int_{-1}^{1}(1-x^2)dx = \int_{-1}^{1}(1-x^2)dx = [x-\frac{x^3}{3}]_{-1}^{1} = \frac{4}{3}$. So $I_{xx} = \rho 2\pi(a^5/4).\frac{4}{3} = \frac{2}{5}Ma^2$. By *symmetry*, $I_{xx} = I_{yy} = I_{zz}$. And by symmetry, *all products of inertia* are zero.

# Derivation of Lagrange's Equations

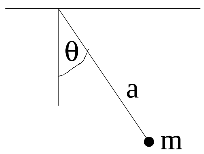
**Theorem**: Suppose that a mechanical system with *fixed holonomic constraints* is described by f independent **generalised** co-ordinates $q_1,\ldots,q_f$, with associated generalised velocities $\dot{q}_1,\ldots,\dot{q}_f$. Let $T = T(q_1,q_2,\ldots,q_f, \dot{q}_1,\dot{q}_2,\ldots,\dot{q}_f)$ be the K.E. Then the **equations** of motion of the system can be *written* as $\frac{d}{dt}(\partial T/\partial\dot{q}_\alpha)-\partial T/\partial q_\alpha = Q_\alpha$, ($\alpha = 1,\ldots,f$), where the $Q_\alpha$ are the *generalised* forces. **Proof** omitted. (It starts from Newton's *law of motion* for the $i^{th}$ particle: $m\ddot{\mathbf{r}}_i = F_i$)

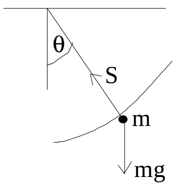
Notes: (i) **Lagrange's** equation is made up of *f equations*. (ii) T is a *quadratic* function of the $\dot{q}_\alpha$, therefore $\partial T/\partial\dot{q}_\alpha$ is a *linear function* of the $\dot{q}_\alpha$, so that $\frac{d}{dt}(\partial T/\partial\dot{q}_\alpha)$ is a *linear function* of the $\ddot{q}_\alpha$.

Rules for *writing down Lagrange's equation:* (1) Decide how many **degrees of freedom** there are, f. (2) Choose a set of *generalised* co-ordinates $q_1,\ldots,q_f$. (This is not a *unique* choice). (3) Find the K.E. T in terms of the $\dot{q}_\alpha$ and the $q_\alpha$. (4) Hence *find* $\partial T/\partial q_\alpha$ and $\frac{d}{dt}(\partial T/\partial\dot{q}_\alpha)$ for $\alpha = 1,\ldots,f$. (5) Find the $Q_\alpha$ by using $\delta W = \Sigma^f_{\alpha=1} Q_\alpha\, \delta q_\alpha$. (6) Substitute into *Lagrange's* equations of motion. (7) Solve if possible.

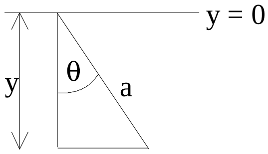
➢ 26th November 1999

Example: A simple pendulum. This is a particle on a *light inextensible string*. Assume that motion remains in a plane. The particle is constrained to move in a **circular arc**, hence f = 1. Take $q_1 = \theta$. (3) The speed of the particle is $v = a\dot{\theta}$, so $T = ½m(a\dot{\theta})^2 = ½ma^2\dot{\theta}^2$. (4) Lagrange's equation for this *system* (f = 1, so there is 1 equation) is the following equation: $\frac{d}{dt}(\partial T/\partial\dot{\theta})-\frac{\partial T}{\partial\theta} = Q_\theta$. Extra: $\frac{\partial T}{\partial\theta} = 0$, and $\frac{d}{dt}(\partial T/\partial\dot{\theta}) = \frac{d}{dt}(\frac{1}{2}ma^2 2\dot{\theta}) = ma^2\frac{d}{dt}\dot{\theta} = ma^2\ddot{\theta}$.




(5) To find the **generalised** force $Q_\theta$, there are 2 forces acting on the particle: the force due to *gravity*, mg, and the tension in the string, S. Note that S does **no** work in a virtual displacement $\delta\theta$. The work done by *gravity* in a virtual displacement $\delta\theta$ is $\delta W = -mg\delta y$, where $\delta y$ is the *virtual displacement* in the vertical direction (corresponding to $\delta\theta$).



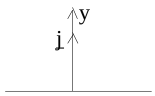

**Now** $y = -a\cos\theta$, and $\delta y = \frac{dy}{d\theta}\delta\theta$, so $\delta y = a\sin\theta\delta\theta$. Therefore, $\delta W = -mga\sin\theta\delta\theta$. The *generalised* force $Q_\theta$ is defined by $\delta W = Q_\theta\delta\theta$. Comparing the *two expressions* for $\delta W$, we get $Q_\theta = -mga\sin\theta$. (6) Substitute into *Lagrange's* equation, giving $ma^2\ddot{\theta}-0 = -mga\sin\theta$, or $a\ddot{\theta} + \frac{g}{a}\sin\theta = 0$ (*a non-linear* oscillator) as the equation of motion.

(**Linearising**, $a\ddot{\theta}+\frac{g}{a}\theta = 0$ (assume small $\theta$). *This is a linear* oscillator, with $\omega = \sqrt{(g/a)}$). This is the *usual* equation for the motion of a single pendulum. **Note** that $\dot{\theta}$ is not a *true* velocity (the true velocity is $a\dot{\theta}$). **Also** note that $Q_\theta$ is not a true force: it is a *moment* of a force: (a torque): $Q_\theta = -mga\sin\theta$, where $-mg$ is a *force*, and $a\sin\theta$ is a *distance*. But the **product** $Q_\theta\delta\theta$ has the dimension of true work, $\delta W = Q_\theta\delta\theta$.

➢ 30th November 1999

# Lagrange's Equations for Conservative Systems

Recall that a *conservative mechanical system* is one in which the forces that do work can be derived from a scalar potential, and that V, the **potential energy** of the system, is the *sum* of these potentials. ($V = mgy$ and $F = -\nabla V \Rightarrow F = -mg\mathbf{j}$).

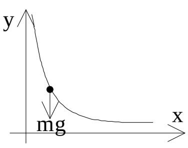



**Proposition**: For a conservative system with *holonomic* constraints, the generalised forces $Q_\alpha$ are defined as $Q_\alpha = -\partial V/\partial q_\alpha$. Proof *omitted*. For conservative systems, Lagrange's equation becomes ${}^{d}/_{dt}(\partial T/\partial \dot{q}_\alpha) - \partial T/\partial q_\alpha = -\partial V/\partial q_\alpha$. We can simplify this equation by defining a *function* L as follows: $L = T-V$.

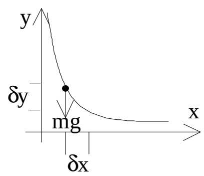
**Definition**: The Lagrangian is defined as $L = T-V$. Note that L is a function of the $q_1,\ldots,q_f$ and the $\dot{q}_1,\ldots,\dot{q}_f$. Also note that $\partial L/\partial \dot{q}_\alpha = \partial T/\partial \dot{q}_a$, because V is not a *function* of the $\dot{q}_\alpha$. So *Lagrange's* equation may be written as ${}^{d}/_{dt}(\partial L/\partial \dot{q}_a) - \partial L/\partial q_\alpha = 0$, for $\alpha = 1,\ldots,f$.

**Rules** for writing down Lagrange's equation for a conservative system: (1) Decide how many *d.o.f. there are*, f. (2) Choose a set of generalised co-ordinates $q_1,\ldots,q_f$. (3) Find the K.E., T, in terms of the $\dot{q}_\alpha$ and the $q_\alpha$. (4) Find the P.E., V, in *terms* of the $q_\alpha$. (5) Form $L = T-V$, and hence find ${}^{d}/_{dt}(\partial L/\partial \dot{q}_\alpha)$ and $\partial L/\partial q_\alpha$. (6) Substitute into *Lagrange's* equations. (7) Solve, but only *if asked to do so*.

Example: A **bead** of mass m is constrained to move along a smooth wire having the shape of a *hyperbola*, $xy = c$, where c is a positive constant. (a) Show that the kinetic energy may be represented as $T = {}^1/_2 m\dot{x}^2(a+(c^2/x^4))$. (b) Obtain the *generalised* force corresponding to x, ($Q_x = mg^c/_{x^2}$), and write down Lagrange’s equation. (c) Derive Lagrange’s equation by using the *potential* energy, V.




A: The particle is **constrained** to move on the curve $xy = c$, so $f = 1$. Choose generalised co-ordinate $q_1 = x$. Find an expression for T: $T = {}^1/_2 mv^2 = {}^1/_2 m(\dot{x}^2+\dot{y}^2)$. But $y = {}^c/_x$, so $\dot{y} = -{}^c/_{x^2}\dot{x}$; $\dot{y}^2 = (c^2/x^4)\dot{x}^2$. So $T = {}^1/_2 m(\dot{x}^2+\dot{x}^2(c^2/x^4)) = m\dot{x}^2/2(1+(c^2/x^4))$. Next, we find $Q_x$.




The work done during a *virtual* displacement $\delta x$ with corresponding virtual displacement $\delta y$ is $\delta W = -mg\delta y$. But $y = {}^c/_x$, so $\delta y = -{}^c/_{x^2}\delta x$. Hence $\delta W = -mg(-{}^c/_{x^2})\delta x = mg({}^c/_{x^2})\delta x$. But $Q_x$ is defined by $\delta W = Q_x\delta x$, therefore $Q_x = mg{}^c/_{x^2}$.

To obtain *Lagrange's* equation for this system, we use ${}^{d}/_{dt}(\partial T/\partial \dot{x}) - {}^{\partial T}/_{\partial x} = Q_x$. First, we *obtain* ${}^{d}/_{dt}(\partial T/\partial \dot{x})$ like this: ${}^{d}/_{dt}(\partial T/\partial \dot{x}) = {}^{d}/_{dt}({}^1/_2 m2\dot{x}(1+(c^2/x^4)) = {}^{d}/_{dt}(m\dot{x}(1+c^2/x^4)) = m\ddot{x}(1+(c^2/x^4)) + m\dot{x}((-4c^2/x^5)\dot{x}) = m\ddot{x}(1+(c^2/x^4)) - 4m\dot{x}^2(c^2/x^5)$. Secondly, ${}^{\partial T}/_{\partial x} = {}^1/_2 m\dot{x}^2(-4c^2/x^5) = -2(mc^2/x^5)\dot{x}^2$. So *Lagrange's equation* is $m\ddot{x}(1+(c^2/x^4))-2(mc^2/x^5)\dot{x}^2 = {}^{mgc}/_{x^2}$.

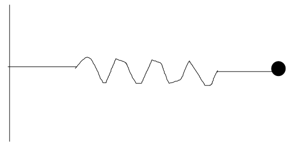
Find the **potential**, and hence write down *Lagrange’s equation* for the system. $V = mgy = mg{}^c/_x$. Also, ${}^{d}/_{dt}(\partial T/\partial \dot{x})-{}^{\partial T}/_{\partial x} = -{}^{\partial V}/_{\partial x}$; so ${}^{d}/_{dt}(\partial L/\partial \dot{x})-{}^{\partial L}/_{\partial x} = 0$, where $L = T-V$. Here, we have $L = {}^1/_2 m\dot{x}^2(1+(c^2/x^4))-mg{}^c/_x$. So Lagrange’s equation for the *system* is ${}^{d}/_{dt}(\partial L/\partial \dot{x})-{}^{\partial L}/_{\partial x} = 0$. This **implies** the same equation of motion as obtained using the *generalised* force, $Q_x$.

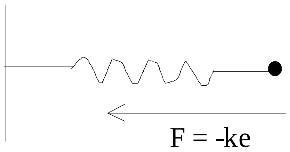
➢ 2nd December 1999

# Notes on the Determination of T and V for Particular Systems

(1) The P.E. due to **gravity** is given by $\Sigma_k m_k g h_k$, where $h_k$ is the height of the $k^{th}$ particle above some *reference* point. (2) The P.E. of a rigid body is MgH, where M is the **total** mass, and H is the height of the **centre** of mass above some reference point. (3) The K.E. of a rigid body is given by $\frac{1}{2}Mv^2+\frac{1}{2}I\dot{\theta}^2$, where M is the **total** mass; v is the speed of the *centre of mass*; and I is the moment of <u>inertia</u> about its axis of rotation.

(4) P.E. due to the **extension** of a spring. Denote the *unextended* natural length of the spring by $l$. Let k be the spring **stiffness**, and let λ be the spring modulus, where $k = {}^{\lambda}/_{l}$. Let the *displacement* of the string from its natural position be e. Then $V = \frac{1}{2}ke^2$, or $V = \frac{1}{2}\frac{\lambda}{l}e^2$. Therefore, $V = -\int F dv = \int ke\, de = \frac{1}{2}ke^2$. So if the spring is *extended* to length $l$, then $V = \frac{1}{2}k(l_1-l)^2$.






# Small Oscillations and Normal Modes

**<u>Proposition</u>**: The solution of *Lagrange's equation* for small oscillations is given by $q_\alpha = a_\alpha\cos(\omega t+\varepsilon)$ (for $\alpha = 1,..,f$), or $\underline{q} = \underline{a}\cos(\omega t+\varepsilon)$ (where $\underline{q} = (q_1,...,q_f)^t$ and $\underline{a} = (a_1,...,a_f)^t$). Here, $\Sigma^{\alpha}_{\beta=1} (V_{\alpha\beta}-\omega^2 T_{\alpha\beta})a_\beta = 0$ (for $\alpha = 1,..,f$), or $(V_{\sim}-\omega^2 T_{\sim})\underline{a} = \underline{0}$.

**<u>Proof</u>**: (*outline*): substitute a trial solution $q_\alpha = a_\alpha e^{\lambda t}$ into Langrange's *equation*, where $a_\alpha$ & λ are constants. But as the **equilibrium** position is stationary, λ must be imaginary, so the *trial* solution must be of the **form** $q_\alpha = a_\alpha\cos(\omega t+\varepsilon)$.

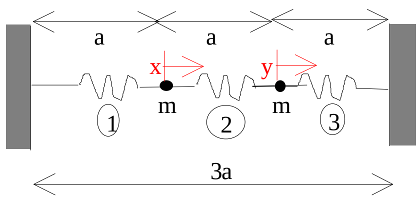
**<u>Note</u>**: This is a generalised *eigenvalue* problem. The natural frequencies of the system are the eigenvalues, and the <u>normal</u> modes are the eigenvectors. The natural frequencies satisfy the *characteristic* equation $\det(V_{\sim}-\omega^2 T_{\sim}) = \underline{0}$. There are as many **normal** modes as there are **degrees** of freedom.

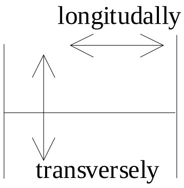
The *natural frequencies* are $\omega_1,...,\omega_f$. In a single normal mode, **all** the particles oscillate with the same frequency. The radius of the amplitude of *different* particles is given by the normal mode shape. The general motion of the system is a **superposition** of all the normal modes.

➢ 3rd December 1999

# Example

Q: 2 particles of **equal** mass m are attached at the point of trisection of an elastic spring of stiffness k and natural length $3l$, which is attached to two *points* 3a apart. The masses are constrained to move longitudinally on a smooth surface. By a Lagrangian method, calculate the natural <u>frequencies</u> of the system, and find the normal modes. Describe the **amplitude** and **phase** of the particle's displacements in each mode.






A: There are **two** degrees of freedom, so f = 2. Choose generalised co-ordinates x & y, which are the *displacements* of the particles from their equilibrium positions. Lagrange's equations for the system can be written down as follows: $\frac{d}{dt}(\partial L/\partial\dot{x})-\frac{\partial L}{\partial x}=0$; $\frac{d}{dt}(\partial L/\partial\dot{y})-\frac{\partial L}{\partial y}=0$, where T = L-V.

For this **system**, $T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2$. To find the *potential energy* of the system, we must calculate the extensions in each of the springs. (see the **table**). So $V=\frac{1}{2}k((a-l)^2+2(a-l)x+x^2+(a-l)^2+2(a-l)(y-x)+(y-x)^2+(a-l)^2+2(a-l)(-y)+y^2)$; $V=\frac{1}{2}k(3(a-l)^2+x^2+y^2-2xy+x^2+y^2)$. Discarding the **additive** constant term gives $V=k(x^2-xy+y^2)$.

| *Spring* | *extension* | *P.E.* |
|---|---|---|
| 1 | a-*l*+x | $\frac{1}{2}k(a-l+x)^2$ |
| 2 | a-*l*-x+y | $\frac{1}{2}k(a-l-x+y)^2$ |
| 3 | a-*l*-y | $\frac{1}{2}k(a-l-y)^2$ |

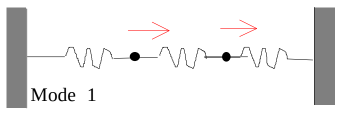
*Hence* $L=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2-k(x^2-xy+y^2)$. Now $\frac{d}{dt}(\partial L/\partial\dot{x})=\frac{d}{dt}(\frac{1}{2}m2\dot{x})=m\ddot{x}$; $\frac{\partial L}{\partial x}=-k(2x-y)$; $\frac{d}{dt}(\partial L/\partial\dot{y})=\frac{d}{dt}(\frac{1}{2}m2\dot{y})=m\ddot{y}$; and $\frac{\partial L}{\partial y}=-k(-x+2y)$. So **Lagrange's** equations are as follows: $m\ddot{x}+k(2x-y)=0$, and $m\ddot{y}+k(2y-x)=0$. To find the **modes** of oscillation, substitute a *trial* solution: $x=A\cos(\omega t+\varepsilon)$; $y=B\cos(\omega t+\varepsilon)$.

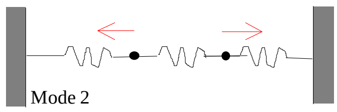
These expressions *give* $\dot{x}=-A\omega\sin(\omega t+\varepsilon)$, and $\ddot{x}=-A\omega^2\cos(\omega t+\varepsilon)$. Similarly, $\ddot{y}=-B\omega^2\cos(\omega t+\varepsilon)$. So $-A\omega^2\cos(\omega t+\varepsilon)+\frac{k}{m}(2A\cos(\omega t+\varepsilon)-B\cos(\omega t+\varepsilon))=0$; $-A\omega^2+\frac{k}{m}(2A-B)=0$; $(\frac{2k}{m}-\omega^2)A-\frac{k}{m}B=0$. Further, $-B\omega^2\cos(\omega t+\varepsilon)+\frac{k}{m}(2B\cos(\omega t+\varepsilon)-A\cos(\omega t+\varepsilon))=0$; $-B\omega^2+\frac{k}{m}(2B-A)=0$; $-\frac{k}{m}A+(\frac{2k}{m}-\omega^2)B=0$.

This can be **placed** in a matrix as shown. For *non-trivial* solutions, the determinant of the first matrix must be **zero**, i.e. $(\frac{2k}{m}-\omega^2)^2-(\frac{k}{m})^2=0$; $4(\frac{k}{m})^2-4\frac{k}{m}\omega^2+(\omega^2)^2-(\frac{k}{m})^2=0$; $(\omega^2)^2-4\frac{k}{m}\omega^2+3(\frac{k}{m})^2=0$; $(w^2-\frac{k}{m})(w^2-\frac{3k}{m})=0$.

$$\begin{pmatrix} 2\frac{k}{m}-\omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m}-\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the *natural frequencies* (the eigenvalues) of the system are $\omega^2=\frac{k}{m}$ — so $\omega=\sqrt{(\frac{k}{m})}$; and $\omega^2=\frac{3k}{m}$ — so $\omega=\sqrt{(\frac{3k}{m})}$. To obtain the *natural modes* (the eigenvectors), substitute back into the matrix equation for $\omega^2$. Firstly, for $\omega^2=\frac{k}{m}$: $(2\frac{k}{m}-\frac{k}{m})A-\frac{k}{m}B=0$; $\frac{k}{m}A-\frac{k}{m}B=0$. Therefore, A = B. Secondly, for $\omega^2=3\frac{k}{m}$: $(2\frac{k}{m}-3\frac{k}{m})A-\frac{k}{m}B=0$; $-\frac{k}{m}A-\frac{k}{m}B=0$. Therefore, A = -B.




In mode 1, $\omega=\sqrt{(\frac{k}{m})}$, and the two particles are oscillating with **equal** amplitude exactly **in** phase. In mode 2, $\omega=\sqrt{(\frac{3k}{m})}$, and the two particles are oscillating with **equal** amplitude exactly **out** of phase. What about if we have *different* masses or more particles — in transverse oscillation, different masses would mean different **peaks** for each mass' oscillation, while increasing the amount of particles to infinity would then model a string.

**EXAM**: In *Section 1*, expect 1 question on **Orbits** and another on **Solid Body Mechanics.** In *Section 2*, expect questions on **Orbits; Solid Body Mechanics; Oscillations (Lagrange)** and **Relativity**.

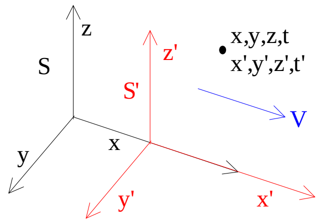
➢ 7th December 1999

# The Theory of Special Relativity

Newtonian (*classical*) mechanics fails in two main situations: (1) At a **small** scale, when typical length × momentum is ≤ h ($6.6\times10^{-34}$). Example: an electron in orbit about a nucleus has angular momentum approximately h. We need quantum mechanics. (2) Where velocities relative to the observer are **high** (approximately c, the velocity of light, $3\times10^{8}ms^{-1}$). We need *relativity theory*. If both (1) & (2) occur together, we require *relativistic quantum mechanics.*

**Special Relativity: Einstein, 1905**. This applies when gravitational forces are *unimportant* and the frames of references are *inertial*. **General Relativity: Einstein, 1916.** This is the relativistic theory of gravitation, and deals with *non-inertial* frames.

**Standard Configuration**. Consider two *inertial* frames S & S' moving with relative velocity V. It is always possible to align the co-ordinate axes so that the relative velocity is along the Ox and O'x' axes. This is known as the standard configuration.




Let an event occur at a time and a place in frame S. It is **specified** by x, y, z and t in frame S; and x', y', z' and t' in frame S'. The Gallileon transformation (which is used in *classical* mechanics) is x' = x-Vt; y' = y; z' = z; and t' = t. The *implicit* assumptions are: (1) the existence of a universal time; (2) the **distance** between two particles is independent of the observer: $\sqrt{(x_1'-x_2')^2+(y_1'-y_2')^2+(z_1'-z_2')^2} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$, where $(x_1,y_1,z_1)$ and $(x_2,y_2,z_2)$ are the positions of the *first and second* particles in frame S; and $(x_1',y_1',z_1')$ and $(x_2',y_2',z_2')$ are the positions of the *first and second* particles in frame S'.

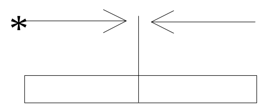
# The Michelson Morley Experiment (1887)

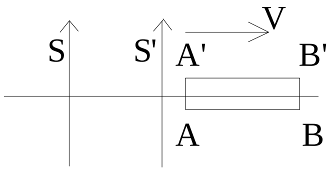
The experiment tried to measure the speed of the Earth *relative* to something called "ether". The experiment could not find a "*preferred direction*", which is implicit in the **Galilean** transformation when applied to electromagnetics. Postulates of Special Relativity: (1) The laws of nature take the same form in **all** inertial frames; (2) In any inertial frame, the velocity of light is the same in *all directions*, independently of the *velocity* of the light source.

➢ 9th December 1999

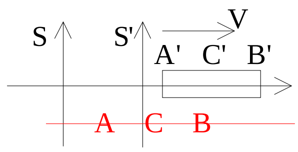
# The Relative Nature of Simultaneity

Two events at different places in an *inertial* frame are simultaneous if light signals announcing these events reach the mid point at the same time, as shown on the right. Now consider a rod A'B' in frame S' (at rest in frame S'). Suppose at a certain instant, the rod is AB in *frame* S, as shown on the left.





At this instant, light flashes are *emitted* from the two ends, which appear simultaneously according to S (i.e. the two light signals reach the midpoint of AB at the **same** time — denote this position as C). Let us denote the mid point of the rod in *frame* S' as C'. To the observer in S, the light flashes have just reached C. This means that the light flash from B has *passed* C', and the light from A has not reached C'.




So the two light flashes do **not** reach C' simultaneously. An observer in S' would say that the two events did *not* occur simultaneously.

# Lorentz Transformation

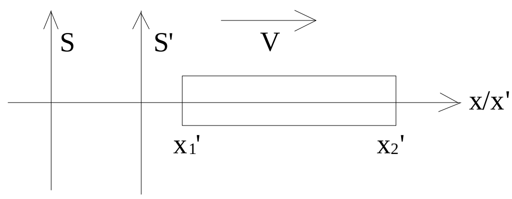
We want a **transformation** between x, y, z and t (which describes an event according to the frame S) and x', y', z' and t' (which describes the *same* event in frame S'). The derivation of the transformation assumes the following: (1) the transformation is linear; (2) we are in standard configuration, i.e. y' = y, z' = z; (3) we suppose that a light flash is emitted from the origin of S just as it *coincides* with the origin of S'. This *eventually* gives us the relationship $x'^2-c^2t'^2 = x^2-c^2t^2$.

**Lorentz Transformation**: $x' = (x-Vt)/\sqrt{1-(V^2/c^2)}$, y' = y, z' = z, and $t' = (t-(Vx/c^2))/\sqrt{1-(V^2/c^2)}$. *Inverse* Lorentz Transformation: $x = (x'+Vt')/\sqrt{1-(V^2/c^2)}$, y = y', z = z', and $t = (t'+(Vx'/c^2))/\sqrt{1-(V^2/c^2)}$. Note: As $c \to \infty$, the Lorentz transformation becomes the *same* as the Galilean transformation. Thus, unless V is near to the speed of light, the Lorentz transformation is very **close** to the Galilean transformation.

**Immediate consequences of the Lorentz transformation**: (1) *Simultaneity*. Consider the two events $E_1$ and $E_2$, with *co-ordinates* $z_1,y_1,z_1,t_1$ and $x_2,y_2,z_2,t_2$ in frame **S**; and co-ordinates $x'_1,y'_1,z'_1,t'_1$ and $x'_2,y'_2,z'_2,t'_2$ in frame **S'**. According to the *Lorentz* transformation, (event 1) $t_1 = (t_1'+(Vx_1'/c^2))/\sqrt{1-(V^2/c^2)}$, and, (event 2), $t_2 = (t_2'+(Vx_2'/c^2))/\sqrt{1-(V^2/c^2)}$.

Let us denote the *time interval* between the **two** events as $\Delta t = t_2-t_1$ in frame S, and $\Delta t' = t_2'-t_1'$ in frame S'. *Subtracting* the above two equations gives $t_2-t_1 = \frac{t_2'-t_1'+(v/c^2)(x_2'-x_1')}{\sqrt{1-(V^2/c^2)}}$. Note: consider two *simultaneous* events in frame S', i.e. $\Delta t' = 0$. Then $\Delta t \neq 0$ unless $\Delta x' = 0$.

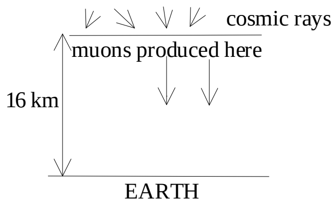
(2) **Lorentz-Fitzgerald** contraction. Consider a rod at rest on the x-axis in the S' frame. An observer *measures* its ends as being at $x_1'$ and at $x_2'$. The length of the rod as measured by an observer in S' is $l_0 = x_2'-x_1'$ (the *proper* length of the rod).




Suppose an **observer** in S finds the ends of the rod to be at *positions* $x_1$ and $x_2$ at time t, and hence measures its length to be $l = x_2-x_1$. The *Lorentz transformation* gives $x_1' = (x_1-Vt)/\sqrt{1-(V^2/c^2)}$ and $x_2' = (x_1-Vt)/\sqrt{1-(V^2/c^2)}$. *Subtracting* gives $x_2'-x_1' = (x_2-x_1)/\sqrt{1-(V^2/c^2)}$. So $l_0 = 1/\sqrt{1-(V^2/c^2)}$; $l_0 = 1/\sqrt{1-(V^2/c^2)}$, or $l = l_0\sqrt{1-(V^2/c^2)}$. Thus the length measured *in S* is smaller than the length measured in S'. The length as measured in the frame in which the rod is at rest is called its **proper** length.

(3) **<u>Time Dilation</u>**. Consider a clock fixed in S. Suppose that *two* readings of the clock are $t_1$ and $t_2$. These two events occur at the **same** position in frame S, i.e. $x_1 = x_2$. Suppose the times assigned by S' are $t_1'$ and $t_2'$. The *Lorentz* transformation gives $t_1' = \frac{t_1-(Vx_1/c^2)}{\sqrt{1-(V^2/c^2)}}; t_2' = \frac{t_2-(Vx_2/c^2)}{\sqrt{1-(V^2/c^2)}}$. *Let* $\Delta t' = t_2'-t_1'$, and let $\Delta t = t_2-t_1$, — then $\Delta t' = \Delta t/\sqrt{(1-(V^2/c^2))}$. Thus $\Delta t = \Delta t'\sqrt{(1-(V^2/c^2))}$, and therefore a clock *fixed* in S goes <u>slow</u> compared to the one in S'.

**<u>Question</u>**: Muons enter the Earth's atmosphere at a height of 16km, and have a lifetime of $2.2\times10^{-6}$s. Explain why many muons reach the Earth's *surface*, having velocities of about 0.999c. A: Using **classical** theory, $S = ut = (0.99c)(2.2\times10^{-6}) = (3\times10^8)(2.2\times10^{-6}) \approx 660$m. We may explain the observation in **two** ways:




(1) *Time dilation*: According to an observer on Earth, the muon clock goes <u>slow,</u> and the lifetime of a moving muon is $2.2\times10^{-6}/\sqrt{(1-V^2/c^2)} = 4.92\times10^{-5}$s. As it is *travelling* at 0.999c, it travels $0.999c\times4.92\times10^{-5}$m = 14745m. Thus to an observer on Earth, this *allows* enough time for the muons to reach the Earth. (2) *Lorentz-Fitzgerald* contraction: In the muon frame, the **lifetime** is $2.2\times10^{-6}$s, and the Earth is approaching at 0.99c. The *distance* to Earth, as measured by a muon, is $16\sqrt{(1-V^2/c^2)} \approx 0.66$km.

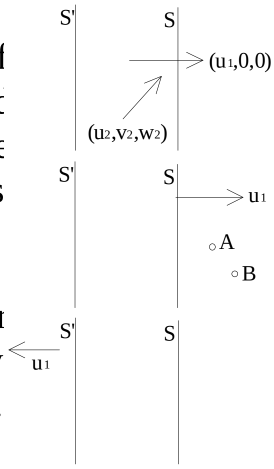
➢ <u>14th December 1999</u>

# <u>Transformation of Velocities</u>

Consider a *particle* with co-ordinates x,y,z,t in S, and x',y',z',t' in S'. The velocity in S is $(u,v,w) = (dx/dt, dy/dt, dz/dt)$. In S', it is $(u',v',w') = (dx'/dt', dy'/dt', dz'/dt')$. From the *Lorentz* transformation, $dx = \frac{dx'+Vdt'}{\sqrt{(1-(V^2/c^2))}}$; $dy = dy'$; $dz = dz'$; and $dt = \frac{dt'+(Vdx'/c^2)}{\sqrt{(1-(V^2/c^2))}}$. Now $u = \frac{dx}{dt} = \frac{dx'+Vdt'}{dt'+(Vdx'/c^2)} = \frac{(dx'/dt')+V}{1+(V/c^2)(dx'/dt')}$. Therefore, $u = \frac{u'+V}{1+(V/c^2)u'}$. Further, $v = \frac{dy}{dt} = \frac{dy'\sqrt{(1-(V^2/c^2))}}{dt'+(Vdx'/c^2)} = \frac{v'\sqrt{1-(V^2/c^2))}}{1+(Vu'/c^2)}$. And $w = \frac{dz}{dt} = \ldots = \frac{w'\sqrt{1-(V^2/c^2)}}{1+(Vu'/c^2)}$.

# <u>Relative Velocity</u>

The velocity of B *relative* to A is the velocity of B in the frame of **reference** in which A is at rest. Suppose that in S', A has *velocity* $(u_1,0,0)$ and B has *velocity* $(u_2,v_2,w_2)$. Choose S such that A is at *rest* in it. Then the velocity of S relative to S' is $(u_1,0,0)$. So the velocity of S' relative to S is $(-u_1,0,0)$.



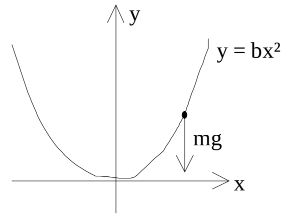

The velocity of B *relative* to S is given by the transformation laws for velocity (derived above), with $V = -u_1$. Special Case: suppose that the velocity of B in S' is $(u_2,0,0)$. Then, the velocity of B relative to A is $\left(\frac{u_2-u_1}{1-(u_1u_2/c^2)}, 0, 0\right)$.

# <u>Minkowski Space Time</u>

Let us define $\tau = ict$, then under the *Lorentz transformation*, $x^2+y^2+z^2+\tau^2 = x'^2+y'^2+z'^2+\tau'^2$. It may be seen that the Lorentz transformation is a rotation in the **four** dimensional space time *continuum*.

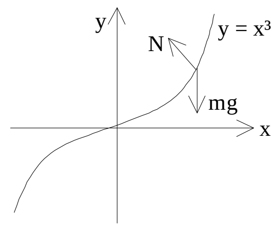
# Exercises 4: Analytical Mechanics

Q: A bead of mass m is free to slide on a *smooth parabolic wire* of shape $y = ax^2$, set up in a vertical plane so that the force of gravity acts in the negative y-direction. Determine the **Lagrangian**, and hence derive Lagrange's equations for the system. A: f = 1, and take the *generalised co-ordinate* to be x. The K.E. T is $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$. But $y = bx^2$, so $\dot{y} = 2bx\dot{x}$.




**Therefore,** $T = \frac{1}{2}m(\dot{x}^2+(2bx\dot{x})^2) = \frac{1}{2}m\dot{x}^2(1+4b^2x^2)$. The P.E. is $V = mgh = mgy = mgbx^2$. **The Lagrangian is** $L = T-V = \frac{1}{2}m\dot{x}^2(1+4b^2x^2) - mgbx^2$. Lagrange's Equation is $\frac{d}{dt}(\partial L/\partial\dot{x})-\frac{\partial L}{\partial x} = 0$. Now we have $\frac{d}{dt}(\partial L/\partial\dot{x}) = m\ddot{x}(1+4b^2x^2)+8mb^2x\dot{x}^2$; and $\frac{\partial L}{\partial x} = \frac{1}{2}m\dot{x}^2(8b^2x)-2mgbx = 4mb^2x\dot{x}^2 - 2mgbx$. So *Lagrange's* equation is $m\ddot{x}(1+4b^2x^2) + 8mb^2x\dot{x}^2 - 4mb^2x\dot{x}^2 + 2mgbx = 0$. *Therefore,* $m\ddot{x}(1+4b^2x^2)+4mb^2x\dot{x}^2+2mgbx = 0$.

Q: A bead of mass m is free to slide on a smooth parabolic wire of shape $y = x^3$, set up in a vertical plane so that the force of gravity acts in the *negative* y-direction. Determine the generalised force corresponding to x, and derive the corresponding Lagrangian equation of motion. Determine the *Lagrangian,* and hence derive Lagrange's equation of motion for the system.



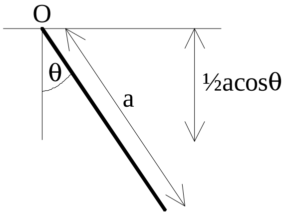

A: f = 1, and take $q_1$ to be x. **Lagrange's** equation is $\frac{d}{dt}(\partial T/\partial\dot{x})-\frac{\partial T}{\partial x} = Q_x$. The forces acting on the *mass* are its weight, acting downwards in the **negative** y-direction; and the normal reaction from the wire, which does no work. So in a virtual *displacement* $\delta x$, with corresponding $\delta y$, $\delta W = -mg\delta y$.

Now $y = x^3$, so $\delta y = \frac{dy}{dx}\delta x = 3x^2\delta x$. Therefore, $\delta W = -3mgx^2\delta x$. By the *definition* of the generalised force, $\delta W = Q_x\delta x$. Therefore, $Q_x = -3mgx^2$. Now $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$. But $y = x^3$, so $\dot{y} = 3x^2\dot{x}$. So $T = \frac{1}{2}m\dot{x}(1+9x^4)$. Now $\frac{d}{dt}(\partial T/\partial\dot{x}) = \frac{d}{dt}(m\dot{x}(1+9x^4)) = m\ddot{x}(1+9x^4)+36m\dot{x}^2x^3$. And $\frac{\partial T}{\partial x} = \frac{1}{2}m\dot{x}^2(36x^3) = 18m\dot{x}^2x^3$. So *Lagrange's* equation becomes $m\ddot{x}(1+9x^4) + 36m\dot{x}^2x^3 - 18m\dot{x}^2x^3 = -3mgx$. Therefore, $\ddot{x}(1+9x^4)+18\dot{x}^2x^3+3gx^2 = 0$.

The Lagrangian is L = T-V. For the *mass,* $V = mgh = mgy = mgx^3$. Therefore, we have $L = \frac{1}{2}m\dot{x}^2(1+9x^4)-mgx^3$. *Lagrange's* equation for a conservative system is $\frac{d}{dt}(\partial L/\partial\dot{x}) - \frac{\partial L}{\partial x} = 0$. Here, $\frac{d}{dt}(\partial L/\partial\dot{x}) = \frac{d}{dt}(m\dot{x}(1+9x^4)) = m\ddot{x}(1+9x^4)+36m\dot{x}^2x^3$. Further, $\frac{\partial L}{\partial x} = \frac{1}{2}m\dot{x}^2(36x^3)-3mgx^2 = 18m\dot{x}^2x^3-3mgx^2$. So *Lagrange's equation* becomes $m\ddot{x}(1+9x^4)+36m\dot{x}^2x^3-18m\dot{x}^2x^3+3mgx^2 = 0$. Therefore, $\ddot{x}(1+9x^4)+18\dot{x}^2x^3+3gx^2 = 0$, as before.

Q: A rigid rod pendulum is suspended by one of its ends from a **fixed** point O, and swings in a vertical plane subject to the force due to gravity, g. Derive the Lagrangian equation for the rod, in terms of its total mass, M, and its moment of *inertia* about its end, I. Calculate M & I for the rod, in terms of its length a and its density ρ (mass per unit length). Substitute the derived expressions into the **equation** of motion.

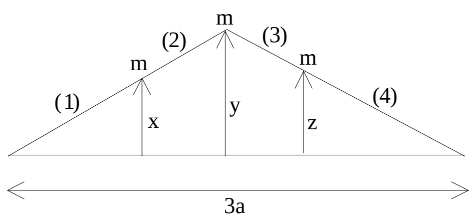
A: f = 1, and take *generalised* co-ordinate θ (the angle of the rod from the vertical). The K.E. is $T = \frac{1}{2}I\dot{\theta}^2$, where I is the *moment* of inertia about O. The P.E. is $V = Mgh = -Mg\frac{1}{2}a\cos\theta = -\frac{1}{2}Mga\cos\theta$. Therefore, we have the following: $L = T-V = \frac{1}{2}I\dot{\theta}^2+\frac{1}{2}Mga\cos\theta$. *Lagrange's* equation is $\frac{d}{dt}(\partial L/\partial\dot{\theta})-\frac{\partial L}{\partial\theta}$, which gives $I\ddot{\theta}+\frac{1}{2}Mga\sin\theta = 0$. Now $I = \Sigma mr^2 = \int_0^a \rho r^2 dr = \rho[\frac{r^3}{3}]_0^a = \rho\frac{a^3}{3}$. Also, $M = \rho a$. Therefore, $I = \frac{1}{3}Ma^2$. *Substituting* for I, Lagrange's equation **becomes** $\ddot{\theta}+\frac{3}{2}\frac{g}{a}\sin\theta = 0$.



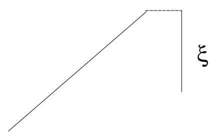

Q: Consider the rod in the *preceding* question. Show that the period of small oscillations about the <u>vertical</u> through O is given by $2\pi\sqrt{(2a/3g)}$. A: From the above, the *equation* of motion is $\ddot{\theta}+\frac{3}{2}g/_a\sin\theta = 0$. For small θ, $\sin\theta \approx \theta$, so the equation *becomes* $\ddot{\theta}+\frac{3}{2}\frac{g}{a}\theta = 0$. This is the equation for *SHM* with angular frequency $\omega = \sqrt{(\frac{3g}{2a})}$. The *period* is $\frac{2\pi}{\omega} = 2\pi\sqrt{(\frac{2a}{3g})}$.

Q: 3 particles of equal mass m are attached at *equal* intervals along a string of stiffness k and natural length 4*l*, which has been extended and attached to two points 4a apart. The masses are constrained to move **transversely** on a smooth surface. Calculate the natural frequencies of the system, and find the normal modes. Describe the amplitude and phase of the particle's displacements in each mode.

A: f = 3. (*Take generalised* co-ordinates x, y and z). The K.E. is $T = \frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2+\frac{1}{2}m\dot{z}^2$. P.E.: Take the P.E. correct up to *second* order (as in exercise 2 in lecture notes). The P.E. of one **section** of string is given by $\frac{1}{2}K\xi^2$, where $K = \frac{k(a-l)}{a}$; k is the spring stiffness; and ξ is the *transverse displacement* difference. Therefore, $V = \frac{1}{2}K(x^2+(x-y)^2+(y-z)^2+z^2\} = K(x^2-xy+y^2-yz+z^2)$.

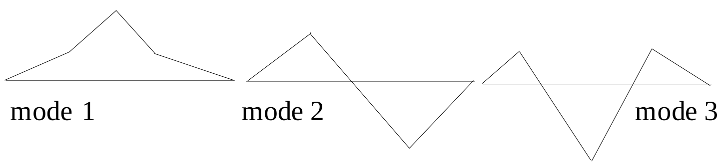



| *Section* | *Displacement* | *PE* |
|---|---|---|
| 1 | x | $\frac{1}{2}Kx^2$ |
| 2 | \|x-y\| | $\frac{1}{2}K(x-y)^2$ |
| 3 | \|y-z\| | $\frac{1}{2}K(y-z)^2$ |
| 4 | z | $\frac{1}{2}Kz^2$ |



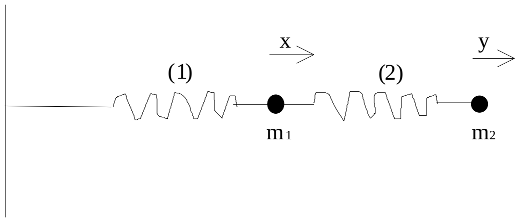

For **eigenvalue** analysis, rewrite T & V in matrix form: $T = \frac{1}{2}m(\dot{x}\ \dot{y}\ \dot{z})\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix}$, so $T_0 = \begin{pmatrix}m&0&0\\0&m&0\\0&0&m\end{pmatrix}$. And $V = \frac{1}{2}K(x\ y\ z)\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$, so $V_0 = \begin{pmatrix}2k&-k&0\\-k&2k&-k\\0&-k&2k\end{pmatrix}$. Solving a *general* eigenvalue problem, we obtain eigenvalues $\omega^2 = \frac{k}{m}(2-\sqrt{2})$; $\omega^2 = \frac{2k}{m}$; and $\omega^2 = \frac{k}{m}(2+\sqrt{2})$.

Substitute back to obtain the *normal* modes: Mode 1: A:B:C = 1:√2:1, meaning that all particles are in phase. Mode 2: A:B:C = 1:0:-1, meaning that the middle man is stationary, but other masses oscillate with **equal** amplitude, out of phase. Mode 3: A:B:C = 1:-√2:1, meaning that the outer masses have equal amplitude and are in phase; and that the middle mass has *opposite* phase and larger amplitude.

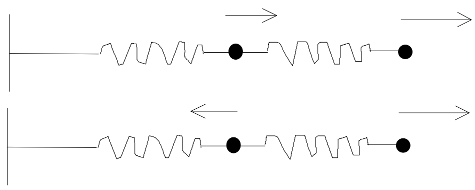



Q: 2 particles of masses $m_1$ and $m_2$ are attached to 2 identical springs, such that one spring connects $m_1$ to a fixed point, and the other connects mass $m_1$ to mass $m_2$. The masses are constrained to move *longitudinally* on a smooth surface. Calculate the natural frequencies of the system; and find the **normal modes** and interpret them.

A: d.o.f. $f = 2$, and take *generalised* co-ordinates x & y from the **unextended** equilibrium position. Denote the *spring constant* as k. The K.E. is $T = \frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{y}^2$. The P.E. is $V = \frac{1}{2}k(x^2+(x-y)^2) = \frac{1}{2}k(2x^2-2xy+y^2)$. For *normal* mode analysis, write $T = \frac{1}{2}(\dot{x}\ \dot{y})\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, so that $T_0 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$. Further, $V = \frac{1}{2}k(x\ y)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so that $V_0 = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$. So the *generalised* eigenproblem is $\begin{pmatrix} 2k-m_1\omega^2 & -k \\ -k & k-m_2\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where A and B are the *amplitudes of vibration* of x and y.




For **non** trivial solution, $\begin{vmatrix} 2k-m_1\omega^2 & -k \\ -k & k-m_2\omega^2 \end{vmatrix} = 0$. Therefore, $(2k-m_1\omega^2)(k-m_2\omega^2)-k^2 = 0$; ...; $\omega^2 = \frac{k}{2m_1m_2}[(m_1+2m_2) \pm \sqrt{(m_1^2+4m_2^2)}]$. Substitute for $\omega^2$ to *obtain* the normal modes Mode 1: $\frac{B}{A} = 1-\frac{m_1}{2m_2} + \sqrt{((\frac{m_1}{2m_2})^2+1)}$. Mode 2: $\frac{B}{A} = 1-\frac{m_1}{2m_2}-\sqrt{((\frac{m_1}{2m_2})^2+1)}$. The *relative amplitude* of $m_1$ and $m_2$ depends on the **magnitude** of the masses.




For *mode 1*, since $\frac{m_1}{2m_2} < \sqrt{((\frac{m_1}{2m_2})^2+1)}$, then $\frac{B}{A} > 0$, so masses oscillate *in phase*. For mode 2, since $\sqrt{((\frac{m_1}{2m_2})^2+1)} > 1$, then $\frac{B}{A} < 0$, so the masses oscillate *out of phase*. **Examples**: (a) $m_1 = m_2 = m$, say. Then $\omega^2 = \frac{k}{2m}(3\pm\sqrt{5})$; $\frac{B}{A} = \frac{1}{2}(1\pm\sqrt{5})$. (b) $m_1 = 2m_2$: $\omega^2 = \frac{2k}{m_1}(2\mp\sqrt{2})$; $\frac{B}{A} = \pm\sqrt{2}$. (c) For mode 2 *(the higher frequency mode)*, $m_1 < \frac{3}{2}m_2 \Rightarrow |B| < |A|$; $m_1 = \frac{3}{2}m_2 \Rightarrow |B| = |A|$; and $m_1 > \frac{3}{2}m_2 \Rightarrow |B| > |A|$.

➢ 16th December 1999

# Tutorial

Q: Two particles approach each other with *speeds* 0.9c in some **inertial** frame. What is their relative speed? A: Relative case: $(\frac{u_2-u_1}{(1-(u_1u_2/c^2))})$. (The relative speed of 2 particles moving towards each other along the *x-axis)*. So the relative speed is $\frac{0.9c-(-0.9c)}{(1-((0.9c)(-0.9c)/c^2))} = \frac{1.9c}{1+0.81} = \frac{1.8c}{1.81} = 0.9945c$.

Q: A particle is created in the laboratory, and has a *mean life* of $10^{-5}$s in a frame in which it is at rest. If it has speed $2.7\times10^8\text{ms}^{-1}$ relative to the laboratory, (a) what is the life time measured by an observer in the *laboratory*; (b) what distance does it travel, on average, before disintegrating; and (c) what would be the distance calculated if **relativistic** effects were ignored?

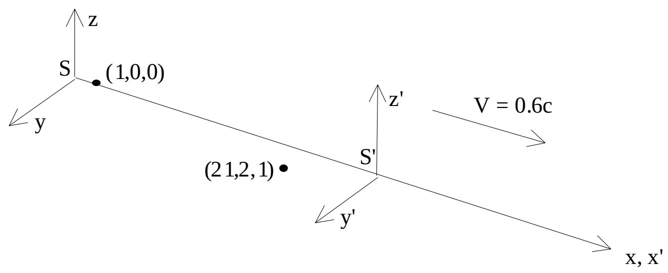
A: (a) Using the formula for *time dilation*, $\Delta t' = \frac{\Delta t}{\sqrt{(1-(v^2/c^2))}} = 1\times10^{-5}/\sqrt{(1-(2.7\times10^8)^2/c^2)} = ... = 2.294\times10^{-5}$s. This is the mean life time in the *laboratory*. (b) In the laboratory frame, $u = 2.7\times10^8\text{ms}^{-1}$ and $t = 2.294\times10^{-5}$s, so the *distance travelled* is $2.7\times2.294\times10^3$m = 6193.8m. (c) Ignoring the time *dilation*, $u = 2.7\times108\text{ms}^{-1}$ and $t = 10^{-5}$s, so the distance travelled is $2.7\times10^8\times10^{-5}$ = 2700m.

# Assignment 4

Q: Calculate the *speed* with which a car must move so that its length may be shortened to **half** its proper length. A: Let the *proper* length of the car be *l*. According to the Lorentz-Fitzgerald contraction, the **stationary** observer would measure the length to be *l'*, where $l' = l\sqrt{(1-(\frac{v^2}{c^2}))}$. We want $\sqrt{(1-(\frac{v^2}{c^2}))}$ to be a half, so $1-\frac{v^2}{c^2} = \frac{1}{4}$; $\frac{v^2}{c^2} = \frac{3}{4}$; $v^2 = \frac{3c^2}{4}$; $v = \frac{c\sqrt{3}}{2}\text{ ms}^{-1}$.

Q: A car moves with a speed of 108 km per hour. If the *length* of the car is 2.4m, calculate the decrease in its length as noted by a **stationary** observer. A: The car is travelling at $30ms^{-1}$. We can calculate the length of the car as measured by the *observer* in S' using $l' = l_0\sqrt{(1-(^{v^2}/_{c^2}))}$, so $l' = 2.4\sqrt{1-(30^2/(3\times10_8)^2)}$; $l' = 2.4\sqrt{(1-(1\times10^{-14}))}$. The observed *decrease* in length is the proper length - the observed length = $2.4-2.4\sqrt{(1-(1\times10^{-14}))} = 2.4(1-\sqrt{(1-(1\times10^{-14}))})$. This is a very small length, not *noticeable* at all.

Q: 2 events occur *simultaneously* at points (21,2,1) and (1,0,0) of a frame S. Determine the time interval between them in a frame S' moving with a speed 0.6c **relative** to S, and along the <u>positive</u> direction of their common x-axis. A
Let the *event* at (21,2,1) be $E_1$, with co-ordinates $(x_1,y_1,z_1,t_1) = (21,2,1,t_1)$ in S; and $(x_1',y_1',z'_1,t'_1) = (x_1',2,1,t_1')$ in S'. Let the event at (1,0,0) be $E_2$, with co-ordinates $(x_2,y_2,z_2,t_2) = (1,0,0,t_2)$ in S; and $(x_2',y_2',z_2',t_2') = (x_2',0,0,t_2')$ in S'.




According to the *Lorentz* transformation, $t_1' = {}^{t_1-(Vx_1/c^2)}/_{\sqrt{(1-(v^2/c^2))}}$, and $t_2' = {}^{t_2-(Vx_2/c^2)}/_{\sqrt{1-(v^2/c^2))}}$. Subtracting the *above* two equations gives the time interval between the two events in **frame** S', which is exactly what we want. So $t_2'-t_1' = {}^{t_2-t_1+(V/c^2)(x_2-x_1)}/_{\sqrt{(1-(v^2/c^2))}}$; $t_2'-t_1' = {}^{t_2-t_1\ +\ (0.6c/c^2)(21-2)}/_{\sqrt{(1-((0.6c)^2/c^2))}}$. But we *know* that $t_2 = t_1$, (they are simultaneous events in frame S), so we have the following: $t_2'-t_1' = \frac{\frac{0.6c}{c^2}(21-1)}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = \frac{\frac{0.6}{c}(20)}{\sqrt{1-\frac{0.36c^2}{c^2}}} = \frac{12/c}{\sqrt{1-0.36}} = \frac{12}{c\sqrt{0.64}} = \frac{12}{0.8c} = \frac{15}{c} = \frac{15}{3\times10^8} = 5\times10^{-8}$.

# Exam Paper: January 2000

## SECTION 1 (Compulsory)

(1) (a) (i) A particle moves under the influence of a central force. Write down the vector equation of motion. Hence write down the value of $\boldsymbol{r} \times \ddot{\boldsymbol{r}}$ and deduce that the motion lies in a plane and that the angular momentum per unit mass h is a constant. **[5 marks]**

(ii) Derive the radial and transverse components of velocity $\dot{\boldsymbol{r}}$ and acceleration $\ddot{\boldsymbol{r}}$ in plane polar coordinates (r, θ). **[4 marks]**

(b) A square lamina of side length a has uniform density σ per unit area and total mass M. Calculate the components of the inertia tensor relative to a set of axes Oxyz fixed in the lamina such that O is at the bottom left corner and Ox and Oy are along two sides of the square. Use the parallel axis theorem to obtain the moment of inertia about the z-axis relative to a set of axes with origin at the centre of the square. **[8 marks]**

## SECTION 2 (Answer 2 out of 4 questions)

(2) (a) A particle moves under the influence of a central force. By referring to the equation of motion in plane polar coordinates (r, θ) as derived in Question 1 and introducing reciprocal place polar coordinates (u, θ), derive expressions for $\dot{r}$ and $\ddot{r}$ in terms of h, u, $du/d\theta$ and $d^2u/d\theta^2$. Hence derive the equation of orbit. **[6 marks]**

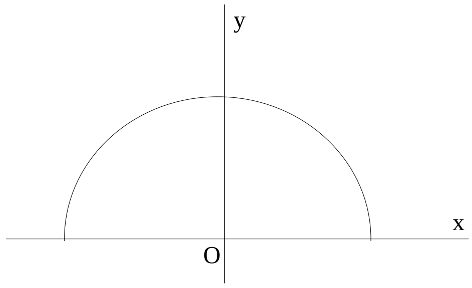
(b) Consider an inverse square law with force per unit mass given by $P = -\mu/r^2$, where μ is a positive constant. Write down the equation of orbit for this force. **[1 mark]**

(c) Find the angular momentum per unit mass and the constant speed v for which the particle can describe the circle r = a. **[3 marks]**

(d) By obtaining a general solution to the equation of orbit for the inverse square law of force, deduce that the orbit is a conic section with focus at the centre of force.

**[5 marks]**

(3) A lamina is in the shape of a semi-circular disc of radius a and has constant surface density $\sigma$ per unit area.




(a) Using place polar coordinates, calculate the components of the inertia tensor relative to a set of axes Oxyz aligned as shown in the above diagram. **[7 marks]**

(b) Calculate the angular momentum and kinetic energy of the lamina if it rotates with angular speed $\Omega$ about (i) the x-axis, (ii) the z-axis, (iii) the line in the xy-plane through the origin at an angle $^{\pi}/_{4}$ to Ox, (iv) the line perpendicular to the lamina through its centre of mass G. **[8 marks]**

(4) Two particles of equal mass m are attached at the points of trisection of an elastic spring of stiffness k and natural length $3l$ which is attached to two points 3a apart. The masses are constrained to move longitudally on a smooth surface. By a Lagrangian method, calculate the natural frequencies of the system and find the norm modes. Describe the amplitude and phase of the particles' displacements in each mode. **[15 marks]**

(5) (a) State the two basic postulates for Einstein's Special Theory of Relativity. **[1 mark]**

(b) Consider two inertial frames S and S' which are in standard configuration with relative velocity V along the x-axis. State the Galilean and Lorentz transformations relating the description of an event at x, y, z, t in frame S with its description x', y', z', t' in frame S'. Discuss and compare the two transformations. **[4 marks]**

(c) Derive an expression for the Lorentz-Fitzgerald contraction and an expression for time dilation. **[6 marks]**

(d) A car moves with a speed of 108 km per hour. If the length of the car is 3.4m, calculate the relative decrease in its length as noted by a stationary observer. (Take the speed of light to be $3\times10^{8}ms^{-1}$). **[2 marks]**

(e) Calculate the speed with which the car must move in order that its length be shortened to half its proper length. **[2 marks]**

<u>(Questions done: 1, 3, 4)</u>