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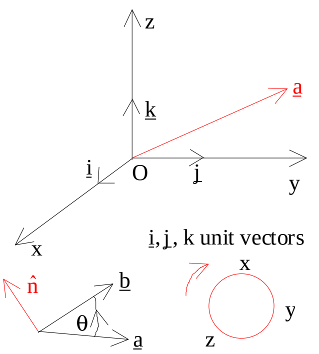
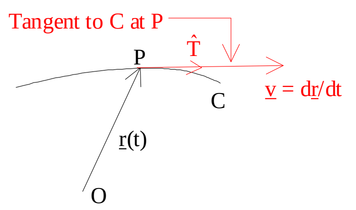
Section 1: Vector Calculus



1.1. Revision of vector algebra. (i) $\underline{a} = a_x\underline{i}+a_y\underline{j}+a_z\underline{k}$. $|\underline{a}|$ = *magnitude* of $\underline{a}$ = a = $+\sqrt{(a_x^2+a_y^2+a_z^2)}$ (---(1)). $\hat{\underline{a}}$ = unit vector in the direction of $\underline{a} = \underline{a}/a$ (---(2)). (ii) If $\underline{b} = b_x\underline{i}+b_y\underline{j}+b_z\underline{k}$, then $\underline{a}.\underline{b} = ab\cos\theta = a_xb_x+a_yb_y+a_zb_z = \underline{b}.\underline{a}$ (---(3)). (iii) $\underline{a}\times\underline{b} = ab\sin(\theta)\hat{n}$ = the *determinant* shown on the **right** = $\underline{i}(a_yb_z-a_zb_y)-\underline{j}(a_xb_z-a_zb^x)+\underline{k}(a_xb_y-a_yb_x)$ $[+\underline{j}(a_zb_x-a_xb_z)]$ = $-\underline{b}\times\underline{a}$ (---(4)). (iv) $\underline{a}.(\underline{b}\times\underline{c})$ is the *Scalar Triple Product* = the *determinant* shown on the **left** (---(5)). (v) $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}.\underline{c})\underline{b}-(\underline{a}.\underline{b})\underline{c}$ is the *Vector Triple Product*.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$



1.2 Vector function of a single real variable. $\frac{d\boldsymbol{r}}{dt} = \lim_{\delta t\to 0}\left\{\frac{\boldsymbol{r}(t+\delta t)-\boldsymbol{r}(t)}{\delta t}\right\}$ (---(1)). $\underline{v}$ = velocity. $|\underline{v}|$ = speed = v. $\underline{V} = V\hat{T}$ ($\hat{T}$ = the *unit tangent vector* to C at P, in the **direction** of motion). **Recall**: $^{d}/_{dt}(\phi(t)\underline{r}(t))$ ($\phi(t)$ is a **scalar** function of t; $\underline{r}(t)$ is a **vector** function of t) $=(\frac{d\phi(t)}{dt})\boldsymbol{r}(t)+\phi(t)(\frac{d(\boldsymbol{r}(t))}{dt})$ (---(2a)). We can deduce also the following: $\frac{d}{dt}(\boldsymbol{r}_1.\boldsymbol{r}_2)=(\frac{d\boldsymbol{r}_1}{dt}).\boldsymbol{r}_2+\boldsymbol{r}_1.\frac{d}{dt}(\boldsymbol{r}_2)$ (---(2b)); and $\frac{d}{dt}(\boldsymbol{r}_1\times\boldsymbol{r}_2)=(\frac{d\boldsymbol{r}_1}{dt})\times\boldsymbol{r}_2+\boldsymbol{r}_1\times(\frac{d\boldsymbol{r}_2}{dt})$ (---(2c)).

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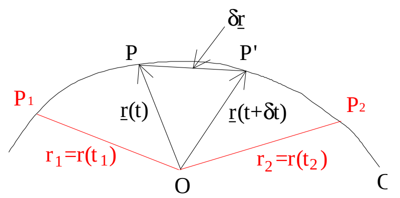
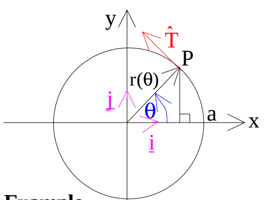


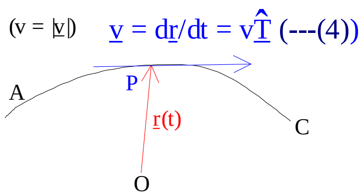
Arc Length. Define $\overrightarrow{PP^{-1}} = \underline{r}(t+\delta t)-\underline{r}(t) = \delta\underline{r}$. So $\Sigma PP^{-1} = \Sigma\,|d\underline{r}|$ Define Arc Length, L, as the *distance* from P_1 to P_2 along C = $\int_{P_1}^{P_2}|d\underline{r}|$ But $|d\underline{r}| = |(^{dx}/_{dt}, {}^{dy}/_{dt}, {}^{dz}/_{dt})dt|$ (*where* $\underline{r} = x\underline{i}+y\underline{j}+z\underline{k} = (x,y,z)$) = $\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt \Rightarrow L = \int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$. (---(3)).



Example

Example: r = $a\cos\theta\underline{i}+a\sin\theta\underline{j}+0\underline{k}$ for $0\le\theta\le 2\pi$. So $L = \int^{2\pi}_0 d\underline{r}$ = (by (3)) = $\int^{2\pi}_0\sqrt{[(-a\sin\theta)^2+(a\cos\theta)^2+0^2]}d\theta = \int^{2\pi}_0 a\sqrt{(\sin^2\theta+\cos^2\theta)}d\theta = \int^{2\pi}_0 a\,d\theta = a[\theta]^{2\pi}_0 = 2\pi a$. **Also,** $d\underline{r}/d\theta = (-a\sin\theta, a\cos\theta, 0) \Rightarrow \hat{\mathbf{T}}$ = (unit tangent vector) = $\frac{d\boldsymbol{r}/d\theta}{|d\boldsymbol{r}/d\theta|} = \frac{(-a\sin\theta,\, a\cos\theta,\, 0)}{\sqrt{(-a\sin\theta)^2+(a\cos\theta)^2+0^2}} = {}^1/_a(-a\sin\theta, a\cos\theta, 0) = (-\sin\theta, \cos\theta, 0)$.

Curvature and Torsion of a Curve

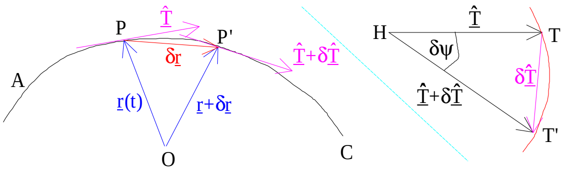


Let *v = 1*, so that t can be **identified** with the distance s *along the curve c*, from some fixed point A. This implies that $\underline{v} = d\underline{r}/ds = v\hat{\mathbf{T}}$ = (*because* v = 1) = $\hat{\mathbf{T}} \Rightarrow \hat{\mathbf{T}} = d\underline{r}/ds$ (---(5)). Since $\hat{\mathbf{T}}$ is of unit length, we have $\hat{\mathbf{T}}.\hat{\mathbf{T}} = 1$. *Differentiating*, $^{d}/_{ds}(\hat{\mathbf{T}}.\hat{\mathbf{T}}) = {}^{d}/_{ds}(1) = 0$. But $^{d}/_{ds}(\hat{\mathbf{T}}.\hat{\mathbf{T}}) = (d\hat{\mathbf{T}}/ds).\hat{\mathbf{T}}+\hat{\mathbf{T}}.(d\hat{\mathbf{T}}/ds) = 2((d\hat{\mathbf{T}}/ds).\hat{\mathbf{T}})$; so $(d\hat{\mathbf{T}}/ds).\hat{\mathbf{T}} = 0 \Rightarrow d\hat{\mathbf{T}}/ds$ is $\perp^r$ to $\hat{\mathbf{T}}$, as $\hat{\mathbf{T}} \ne \underline{0}$ and $(d\hat{\mathbf{T}}/ds) \ne \underline{0}$ by *assumption*. So $d\hat{\mathbf{T}}/ds$ is **normal** to the curve at P; and it follows that $d\hat{\mathbf{T}}/ds = \kappa\hat{\mathbf{N}}$ (---(6)).

In the above, $\hat{\mathbf{N}}$ is the **unit** vector in the normal direction, and κ is a *non-negative scalar*. The straight line through P in the **direction** of $\hat{\mathbf{N}}$ is called the principal normal to the curve at P, and $\hat{\mathbf{N}}$ is called the unit principal normal vector at P. The *plane* containing $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ at P is called the osculating plane, or the plane of curvature at P. (6) $\Rightarrow \kappa = |d\hat{\mathbf{T}}/ds|$ (---(7)), where κ ("*kappa*") is called the **curvature** of C at P. The **radius of curvature**, ρ, is defined to be $^1/_\kappa$, so that $\rho = {}^1/_\kappa$. ($\kappa \ne 0$) (---(8)).

Geometrical Meaning of κ

$\hat{T}$ and $\hat{T}+\delta\hat{T}$ give the *unit tangent vectors* at P and P' respectively. Since $|\hat{T}| = 1$ and $|\hat{T}+\delta\hat{T}| = 1$, then HT = HT' = 1. Therefore, $\delta\psi = T\hat{H}T'$ = arc TT' of the *unit circle* centred at H. (ψ is measured in *radians*). So $\kappa = |\frac{d\hat{T}}{ds}| = \lim_{\delta s\to 0} |\frac{\delta\hat{T}}{\delta s}| = \lim_{\delta s\to 0}|\frac{\delta\hat{T}}{\delta\psi}\times\frac{\delta\psi}{\delta s}| = \lim_{\delta s\to 0}(\frac{TT'}{\delta\psi}\times\frac{\delta\psi}{\delta s}) = \frac{d\psi}{ds}$ **as** $\lim_{\delta s\to 0}({}^{TT'}/_{\delta\psi}) = 1$. So $\kappa = {}^{\delta\psi}/_{\delta s}$.



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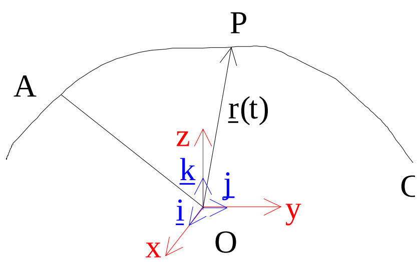
Since $\hat{T}$ and $\hat{N}$ are $\perp^r$ unit vectors, it follows that $\hat{B} = \hat{T}\times\hat{N}$ (---(9)) is also a unit vector. $\hat{B}$ is called the unit binormal vector at P. So $\hat{T}$, $\hat{N}$ and $\hat{B}$ form a *right-handed system* of mutually **perpendicular** unit vectors. Since $\hat{B}$ is a unit vector of *constant* length, then $(d\hat{B}/ds)$ is $\perp^r$ to $\hat{B}$. Also, *differentiating* the equation $\hat{T}.\hat{B} = 0 \Rightarrow (d\hat{T}/ds).\hat{B} + \hat{T}.(d\hat{B}/ds) = 0$ (---(10)).

(6) & (10) $\Rightarrow \kappa\hat{N}.\hat{B} + \hat{T}.(d\hat{B}/ds) = 0$. (Because $\hat{N}.\hat{B} = 0$) (---(11)). And $\hat{N}.\hat{B} = 0$ (---(12)). (12) in (11) $\Rightarrow$ $(d\hat{B}/ds)$ is $\perp^r$ to $\hat{T}$. But $(d\hat{B}/ds)$ is also $\perp^r$ to $\hat{B}$; so $(d\hat{B}/ds)$ must be *parallel* to $\hat{N}$. We write $(d\hat{B}/ds) = -\tau\hat{N}$ (---(13)). (**Note**: the *minus* sign is by convention, and τ can be -ve, +ve or zero). τ is known as the torsion of the curve at P.

The torsion is the *arc-rate of rotation* of the plane of curvature, since $\hat{B}$ is always $\perp^r$ to this plane. The **radius of torsion**, σ, is *defined* by $\sigma = {}^{1}/_{|\tau|}$ for $\tau \neq 0$. Using (6) & (13), we can *find* $(d\hat{N}/ds)$. Now $\hat{N} = \hat{B}\times\hat{T} \Rightarrow (d\hat{N}/ds) = {}^{d}/_{ds}(\hat{B}\times\hat{T}) = (d\hat{B}/ds)\times\hat{T} + \hat{B}\times(d\hat{T}/ds)$ = (by (6) and (13)) $= (-\tau\hat{N})\times\hat{T}+\hat{B}\times\kappa\hat{N} = (-\tau)(-\hat{B})+\kappa(-\hat{T})$. So $(d\hat{N}/ds) = -\kappa\hat{T}+\tau\hat{B}$ (---(14)). (6), (13) and (14) are called **Frenes** formulae: $(d\hat{T}/ds) = \kappa\hat{N}$; $(d\hat{B}/ds) = -\tau\hat{N}$; and $(d\hat{N}/ds) = -\kappa\hat{T}+\tau\hat{B}$ (---(set 15)).

Formulae for Computing κ & τ

Let C be given by $\underline{r} = \underline{r}(t) = x(t)\underline{i}+y(t)\underline{j}+z(t)\underline{k}$ (---(16)), where t denotes a parameter which is *not necessarily* the distance s along the curve from some **fixed** point A. Now (16) $\Rightarrow \dot{\mathbf{r}} = d\underline{r}/dt = (d\underline{r}/ds)\times(ds/dt)$. (**Note**: $d\underline{r}/ds$ is $\hat{T}$ from (5)). So $\dot{\mathbf{r}} = \hat{T}\dot{s}$ (---(17)). Now (17) $\Rightarrow \ddot{\mathbf{r}} = {}^{d}/_{dt}(\hat{T}\dot{s}) = (d\hat{T}/dt)\dot{s} + \hat{T}\ddot{s} = ((d\hat{T}/ds)(ds/dt))\dot{s} + \hat{T}\ddot{s}$ = (6) $= \kappa\hat{N}\dot{s}^2 + \hat{T}\ddot{s}$. So we have $\ddot{\mathbf{r}} = \ddot{s}\hat{T} + \kappa\dot{s}^2\hat{N}$ (---(18)). (17) & (18) $\Rightarrow \dot{\mathbf{r}}\times\ddot{\mathbf{r}} = (\hat{T}\dot{s})\times(\ddot{s}\hat{T}+\kappa\dot{s}^2\hat{N}) = \dot{s}\ddot{s}(\hat{T}\times\hat{T}) + \kappa\dot{s}^3(\hat{T}\times\hat{N})$. *Because* $\hat{T}\times\hat{T} = 0$, and because $\hat{T}\times\hat{N} = \hat{B}$, we have $\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \kappa\dot{s}^3\hat{B}$ (---(19)). **Now** (19) $\Rightarrow \kappa = (|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|)/\dot{s}^3$ (---(20)). ($|\hat{B}| = 1$). But (17) $\Rightarrow \dot{s} = |\dot{\mathbf{r}}|$ (---(21)). ($|\hat{T}| = 1$). (21) in (20) $\Rightarrow \kappa = (|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|)/(|\dot{\mathbf{r}}|^3)$. (---(22)).



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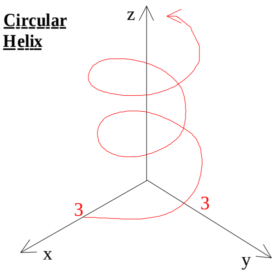
To find the *torsion* τ, consider $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}.\dddot{\mathbf{r}}$. From (18), $\ddot{\mathbf{r}} = \ddot{s}\hat{T} + \kappa\dot{s}^2\hat{N}$. *Differentiating*, we obtain $\dddot{\mathbf{r}} = {}^{d}/_{dt}(\ddot{s}\hat{T}) + {}^{d}/_{dt}(\kappa\dot{s}^2\hat{N})$; $\dddot{\mathbf{r}} = \dddot{s}\hat{T} + \ddot{s}(d\hat{T}/dt) + {}^{d}/_{dt}(\kappa\dot{s}^2)\hat{N} + \kappa\dot{s}^2(d\hat{N}/dt)$; $\dddot{\mathbf{r}} = \dddot{s}\hat{T} + \ddot{s}(d\hat{T}/ds)\dot{s} + d/dt(\kappa\dot{s}^2)\hat{N} + \kappa\dot{s}^2(d\hat{N}/ds)\dot{s}$ = (15) $= \dddot{s}\hat{T} + \ddot{s}\kappa\hat{N}\dot{s} + {}^{d}/_{dt}(\kappa\dot{s}^2)\hat{N} + \kappa\dot{s}^3(-\kappa\hat{T}+\tau\hat{B})$; $\dddot{\mathbf{r}} = (\dddot{s}-\kappa^2\dot{s}^3)\hat{T} + (\ddot{s}\kappa\dot{s} + {}^{d}/_{dt}(\kappa\dot{s}^2))\hat{N} + (\kappa\dot{s}^3\tau)\hat{B}$ (---(23)). Now (19) $\Rightarrow \dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \kappa\dot{s}^3\hat{B}$, and as $\hat{B}.\hat{T} = 0 = \hat{B}.\hat{N}$, and $\hat{B}.\hat{B} = 1$, then (19) & (23) $\Rightarrow (\dot{\mathbf{r}}\times\ddot{\mathbf{r}}).\dddot{\mathbf{r}} = (\kappa\dot{s}^3)^2\tau$ (---(24)). Finally, (20) & (24) $\Rightarrow \tau = \frac{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}).\dddot{\mathbf{r}}}{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}^2)^2}$ (---(25)).

Example: Sketch the *space curve* $\underline{r}(t) = x\underline{i}+y\underline{j}+z\underline{k}$, where $x = 3\cos(t)$, $y = 3\sin(t)$, and $z = 4t$ (---(1)), and find (a) $\hat{T}$, (b) $\hat{N}$, (c) κ, ρ, (d) $\hat{B}$, (e) τ, σ. A: (1) $\Rightarrow x^2+y^2 = 3^2 \Rightarrow$ curve lies on the *surface of the cylinder* $x^2+y^2 = 9$ (---(2)). (a) $\underline{r} = 3\cos(t)\underline{i}+3\sin(t)\underline{j}+4t\underline{k}$. So $\dot{\mathbf{r}} = -3\sin(t)\underline{i}+3\cos(t)\underline{j}+4\underline{k}$; $^{ds}/_{dt} = |\dot{\mathbf{r}}| = \sqrt{((-3\sin(t))^2+(3\cos(t))^2+4^2)} = \sqrt{(3^2+4^2)} = 5$ (---(3)). Therefore, $\hat{T} = {}^{d\mathbf{r}}/_{ds} = {}^{d\mathbf{r}/dt}/_{ds/dt} = -{}^3/_5\sin(t)\underline{i}+{}^3/_5\cos(t)\underline{j}+{}^4/_5\underline{k}$ (---(4)).

Circular Helix



Now $^d/_{dt}(\hat{T})$ = (4) = $-{}^3/_5\cos(t)\underline{i}-{}^3/_5\sin(t)\underline{j}+\underline{0}$. As $(d\hat{T}/ds) = (d\hat{T}/dt)(dt/ds) = -{}^3/_{25}\cos(t)\underline{i}-{}^3/_{25}\sin(t)\underline{j}$ (---(5)), then *since* $(d\hat{T}/ds) = \kappa\hat{N}$, we have $\kappa = |(d\hat{T}/ds)|$ $(\kappa \geq 0)$ = (5) = $|-{}^3/_{25}\cos(t)\underline{i}-{}^3/_{25}\sin(t)\underline{j}| = {}^3/_{25}$. **Now** $\rho = {}^1/_\kappa$ $(\kappa \neq 0) = {}^{25}/_3$ (---(6)). So (5) & (6) $\Rightarrow \hat{N} = {}^1/_\kappa(d\hat{T}/ds) = ({}^{25}/_3)(-{}^3/_{25}\cos(t)\underline{i}-{}^3/_{25}\sin(t)\underline{j}) = -(\cos(t)i+\sin(t)j)$ (---(7)).

Now $\hat{B} = \hat{T}\times\hat{N}$ = the *determinant* shown on the right = ${}^4/_5\sin(t)\underline{i}-{}^4/_5\cos(t)\underline{j}+{}^3/_5\underline{k}$ (---(8)). **Further,** (8) $\Rightarrow (d\hat{B}/ds) = (d\hat{B}/dt)(dt/ds) = {}^4/_{25}\cos(t)\underline{i}+{}^4/_{25}\sin(t)\underline{j}+\underline{0}$ (---(9)). **Since** $(d\hat{B}/ds) = -\tau\hat{N}$, then (7) & (9) $\Rightarrow {}^4/_{25}\cos(t)\underline{i}+{}^4/_{25}\sin(t)\underline{j} = -\tau(-\cos(t)\underline{i}-\sin(t)\underline{j})$, so $\tau = {}^4/_{25}$ (*Note*: > 0). <u>And</u> $\sigma = {}^1/_{|\tau|} = {}^{25}/_4$.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

We now use *formulae* (22) & (25) to get κ and τ. Now we **know** that $\underline{r} = 3\cos(t)\underline{i}+3\sin(t)\underline{j}+4t\underline{k}$; so that $\dot{\mathbf{r}} = -3\sin(t)\underline{i}+3\cos(t)\underline{j}+4\underline{k}$ (which $\Rightarrow |\dot{\mathbf{r}}| = 5$); $\ddot{\mathbf{r}} = -3\cos(t)\underline{i}-3\sin(t)\underline{j}$; *and* $\dddot{\mathbf{r}} = 3\sin(t)\underline{i}-3\cos(t)\underline{j}$. So $\dot{\mathbf{r}}\times\ddot{\mathbf{r}}$ = the determinant shown on the *left* = $12\sin(t)\underline{i}-12\cos(t)\underline{j}+9\underline{k}$. So $\kappa = {}^{\sqrt{(12^2\sin^2(t)+12^2\cos^2(t)+9^2)}}/_{5^3} = {}^{\sqrt{(12^2+9^2)}}/_{5^3} = {}^{3\sqrt{(4^2+3^2)}}/_{5^3} = {}^3/_{25}$. **Now** $(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}).\dddot{\mathbf{r}} = (12\sin(t))(3\sin(t))+(-12\cos(t))(-2\cos(t))+9\times0 = 36$. And $|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = 3\times5$, so $\tau = {}^{36}/_{15^2} = {}^4/_{25}$. Both are *verified*.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix}$$

<u>Notation Note:</u> From now on, I will remove the "*hats*" from the tops of letters and use **BOLD** type instead. As an example, $\hat{T}$ will now be *denoted* simply as **T**.

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<u>Tutorial</u>

<u>General Notes.</u> Find the magnitude: If $\underline{a} = 3\underline{i}+4\underline{k}$, then $|\underline{a}| = \sqrt{(3^2+4^2)} = 5$; the *unit vector* in the direction of $\underline{a}$ is $\underline{a}/|\underline{a}|$. (= $\hat{\underline{a}}$). *Remember* that $\underline{a}.\underline{b}$ = multiply **components**, or use $ab\cos\theta$. <u>Orthogonal:</u> $\cos\theta = 0$. To calculate $\underline{a}\times\underline{b}$, use the *determinant*. Be careful to use the correct formula in **computing** e.g. $\underline{a}\times(\underline{a}\times\underline{b})$. The *unit normal* to the plane containing $\underline{a}$ and $\underline{b}$ is $\underline{a}\times\underline{b}/|\underline{a}\times\underline{b}|$. Be careful when **doing** e.g. $^d/_{dt}(\underline{a}\times\underline{b})$ — *think*!

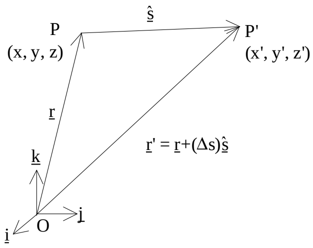
Q: For the **space** curve $\underline{r} = (t-{}^{t^3}/_3)\underline{i}+t^2\underline{j}+(t+{}^{t^3}/_3)\underline{k}$, find **T**, κ, **N**, **B** and τ. A: $\dot{\mathbf{r}} = (1-t^2)\underline{i}+2t\underline{j}+(1+t^2)\underline{k}$; and $\dot{s} = |\dot{\mathbf{r}}| = \sqrt{((1-t^2)^2+(2t)^2+(1+t^2)^2)} = \ldots = \sqrt{(2)}(1+t^2)$. So $\mathbf{T} = \dot{\mathbf{r}}/\dot{s} = {}^{(1-t^2)}/_{\sqrt{(2)}(1+t^2)}\underline{i}+{}^{\sqrt{(2)}t}/_{(1+t^2)}\underline{j}+{}^1/_{\sqrt{(2)}}\underline{k}$. *Now* $^{d\mathbf{T}}/_{dt}$ = (use *quotient* rule) = ... = $-{}^{2\sqrt{(2)}t}/_{(1+t^2)^2}\underline{i} + {}^{\sqrt{(2)}(1-t^2)}/_{(1+t^2)^2}\underline{j}$. Further, $^{d\mathbf{T}}/_{ds} = (d\mathbf{T}/dt)/\dot{s} = \ldots = [-2t\underline{i}+(1-t^2)\underline{j}]/(1+t^2)^3$. Now *since* $^{d\mathbf{T}}/_{ds} = \kappa\mathbf{N}$, then $\kappa = |{}^{d\mathbf{T}}/_{ds}| = {}^1/_{(1+t^2)^3}\sqrt{((-2t)^2+(1-t^2)^2)} = {}^1/_{(1+t^2)^2}$.

Now $\mathbf{N} = {}^1/_\kappa{}^{d\mathbf{T}}/_{ds} = \ldots = {}^1/_{(1+t^2)}(-2t\underline{i}+(1-t^2)\underline{j})$. **Now** use $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ (using the *determinant* method) to get $B = {}^1/_{\sqrt{(2)}(1+t^2)}[(t^2-1)\underline{i}-2t\underline{j}+(1+t^2)k]$. *Then* get $^{d\mathbf{B}}/_{dt}$ and $^{d\mathbf{B}}/_{ds}$, and because $^{d\mathbf{B}}/_{ds} = -\tau\mathbf{N}$, we can *obtain* τ *using* $\tau = {}^1/_{(1+t^2)^2}$. Note here that *curvature* = *torsion* for this <u>curve</u>.

1.3: Scalar and Vector Fields

If, to *each* point (x,y,z) of a domain D of space, an unique **scalar** is assigned, then a scalar field is defined in D. **Example**: *Temperature* in a room, T = T(x,y,z). If, to each point (x,y,z) of a domain D of space, an unique **vector** is assigned, then a vector field is defined in D. **Example**: The *velocity* of particles in a fluid, $\underline{v} = \underline{v}(x,y,z) = v_x\underline{i}+v_y\underline{j}+v_z\underline{k}$, where $v_\alpha = v_\alpha(x,y,z)$ and $\alpha = x,y,z$. We shall see that *scalar* fields give rise to *vector* fields and **vice-versa**.

1.4: The Directional Derivative



Let f(x,y,z) be a *scalar* field in some **domain** D. In the diagram, $\underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$, and $\hat{s}$ = the unit vector in *direction* PP'. **Let** $\overrightarrow{PP'} = (\Delta s)\hat{s}$. The *directional derivative*, $^{dF}/_{ds}$, of F(x,y,z) in the *direction* of $\hat{s}$, is defined **by** $^{dF}/_{ds} = \lim_{\Delta s\to 0} (^{\Delta F}/_{\Delta s})$ (---(1)), *where* $\Delta F = F(P')-F(P)$ (---(2)), and $\Delta S = PP'$ (---(3)).

Now $\hat{s} = \cos(\alpha)\underline{i}+\cos(\beta)\underline{j}+\cos(\gamma)\underline{k}$ (---(4)), where α, β and γ are the *angles* between the **direction** of $\hat{s}$ and the *directions* of $\underline{i}$, $\underline{j}$ and $\underline{k}$ *respectively*. Then $\underline{r}' = \underline{r}+(\Delta s)\hat{s} \Rightarrow (x', y', z') = (x,y,z)+(\Delta s)(\cos\alpha, \cos\beta, \cos\gamma) = (x+(\Delta s)\cos\alpha, y+(\Delta s)\cos\beta, z+(\Delta s)\cos\gamma)$ (---(5)). And therefore ΔF = (2) = $F(x+(\Delta s)\cos\alpha, y+(\Delta s)\cos\beta, z+(\Delta s)\cos\gamma)$ (---(6)).

Now, if F *satisfies* the conditions for a total differential, i.e. $^{\partial F}/_{\partial x}$, $^{\partial F}/_{\partial y}$ and $^{\partial F}/_{\partial z}$ are all continuous, then it *can be shown* that $\Delta F = (^{\partial F}/_{\partial x})(\Delta s)\cos\alpha+(^{\partial F}/_{\partial y})(\Delta s)\cos\beta + (^{\partial F}/_{\partial z})(\Delta s)\cos\gamma + \varepsilon_1(\Delta s)\cos\alpha+\varepsilon_2(\Delta s)\cos\beta+\varepsilon_3(\Delta s)\cos\gamma$ (---(7)), *where* ε_1, ε_2 and $\varepsilon_3 \to 0$ as $\Delta S \to 0$. Further, (7) $\Rightarrow$ $^{\partial F}/_{\partial s} = {}^{\partial F}/_{\partial x}\cos\alpha+{}^{\partial F}/_{\partial y}\cos\beta+{}^{\partial F}/_{\partial z}\cos\gamma+\varepsilon_1\cos\alpha+\varepsilon_2\cos\beta+\varepsilon_3\cos\gamma \Rightarrow {}^{dF}/_{ds} = \lim_{\Delta s\to 0} {}^{\Delta F}/_{\Delta s} = {}^{\partial F}/_{\partial x}\cos\alpha+{}^{\partial F}/_{\partial y}\cos\beta+{}^{\partial F}/_{\partial z}\cos\gamma$ (---(8)), **where** α, β and γ are the *direction cosines* of the chosen direction. [**Note**: (8) may be *rewritten as* $(^{\partial F}/_{\partial x}, {}^{\partial F}/_{\partial y}, {}^{\partial F}/_{\partial z}).(\cos\alpha, \cos\beta, \cos\gamma)$].

Assignment 1: Set 11/2; In 25/2; Back 9/3

Q: Find the **length** of the *curve* $\underline{r} = (3\cos(t))\underline{i}+(3\sin(t))\underline{j}+2t^{3/2}\underline{k}$ for $0 \le t \le 3$. A: If $\underline{r}(t) = f(t)\underline{i}+g(t)\underline{j}+h(t)\underline{k}$ for $a \le t \le b$, then the *length* of the smooth curve traced from a to b is $L = \int^b_a \sqrt{[(^{df}/_{dt})^2+(^{dg}/_{dt})^2+(^{dh}/_{dt})^2]}dt$. So in this *example*, $L = \int^3_0\sqrt{[(^d/_{dt}(3\cos(t))^2+(^d/_{dt}(3\sin(t))^2+ (^d/_{dt}(2t^{3/2}))^2]}dt = \int^3_0 \sqrt{[(-3\sin t)^2+(3\cos t)^2+(3t^{1/2})^2]}dt = \int^3_0 \sqrt{[9\sin^2 t+9\cos^2 t+9t]}dt = 3\int^3_0 \sqrt{(\sin^2 t+\cos^2 t+t)}dt = 3\int^3_0 \sqrt{(1+t)}dt = 3[^2/_3(1+t)^{3/2}]^3_0 = ... = 14$ units.

Q: Find the *unit tangent* **T**, the *unit principal normal* **N**, the *unit binormal* **B**, the *curvature* κ and the *torsion* τ at the given *value* of t: $\underline{r} = (3\cosh(2t))\underline{i}+(3\sinh(2t))\underline{j}+6t\underline{k}$; t = ln2. A: To find **T**, we shall use $\mathbf{T} = d\underline{r}/ds = (d\underline{r}/dt)\times(dt/ds)$. Now $d\underline{r}/dt = \dot{\mathbf{r}} = 6\sinh(2t)\underline{i}+6\cosh(2t)\underline{j}+6\underline{k}$. And $ds/dt = |\dot{\mathbf{r}}| = \sqrt{(36\sinh^2(2t)+36\cosh^2(2t)+36)} = 6\sqrt{(\sinh^2(2t)+\cosh^2(2t)+1} = 6\sqrt{(1+2\sinh^2(2t)+1)}$ (as $\cosh^2(2t)-\sinh^2(2t) = 1$) $= 6\sqrt{(2)}\sqrt{(1+\sinh^2(2t))} = 6\sqrt{(2)}\sqrt{(\cosh^2(2t))} = 6\sqrt{(2)}\cosh(2t)$. So $\mathbf{T} = (^1/_{6\sqrt{(2)}\cosh(2t)})[6\sinh(2t)\underline{i}+6\cosh(2t)\underline{j}+6\underline{k}] = {}^1/_{\sqrt{(2)}}[\tanh(2t)\underline{i}+\underline{j}+\text{sech}(2t)\underline{k}]$.

To get **T** when t = ln(2), all we need to *calculate* is sinh(2t) and cosh(2t) for when t = ln(2). Now sinh(2t) = $e^{2t}-e^{-2t}/2$. When t = ln2, sinh(2t) = $e^{2\ln2}-e^{-2\ln2}/2 = e^{\ln4}-e^{-\ln4}/2 = {}^{4-(1/4)}/_{2} = {}^{15}/_{8}$. *Similarly*, cosh(2t) = $e^{2t}+e^{-2t}/2$, and when t = ln(2), cosh(2t) = ${}^{4+(1/4)}/_{2} = {}^{17}/_{8}$. So when t = ln(2), substituting for sinh(2t) and cosh(2t), we *obtain* **T** = $({}^{1}/_{\sqrt{2}})({}^{(15/8)}/_{(17/8)}\underline{i}+\underline{j}+{}^{1}/_{(17/8)}\underline{k}) = ({}^{1}/_{\sqrt{2}})({}^{15}/_{17}\underline{i}+\underline{j}+{}^{8}/_{17}\underline{k})$.

To get κ and **N**, we will use ${}^{d\mathbf{T}}/_{ds} = \kappa\mathbf{N}$, or ${}^{d\mathbf{T}}/_{dt}\times{}^{dt}/_{ds} = \kappa\mathbf{N}$. Now ${}^{d\mathbf{T}}/_{dt} = {}^{d}/_{dt}({}^{1}/_{\sqrt{2}})(\tanh(2t)\underline{i}+\underline{j}+\text{sech}(2t)\underline{k}) = ({}^{1}/_{\sqrt{2}})(2\text{sech}^2(2t)\underline{i}+0\underline{j}-2\text{sech}(2t)\tanh(2t)\underline{k}) = (\sqrt{2})(\text{sech}^2(2t)\underline{i}-\text{sech}(2t)\tanh(2t)\underline{k})$. We already *know* that ${}^{ds}/_{dt} = 6\sqrt{(2)}\cosh(2t)$; so $(\sqrt{2})(\text{sech}^2(2t)\underline{i}-\text{sech}(2t)\tanh(2t)\underline{k})({}^{1}/_{6\sqrt{(2)}\cosh(2t)}) = \kappa\mathbf{N}$; ...; $({}^{\text{sech}^2(2t)}/_{6})(\text{sech}(2t)\underline{i}-\tanh(2t)\underline{k}) = \kappa\mathbf{N}$.

Now $\kappa = |{}^{d\mathbf{T}}/_{dt}\times{}^{dt}/_{ds}| = |{}^{d\mathbf{T}}/_{ds}| = \sqrt{[({}^{\text{sech}^3(2t)}/_{6})^2+({}^{-\text{sech}^2(2t)\tanh(2t)}/_{6})^2]} = ({}^{1}/_{6})\sqrt{[\text{sech}^6(2t)+\text{sech}^4(2t)\tanh^2(2t)]} = ({}^{1}/_{6})\sqrt{(\text{sech}^4(2t)[\text{sech}^2(2t)+\tanh^2(2t)])} = ({}^{1}/_{6})\sqrt{(\text{sech}^4(2t)[1])} = {}^{\text{sech}^2(2t)}/_{6}$. When t = ln(2), $\kappa = {}^{32}/_{867}$. Now we get **N** using $\mathbf{N} = {}^{1}/_{\kappa}{}^{d\mathbf{T}}/_{ds} = {}^{6}/_{\text{sech}^2(2t)}[{}^{\text{sech}^2(2t)}/_{6}](\text{sech}(2t)\underline{i}-\tanh(2t)\underline{k}) = \text{sech}(2t)\underline{i}-\tanh(2t)\underline{k}$. When t = ln(2), $\mathbf{N} = {}^{8}/_{17}\underline{i}-{}^{15}/_{17}\underline{k}$.

We get **B** using **B** = **T**×**N**. It turns *out* that $\mathbf{B} = (-{}^{1}/_{\sqrt{2}})(\tanh(2t)\underline{i}-\underline{j}+\text{sech}(2t)\underline{k})$. When t = ln2, $\mathbf{B} = (-{}^{1}/_{\sqrt{2}})({}^{15}/_{17}\underline{i}-\underline{j}+{}^{8}/_{17}\underline{k})$. Because of the *similarity* between **T** and **B**, we can deduce that ${}^{d\mathbf{B}}/_{ds} = ({}^{-\text{sech}^2(2t)}/_{6})(\text{sech}(2t)\underline{i}-\tanh(2t)\underline{k})$. (Because the $\underline{i}$ and $\underline{k}$ terms *involved* in the above only differ by sign in **T** and **B**).

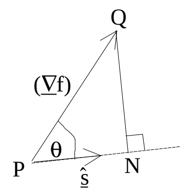
Now as ${}^{d\mathbf{B}}/_{ds} = -\tau\mathbf{N}$, and as we know *everything* apart from τ, we can say that $(-{}^{\text{sech}^2(2t)}/_{6})(\text{sech}(2t)\underline{i}-\tanh(2t)\underline{k}) = -\tau(\text{sech}(2t)\underline{i}-\tanh(2t)\underline{k})$; $(-{}^{\text{sech}^2(2t)}/_{6}) = -\tau$. So $\tau = {}^{\text{sech}^2(2t)}/_{6}$. When t = ln(2), $\tau = {}^{(8/17)^2}/_{6} = {}^{32}/_{867}$. Note that κ = τ.

Q: Find the **gradient** and the **directional derivative** in the direction of $2\underline{i}-\underline{j}-2\underline{k}$ of the *scalar field* f = $x^2yz+4xz^2$ at the *point* (1, -2, 1). A: The *direction of A* is obtained by dividing A by its length. So $\underline{u} = \underline{A}/|\underline{A}| = 2\underline{i}-\underline{j}-2\underline{k}/\sqrt{(2^2+(-1)^2+(-2)^2)} = 2\underline{i}-\underline{j}-2\underline{j}/\sqrt{9} = {}^{2}/_{3}\underline{i}-{}^{1}/_{3}\underline{j}-{}^{2}/_{3}\underline{k}$. The partial derivatives of f are as follows: ${}^{\partial f}/_{\partial x} = 2xyz+4z^2 = 2z(xy+2z)$; ${}^{\partial f}/_{\partial y} = x^2z$; and ${}^{\partial f}/_{\partial z} = x^2y+8xz = x(xy+8z)$.

Therefore, the *gradient* of f at a point (x,y,z) is given by $\underline{\nabla}f = 2z(xy+2z)\underline{i}+x^2z\underline{j}+x(xy+8z)\underline{k}$. At the point (1,-2,1), the **gradient** of f is given by $\underline{\nabla}f = 2(1)(-2+2)\underline{i}+(1)^2(1)\underline{j}+(1)(-2+8)\underline{k} = 2(0)\underline{i}+\underline{j}+6\underline{k} = \underline{j}+6\underline{k}$. The *directional derivative* of f at (1,-2,1) in the direction of $\underline{A}$ is therefore given by $\underline{\nabla}f.\underline{u} = (\underline{j}+6\underline{k}).({}^{2}/_{3}\underline{i}-{}^{1}/_{3}\underline{j}-{}^{2}/_{3}\underline{k}) = (0\times{}^{2}/_{3})+(1\times(-{}^{1}/_{3}))+(6\times(-{}^{2}/_{3})) = 0-{}^{1}/_{3}-{}^{12}/_{3} = -{}^{13}/_{3}$.

1.5: The Gradient Field

Let F(x,y,z) be a *scalar field* where the first **partial** derivatives of f exist in some domain D_0. The *gradient* of the scalar field f, grad f, is defined by grad f = ${}^{\partial f}/_{\partial x}\underline{i}+{}^{\partial f}/_{\partial y}\underline{j}+{}^{\partial f}/_{\partial z}\underline{k}$ (---(1)). **Notation**: grad f = $\underline{\nabla}f$ (‘*del*’ or ‘*nabla*’), where $\underline{\nabla} = \underline{i}{}^{\partial}/_{\partial x}+\underline{j}{}^{\partial}/_{\partial y}+\underline{k}{}^{\partial}/_{\partial z}$. Now $(\underline{\nabla}f)\hat{\mathbf{s}} = {}^{\partial F}/_{\partial x}\cos\alpha+{}^{\partial F}/_{\partial y}\cos\beta+{}^{\partial F}/_{\partial z}\cos\gamma = {}^{df}/_{ds}$ (= the derivative of f in the *direction* of $\hat{\mathbf{s}}$). In the **diagram,** PN = $(\underline{\nabla}f).\hat{\mathbf{s}} = |\underline{\nabla}f||\hat{\mathbf{s}}|\cos\theta = |\underline{\nabla}f|\cos\theta = {}^{df}/_{ds}$. **So** $|\cos\theta| \leq 1 \Rightarrow |{}^{df}/_{ds}| \leq |\underline{\nabla}f \Rightarrow |\underline{\nabla}f|$ is the *maximum* rate of change of f w.r.t. **different** directions.



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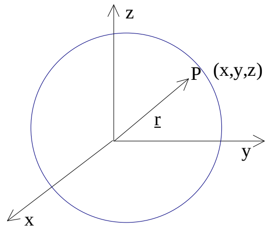
Properties of grad

(i) grad(f+g) = grad(f)+grad(g) (---(7)). (ii) grad(fg) = fgrad(g)+g.grad(f) (---(8)). **Example**: If f(x,y,z) = $x^2+y^2+z^2$, then $^{\partial f}/_{\partial x}$ = 2x, $^{\partial f}/_{\partial}$ = 2y and $^{\partial f}/_{\partial z}$ = 2z ⇒ grad(f) = $^{\partial f}/_{\partial x}\underline{i}+{}^{\partial f}/_{\partial y}\underline{j}+{}^{\partial f}/_{\partial z}\underline{k}$ = 2x$\underline{i}$+2y$\underline{j}$+2z$\underline{k}$ = 2(x$\underline{i}$+y$\underline{j}$+z$\underline{k}$) = 2$\underline{r}$.

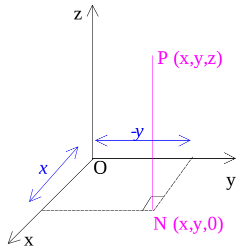


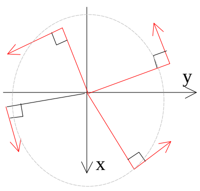
1.6: Divergence of a Vector Field

Let $\underline{v}$ = (v_x, v_y, v_z) be a *vector field* in some domain D. Consider the 3×3 **matrix** shown on the right (---(1)). From (1), construct the *scalar field* div($\underline{v}$) (div = divergence) by the *definition* div($\underline{v}$) = ($\partial v_x/\partial x$)+($\partial v_y/\partial y$)+($\partial v_z/\partial z$) (---(2)) = $\underline{\nabla}.\underline{v}$ = ($^{\partial}/_{\partial x}$, $^{\partial}/_{\partial y}$, $^{\partial}/_{\partial z}$).($v_x$, v_y, v_z). The *divergence* produces a **scalar** field from a **vector** field. **Example**: If $\underline{v}$ = $\underline{r}$ = (x,y,z), then div($\underline{v}$) = (2) = $^{\partial}/_{\partial x}$(x)+$^{\partial}/_{\partial y}$(y)+$^{\partial}/_{\partial z}$(z) = 1+1+1 = 3.

$$\begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Example: Let v = (-y, x, 0). Here, div($\underline{v}$) = $^{\partial}/_{\partial x}$(-y)+$^{\partial}/_{\partial y}$(x)+$^{\partial}/_{\partial z}$(0) = 0+0+0 = 0. *Now let* $\underline{v}$ = ($3x^2$, sinz, cosy). It follows that div($\underline{v}$) = 6x+0+0 = 6x. **Properties of Divergence**: div($\underline{u}$+$\underline{v}$) = div($\underline{u}$)+div($\underline{v}$) (---(3)); div(f$\underline{u}$) = fdiv($\underline{u}$)+(grad f).$\underline{u}$ (---(4)).





Proof of (4): If $\underline{u}$ = $u_x\underline{i}+u_y\underline{j}+u_z\underline{k}$, then f$\underline{u}$ = (fu_x)$\underline{i}$+(fu_y)$\underline{j}$+(fu_z)$\underline{k}$, and so div(f$\underline{u}$) = $^{\partial}/_{\partial x}$(f$u_x$)+$^{\partial}/_{\partial y}$(f$u_y$)+$^{\partial}/_{\partial z}$(f$u_z$) = (*product rule*) = $^{\partial f}/_{\partial x}u_x$+f($\partial u_x/\partial x$)+$^{\partial f}/_{\partial y}u_y$+f($\partial u_y/\partial y$)+$^{\partial f}/_{\partial z}u_z$+f($\partial u_z/\partial z$) = f.div(u)+(grad f).$\underline{u}$.

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The Curl of a Vector Field

The curl of a vector field is *defined* as follows: curl($\underline{v}$) = $(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})\boldsymbol{i}+(\frac{\partial v_y}{\partial z}-\frac{\partial v_z}{\partial x})\boldsymbol{j}+(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial y})\boldsymbol{k}$ = *the determinant* shown in yellow (---(1)) = $\underline{\nabla}\times\underline{v}$. **Example**: If $\underline{v}$ = $\underline{r}$ = (x,y,z), then curl($\underline{v}$) = $\underline{\nabla}\times\underline{v}$ = the same as the determinant *shown*, but with bottom row x, y, 0, giving curl($\underline{v}$) = ($^{\partial}/_{\partial y}$(z)-$^{\partial}/_{\partial z}$(y))$\underline{i}$-($^{\partial}/_{\partial x}$(z)-$^{\partial}/_{\partial z}$(x))$\underline{j}$+($^{\partial}/_{\partial x}$(y)-$^{\partial}/_{\partial y}$(x))$\underline{k}$ = $\underline{0}$. **Example**: If $\underline{v}$ = $\underline{r}$ = (-y,x,0), then curl($\underline{v}$) = $\underline{i}$(0-0)-$\underline{j}$(0-0)+$\underline{k}$(1-(-1)) = 2$\underline{k}$. **Properties of curl**: curl($\underline{u}$+$\underline{v}$) = curl($\underline{u}$)+curl($\underline{v}$) (---(2)); curl(f$\underline{u}$) = f curl($\underline{u}$)+(grad f)×$\underline{u}$ (---(3)).

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

1.8: Combinations of grad, div and curl

Linearity of operators: equations (1.5.7), (1.6.3) and (1.7.2) say that grad(f+g) = grad(f)+grad(g); div($\underline{u}$+$\underline{v}$) = div($\underline{u}$)+div($\underline{v}$); and curl($\underline{u}$+$\underline{v}$) = curl($\underline{u}$)+curl($\underline{v}$) (---(set 1)). These imply that an operation on a *sum* is the *sum of the operation* on the terms. If f = *c*, a constant, then (1.5.1) ⇒ grad f = 0. So (1.5.8), (1.6.4) and (1.7.3) ⇒ grad(*c*g) = *c*grad(g); div(*c*$\underline{u}$) = *c*div($\underline{u}$); and curl(*c*$\underline{u}$) = *c*curl($\underline{u}$) (---(set 2)). (1) & (2) ⇒ grad, div, curl are **linear** operators.

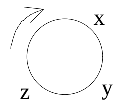
Curl of a Gradient

Let f be any *differentiable* scalar field, then curl(grad f) = the determinant shown in **green** = $\underline{i}(\partial/\partial y\,\partial f/\partial z - \partial/\partial z\,\partial f/\partial y)+(\ldots)\underline{j}+(\ldots)\underline{k}$ = $\underline{i}(\partial^2 f/\partial y\partial z - \partial^2 f/\partial z\partial y)+(\ldots)\underline{j}+(\ldots)\underline{k}$ = 0+0+0 = $\underline{0}$. (**Note**: curl(grad f) = $\underline{\nabla}\times\underline{\nabla}(f) = \underline{0}$, where $\underline{\nabla}\times\underline{\nabla}$ is *where* the 0 comes from). **Converse**: under *additional assumptions*, curl($\underline{v}$) = $\underline{0} \Rightarrow \underline{v}$ = grad(f) for some *scalar* field f.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

Divergence of a Curl

div(curl $\underline{v}$) = $\partial/\partial x[\partial v_z/\partial y - \partial v_y/\partial z] + \partial/\partial y[\partial v_x/\partial z - \partial v_z/\partial x] + \partial/\partial z[\partial v_y/\partial x - \partial v_x/\partial y]$ = 0, since the order of differentiation may be *reversed* if the **second** order partial derivatives exist and are continuous. (The coloured elements cancel out). Remember the *cyclic permutations* as shown. Note: div(curl $\underline{v}$) = $\underline{\nabla}.(\underline{\nabla}\times\underline{v})$ = 0. (Scalar *triple* product; two terms identical $\Rightarrow$ 0). **Converse**: under *additional assumptions*, div($\underline{u}$) = 0 $\Rightarrow$ $\underline{u}$ = curl($\underline{v}$) for some $\underline{v}$. ($\underline{u}$ is *solenoidal*).

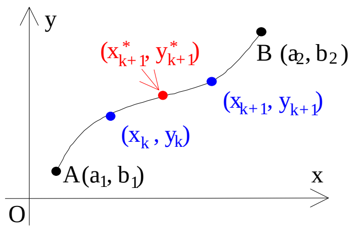


Divergence of a vector product: div($\underline{u}\times\underline{v}$) = $\underline{v}$.curl($\underline{u}$)-$\underline{u}$.curl($\underline{v}$). **Divergence of a gradient:** div(grad f) = $\partial/\partial x(\partial f/\partial x)+\partial/\partial y(\partial f/\partial y)+\partial/\partial z(\partial f/\partial z)$ = $(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2)f$ = $\underline{\nabla}.\underline{\nabla}f = \underline{\nabla}^2 f$, **where** $(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2)$ is the *Laplacian operator*. **Curl of a curl;** curl(curl $\underline{u}$) = grad(div $\underline{u}$) - $((\nabla^2 u_x)\underline{i}+(\nabla^2 u_y)\underline{j}+(\nabla^2 u_z)\underline{k})$. If one defines the *action* of the Laplacian operator on a **vector** $\underline{u}$ by $\nabla^2\underline{u} = (\nabla^2 u_x)\underline{i}+(\nabla^2 u_y)\underline{j}+(\nabla^2 u_z)\underline{k}$, then curl(curl $\underline{u}$) = grad(div $\underline{u}$) - $\nabla^2\underline{u}$.

Section 2: Vector Integral Calculus

2.1: Line Integrals in Space

2-dimensional: Let C be a curve in the *x-y plane* connecting $A(a_1,b_1)$ to $B(a_2,b_2)$. Let P(x,y) and Q(x,y) be single-valued *functions* defined at all points of C. Subdivide C into n parts by choosing n-1 points on it: (x_1,y_1), (x_2,y_2), ..., (x_{n-1},y_{n-1}). Let $\Delta x_k = x_k - x_{k-1}$, and let $\Delta y_k = y_k - y_{k-1}$. (k = 1,...,n, where $(a_1,b_1) = (x_0,y_0)$, and $(a_2,b_2) = (x_n,y_n)$).



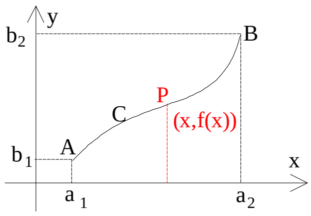
Choose *points* (x^*_{k+1}, y^*_{k+1}) *between* (x_k, y_k) and (x_{k+1}, y_{k+1}). (k = 0,...,n). Then $\lim_{n\to\infty}\sum_{k=1}^{n} P(x^*_k, y^*_k)\Delta x_k + Q(x^*_k, y^*_k)\Delta y_k$ (with *max*$\{\Delta x_k, \Delta y_k\} \to 0$) = $\int_C P(x,y)dx + Q(x,y)dy$ (---(1)), and is called a *line integral along C*. *Sufficient* condition for the existence of (1): P and Q are **continuous** at all points of C. The *value* of (1) depends in general on P, Q, C, A and B.

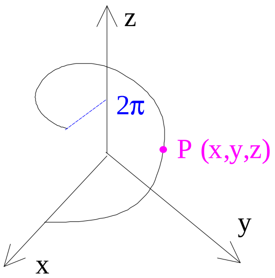
3-dimensional. Equation (1) easily generalises to *3-dimensions*: $\lim_{n\to\infty}\sum_{k=1}^{n}\{A_1(x_k^*,y_k^*,z_k^*)\Delta x_k+A_2(x_k^*,y_k^*,z_k^*)\Delta y_k+A_3(x_k^*,y_k^*,z_k^*)\Delta z_k\}$ (with $\max\{\Delta x_k, \Delta y_k, \Delta z_k\} \to 0$) = $\int_C A_1dx+A_2dy+A_3dx$ (---(2)). **Vector notation:** If A = $A_1\underline{i}+A_2\underline{j}+A_3\underline{k}$, and if $d\underline{r} = (dx)\underline{i}+(dy)\underline{j}+(dz)\underline{k}$, then $\int_C A_1dx+A_2dy+A_3dz = \int_C (A_1\underline{i}+A_2\underline{j}+A_3\underline{k}).((dx)\underline{i}+(dy)\underline{j}+(dz)\underline{k}) = \int_C \underline{A}.d\underline{r}$ (---(3)).

24th February 2000

Evaluation of Line Integrals

2-dimensional. If C is given by $y = f(x)$, then $dy = f'(x)dx$. So (1) $\Rightarrow \int_{a_1}^{a_2} P(x,f(x))dx+Q(x,f(x))f'(x)dx = \int_{a_1}^{a_2}[P(x,f(x))+Q(x,f(x))f'(x)]dx$ (---(4)). *Similarly*, if C is given by $x = g(y)$, then $dx = g'(y)dy$, so (1) $\Rightarrow \int_{b_1}^{b_2} [P(g(y),y)g'(y)+Q(g(y),y)]dy$ (---(5)). Also, if C is given *parametrically* by $x = \phi(t)$ and $y = \varphi(t)$ (so that $dx = \phi'(t)dt$ and $dy = \varphi'(t)dt$), then (1) $\Rightarrow \int_{t_1}^{t_2} [P(\phi(t), \varphi(t))\phi'(t)+Q(\phi(t),\varphi(t))\varphi'(t)]dt$ (---(6)), *where* $(a_i,b_i) = (\phi(t_i), \varphi(t_i))$, and $i = 1,2$.



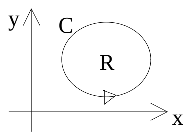


3-dimensional: obvious *generalisation* of the above. **Example**: Evaluate $I = {}_C\int_{(1,0,0)}^{(1,0,2\pi)} zdx+xdy+ydz$, where C is $x = \cos(t)$, $y = \sin(t)$ and $z = t$ (for $0 \le t \le 2\pi$). So $I = \int_{t=0}^{t=2\pi} t(-\sin t)dt+\cos t(\cos t)dt+\sin tdt = \int_{t=0}^{t=2\pi}[\sin t(1-t)+\cos^2 t]dt = \ldots = 3\pi$.

Properties of Line Integrals. (i) $\int_C Pdx+Qdy = \int_C Pdx +\int_C Qdy$ (---(7)). (ii) ${}_C\int_{(a_1,b_1)}^{(a_2,b_2)} Pdx+Qdy = -{}_C\int_{(a_2,b_2)}^{(a_1,b_1)} Pdx+Qdy$ (---(8)). (iii) $\int_{(a_1,b_1)}^{(a_2,b_2)} Pdx+Qdy = \int_{(a_1,b_1)}^{(a_3,b_3)} Pdx+Qdy + \int_{(a_3,b_3)}^{(a_2,b_2)} Pdx+Qdy$ (---(9)), where (a_3,b_3) is some *other point* on C. Similarly for **3-dimensional** line integrals.

Green's Theorem in the Plane

Let P, Q, $^{\partial P}/_{\partial y}$ and $^{\partial Q}/_{\partial x}$ be *single valued* and *continuous* in a *simply connected region R* bounded by a simple closed curve C. Then ${}_C\oint Pdx+Qdy = {}_R\iint(^{\partial Q}/_{\partial x}-^{\partial P}/_{\partial y})dxdy$ (---(10)). **Note**: the *loop* in the integral goes anticlockwise. *Simple closed curve*: a closed non-intersecting curve.



25th February 2000

Tutorial

Q: If $\underline{a}$ is a *constant vector* and we are given the *scalar* and vector *fields* respectively: $f(\underline{r}) = \underline{a}.\underline{r}$; $\underline{v}(\underline{r}) = \underline{a}\times\underline{r}$, *show that* (i) $\underline{\nabla}f = \underline{a}$; (ii) $\underline{\nabla}.\underline{v} = 0$; (iii) $\underline{\nabla}^2 f = 0$; (iv) $\underline{\nabla}^2(f^2) = 2\underline{a}.\underline{a}$; and (v) $\underline{\nabla}.(f\underline{v}) = 0$. Find $\underline{\nabla}\times\underline{v}$, and verify that $\underline{\nabla}\times(f\underline{v}) = f\underline{\nabla}\times\underline{v} + (\underline{\nabla}f)\times\underline{v}$. A: These vector operations are *independent* of the choice of the co-ordinate axes. We can **therefore,** w.l.o.g., take $\underline{a} = (0,0,a)$.

(i) $f(r) = \underline{a}.\underline{r} = (0,0,a).(x,y,z) = 0+0+ax = az$. **So** $\underline{\nabla}f = {}^{\partial}/_{\partial x}(az)\underline{i}+{}^{\partial}/_{\partial y}(az)\underline{j}+{}^{\partial}/_{\partial z}(az)\underline{k} = 0\underline{i}+0\underline{j}+a\underline{k} = (0,0,a) = \underline{a}$. (ii) $\underline{v}(r) = \begin{vmatrix} i & j & k \\ 0 & 0 & a \\ x & y & z \end{vmatrix} = -ay\underline{i}+ax\underline{j}+0\underline{k} = (-ay, ax, 0)$. So $\underline{\nabla}.\underline{v} = ({}^{\partial}/_{\partial x},{}^{\partial}/_{\partial y},{}^{\partial}/_{\partial z}).(-ay,ax,0) = 0+0+0 = 0$. (iii) $\underline{\nabla}^2 f = \underline{\nabla}(\underline{\nabla}f) = \underline{\nabla}(0,0,a) = (0,0,0) = \underline{0}$. (iv) $f^2 = (0,0,a^2z^2)$, and $\underline{\nabla}f^2 = (0,0,2a^2z)$; so $\underline{\nabla}^2 f^2 = (0,0,2a^2) = 2(0,0,a^2) = 2(0,0,a).(0,0,a) = 2\underline{a}.\underline{a}$. (v) $f\underline{v} = (-a^2zy, a^2xz, 0)$, so $\underline{\nabla}.(f\underline{v}) = 0+0+0 = 0$.

Now $\underline{\nabla}\times v = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -ay & ax & 0 \end{vmatrix} = [0+0]\underline{i}-[0+0]\underline{j}+[a+a]\underline{k} = (0,0,2a)$. So $\underline{\nabla}\times f\underline{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -a^2zy & a^2xz & 0 \end{vmatrix} = [0-a^2x]\underline{i}-[0+a^2y]\underline{j}+[a^2z+a^2z]\underline{k} = -a^2x\underline{i}-a^2y\underline{j}+2a^2z\underline{k}$. **Now** $f\underline{\nabla}\times\underline{v} = \begin{vmatrix} i & j & k \\ az(\partial/\partial x) & az(\partial/\partial y) & (az\partial/\partial z) \\ -ay & ax & 0 \end{vmatrix} = [0-0]\underline{i}-[0-0]\underline{j}+[a^2z+a^2z]\underline{k} = 2a^2z\underline{k}$. And $(\underline{\nabla}f\times\underline{v}) = \begin{vmatrix} i & j & k \\ 0 & 0 & a \\ -ay & ax & 0 \end{vmatrix} = [0-a^2x]\underline{i}-[0+a^2y]\underline{j}+0\underline{k} = -a^2x\underline{i}-a^2y\underline{j}+0\underline{k}$. So $f\underline{\nabla}\times\underline{v}+(\underline{\nabla}f\times\underline{v}) = -a^2x\underline{i}-a^2y\underline{j}+2a^2z\underline{k} = \underline{\nabla}\times f\underline{v}$.

Examples 2

Q: Find the *length* of the curve **r** = (2cos(t))**i**+(2sin(t))**j**+t²**k** for $0 \le t \le \pi/4$. A: $L = \int_{t=0}^{t=\pi/4} \sqrt{((-2\sin(t))^2+(2\cos(t))^2+(2t)^2)}dt = 2\int_0^{\pi/4} \sqrt{(1+t^2)}dt$. Consider $I = \int\sqrt{(1+t^2)}dt$. Let t = tanθ, so that dt = sec²θdθ. Therefore, I = ∫secθsec²θdθ = ∫-secθd(tanθ) = (by *parts)* = secθtanθ-∫tanθd(secθ) = secθtanθ-∫tanθsecθtanθdθ = secθtanθ - ∫secθ(sec²θ-1)dθ.

Therefore, I = secθtanθ-∫sec³θdθ+∫secθdθ. Because ∫sec³θdθ = I, it follows that 2I = secθtanθ+∫secθdθ. Now because ∫secθdθ = ln|secθ+tanθ|, then 2I = secθtanθ+ln|secθ+tanθ|, and so $L = 2I = [\sqrt{(1+t^2)}t+\ln|\sqrt{(1+t^2)}+t|]_{t=0}^{t=\pi/4} = {}^{\pi}/_{4}\sqrt{[1+(^{\pi}/_{4})^2]}+\ln(^{\pi}/_{4}+\sqrt{(1+(^{\pi}/_{4})^2)})$.

Q: Find the unit *tangent* **T**, the unit *principal normal* **N**, the unit *binormal* **B**, the *curvature* κ, and the *torsion* τ at the given value of t: **r** = t**i**+$^1/_2e^{2t}$**j** for t = ln2. A: $\dot{\mathbf{r}}$ = **i**+e^{2t}**j**, and $\dot{s}$ = $|\dot{\mathbf{r}}|$ = $\sqrt{(1+e^{4t})}$. So **T** = $\dot{\mathbf{r}}/\dot{s}$ = $\{1/\sqrt{(1+e^{4t})}\}[\mathbf{i}+e^{2t}\mathbf{j}]$. Now κ**N** = $^{d\mathbf{T}}/_{ds}$ = $(^{d\mathbf{T}}/_{dt})/\dot{s}$ = $\{1/\sqrt{(1+e^{4t})}\}[\mathbf{i}\{(-^1/_2)(1+e^{4t})^{-3/2}(4e^{4t})\}+\mathbf{j}\{(-^1/_2)(1+e^{-4t})^{-3/2}(-4e^{-4t})\}$. *(As* $e^{2t}/\sqrt{(1+e^{4t})} = 1/\sqrt{(e^{-4t}+1)}$).

The above *implies* that **T** and **B** lie in the x-y plane <u>for all t</u>. Therefore, **B** = **T**×**N** = **k** for all t, so τ = 0. At t = ln(2), $e^{2t} = e^{2\ln(2)} = e^{\ln(4)} = 4$. Therefore, **T**(0) = $^1/_{\sqrt{17}}$(**i**+4**j**); κ**N** = *(after* substitution) = $^8/_{17\sqrt{17}}\{^{-4\mathbf{i}+\mathbf{j}}/_{\sqrt{17}}\}$, which *implies that* κ = $^8/_{17\sqrt{17}}$ and **N** = $^{-4\mathbf{i}+\mathbf{j}}/_{\sqrt{17}}$.

Assignment 2: Set 25/2; In 10/3; Back 29/3

Q: If <u>a</u> denotes a constant vector, show that (i) div(<u>a</u>×<u>r</u>) = 0; (ii) curl(<u>a</u>×<u>r</u>) = 2<u>a</u>; (iii) provided r ≠ 0, div[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = 0; and (iv) curl[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = $(2/r^4)$(<u>a</u>.<u>r</u>)<u>r</u>. A: Let <u>a</u> be the vector parallel to the *z-axis*, so that <u>a</u> = (0,0,a). Let <u>r</u> be *any point is space*, <u>r</u> = (x,y,z). Then (i) <u>a</u>×<u>r</u> = (using the **matrix** method) = -ay<u>i</u>+ax<u>j</u>+0<u>k</u>.

So div(<u>a</u>×<u>r</u>) = ∇.(<u>a</u>×<u>r</u>) = $(^{\partial}/_{\partial x}, ^{\partial}/_{\partial y}, ^{\partial}/_{\partial z})$.(-ay,ax,0) = $^{\partial}/_{\partial x}(-ay)+^{\partial}/_{\partial y}(ax)+^{\partial}/_{\partial z}(0)$ = 0. QED. (ii) Now curl(<u>a</u>×<u>r</u>) = ∇×(<u>a</u>×<u>r</u>) = (using the *matrix* method) = [0-0]<u>i</u>-[0-0]<u>j</u>+[a--a]<u>k</u> = 0<u>i</u>+0<u>j</u>+2a<u>k</u> = 2(0<u>i</u>+0<u>j</u>+a<u>k</u>) = 2<u>a</u>. QED. (iii) Using a **property**, we can way that div[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = $^1/_{r^2}$div(<u>a</u>×<u>r</u>) + (grad$^1/_{r^2}$).(<u>a</u>×<u>r</u>). We *know* from (i) that div(<u>a</u>×<u>r</u>) = 0, so we **have** div[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = (grad$^1/_{r^2}$).(<u>a</u>×<u>r</u>).

We now need to **find** grad$^1/_{r^2}$. As r = |<u>r</u>| = $\sqrt{(x^2+y^2+z^2)}$, then $r^2 = x^2+y^2+z^2$, and so $^1/_{r^2} = (x^2+y^2+z^2)^{-1}$. *Therefore,* grad($^1/_{r^2}$) = $^{\partial}/_{\partial x}(^1/_{r^2})$<u>i</u>+$^{\partial}/_{\partial y}(^1/_{r^2})$<u>j</u>+$^{\partial}/_{\partial z}(^1/_{r^2})$<u>k</u> = $-1(x^2+y^2+z^2)^{-2}.2x$<u>i</u> $-1(x^2+y^2+z^2)^{-2}.2y$<u>j</u> - $1(x^2+y^2+z^2)^{-2}.2z$<u>k</u> = $-^2/_{(x^2+y^2+z^2)^2}$[x<u>i</u>+y<u>j</u>+z<u>k</u>]. **So** div[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = (grad$^1/_{r^2}$).(<u>a</u>×<u>r</u>) =$-^2/_{(x^2+y^2+z^2)^2}$[x<u>i</u>+y<u>j</u>+z<u>k</u>].(-ay<u>i</u>+ax<u>j</u>+0<u>k</u>) = $-^2/_{(x^2+y^2+z^2)^2}$[-xay+yax+z(0)] = $-^2/_{(x^2+y^2+z^2)^2}$[axy-axy] = 0. <u>QED</u>.

(iv) Using the *property* curl[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = $^1/_{r^2}$curl(<u>a</u>×<u>r</u>) + (grad$^1/_{r^2}$)×(<u>a</u>×<u>r</u>), and (from (i)): curl(<u>a</u>×<u>r</u>) = 2<u>a</u> = 2(0,0,a), and (from (iii)): grad($^1/_{r^2}$) = $-^2/_{(x^2+y^2+z^2)^2}$[x<u>i</u>+y<u>j</u>+z<u>k</u>] = $-(2/r^4)$(x,y,z), we have curl[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = $^1/_{r^2}$[2(0,0,a)] - [$(2/r^4)$(x,y,z)]×(<u>a</u>×<u>r</u>) = $^2/_{r^2}$(0,0,a)-(($2/r^4$)(x,y,z))×(<u>a</u>×<u>r</u>). We calculate the *second* term by **matrix** to get (($2/r^4$)(x,y,z))×(<u>a</u>×<u>r</u>) = $-(2a/r^4)$[xz<u>i</u>+yz<u>j</u>-(x²+y²)<u>k</u>]. Therefore, curl[$^1/_{r^2}$(<u>a</u>×<u>r</u>)] = $^2/_{r^2}$(0,0,a)-($-2a/r^4$)(xz,yz,-x²-y²) = ...= ($2az/r^4$)(x,y,z). To see if this *equals* ($2/r^4$)(<u>a</u>.<u>r</u>)<u>r</u>, let us *evaluate* ($2/r^4$)(<u>a</u>.<u>r</u>)<u>r</u> = ($2/r^4$)[(0,0,a).(x,y,z)](x,y,z) = ($2/r^4$)[0+0+az](x,y,z) = ($2az/r^4$)(x,y,z). <u>QED</u>.

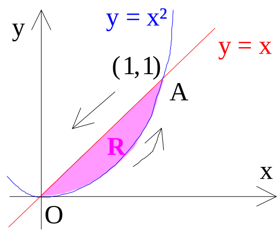
Q: *Show* that $\underline{F} = (y^2\cos(x)+z^3)\underline{i}+(2y\sin(x)-4)\underline{j}+(3xz^2+2)\underline{k}$ is a *conservative force field*. (a) Find a scalar potential function for $\underline{F}$. (b) Show that the work done in moving an **object** in this field from (0,1,-1) to ($^{\pi}/_2$,-1,2) is $15+4\pi$. A: To show that it is conservative, simply **show** that curl$\underline{F} \equiv \underline{0}$, where curl$\underline{F} = \underline{\nabla}\times\underline{v}$ = the matrix shown, which *turns* out to be vectorially **zero**. Therefore, $\underline{F}$ is a *conservative* force.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos(x)+z^3 & 2y\sin(x)-4 & 3xz^2+2 \end{vmatrix}$$

(a) The **above** implies that $\underline{F} = \underline{\nabla}(W)$ for some scalar field W. So $(y^2\cos(x)+z^3)\underline{i} + (2y\sin(x)-4)\underline{j} + (3xz^2+2)\underline{k} = {}^{\partial W}/_{\partial x}\underline{i}+{}^{\partial W}/_{\partial y}\underline{j}+{}^{\partial W}/_{\partial z}\underline{k}$. As $\underline{i}$, $\underline{j}$ & $\underline{k}$ are linearly independent, we can *say* that ${}^{\partial W}/_{\partial x} = y^2\cos(x)+z^3$; ${}^{\partial W}/_{\partial y} = 2y\sin(x)-4$; and ${}^{\partial W}/_{\partial z} = 3xz^2+2$. These *imply* that $W = \int(y^2\cos(x)+z^3)dx = \int(2y\sin(x)-4)dy = \int(3xz^2+2)dz$. So $W = y^2\sin(x)+z^3x + f(y,z) = y^2\sin(x)-4y+g(z,x) = xz^3+2z+h(x,y)$, *where* f(y,z), g(z,x) and h(x,y) are additive "constants" which do not **depend** on x, y or z respectively.

By *inspection*, we choose $f(y,z) = 2z-4y$; $g(z,x) = z^3x+2z$; and $h(x,y) = y^2\sin(x)-4y$, so that $W = y^2\sin(x)+z^3x-4y+2z+c$. (b) The work done is **given** by $W(^{\pi}/_2,-1,2)-W(0,1,-1)$. Now $W(^{\pi}/_2,-1,2) = (-1)^2\sin(^{\pi}/_2)+(2)^3(^{\pi}/_2)-4(-1)+2(2) = \ldots = 9+4\pi$. *Similarly*, $W(0,1,-1) = -6$. So $W(^{\pi}/_2,-1,2)-W(0,1,-1) = (9+4\pi)-(-6) = 15+4\pi$ units. <u>QED</u>.

Q: Verify *Green's Theorem* in the plane for $_C\oint(xy+y^2)dx+x^2dy$ (anticlockwise), where C is the **closed** curve of the region bounded by y = x and y = x². A: In this *question*, $P = (xy+y^2)$, and $Q = x^2$, so we need to **verify** the following: $_C\oint(xy+y^2)dx+x^2dy = \iint_R(2x-(x+2y))dxdy$; $_C\oint(xy+y^2)dx+x^2dy = \iint_R(x-2y)dxdy$. The region involved is as *shown in the diagram on the right*.



Let us first evaluate the L.H.S. The **counter-clockwise** integral can be split into 2 parts: from O to A, and then from A to O. So $_C\oint = {}_{OA}\oint + {}_{AO}\oint$ = (where the **first** part is along $y = x^2$, so $Dy = 2xdx$; and the **second** part is along $y = x$, so $Dy = 1dx$) $= {}_{OA}\oint(xy+y^2)dx+x^2dy + {}_{AO}\oint(xy+y^2)dx+x^2dy$.

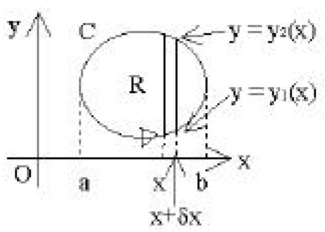
So $_C\oint = \int^{y=1}_{y=0}(xy+y^2)dx+x^2dy$ *(with $y = x^2$)* + $\int^{y=0}_{y=1}(xy+y^2)dx+x^2dy$ *(with $y = x$)* $= \int^{y=1}_{y=0}(xx^2+(x^2)^2)dx+x^2.2xdx + \int^{y=0}_{y=1}(xx+(x)^2)dx+x^2.1dx = \int^{y=1}_{y=0}(x^3+x^4)dx+2x^3dx + \int^{y=0}_{y=1}(x^2+x^2)dx+x^2dx = \int^{y=1}_{y=0}(3x^3+x^4)dx+\int^{y=0}_{y=1}3x^2dx = \int^{y=1}_{y=0}(3x^3+x^4)dx-\int^{y=1}_{y=0}3x^2dx = \int^{y=1}_{y=0}(3x^3+x^4-3x^2)dx = \ldots = -^1/_{20}$.

Now let us evaluate the **RHS**, $\iint_R(x-2y)dxdy$. Looking at the <u>diagram</u>, x goes from 0 to 1 while y goes from x² to x over our *region* R. **Alternatively**, y goes from 0 to 1, while x goes from y to √(y). So $\iint_R(x-2y)dxdy = \int^1_0\int^{\sqrt{y}}_y(x-2y)dxdy = \int^1_0[^{x^2}/_2-2yx]^{\sqrt{y}}_y dy = \int^1_0[^y/_2-2y^{3/2}-(^{y^2}/_2-2y^2)]dy = \int^1_0 {}^{3y^2}/_2-2y^{3/2}+^y/_2dy = \ldots = -^1/_{20}$. QED.

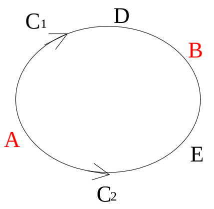
Proof of Green's Theorem

1st March 2000

Consider $\iint_R {}^{\partial P}/_{\partial y}\, dxdy = \int_{x=a}^{x=b}[\int_{y=y_1(x)}^{y=y_2(x)} {}^{\partial P}/_{\partial y} dy]dx = \int_{x=a}^{x=b}[P(x,y)]_{y_1(x)}^{y_2(x)}dx = \int_{x=a}^{x=b}(P(x,y_2)-P(x,y_1))dx = -\int_{x=b}^{x=a}(P(x,y_2)dx) - \int_{x=a}^{x=b}P(x,y_1)dx = -\oint_C Pdx$ (---(11)). (*Anti clockwise* loop in integral) *Similarly*, $\iint_R {}^{\partial Q}/_{\partial x}\, dxdy = \oint_C Qdy$ (---(12)). (Anti clockwise). Now (11) & (12) $\Rightarrow \oint_C Pdx+Qdy = \iint_R ({}^{\partial Q}/_{\partial x}-{}^{\partial P}/_{\partial y})dxdy$.



If P and Q have *continuous partial derivatives* in a simply connected region R, then a **necessary** and **sufficient** condition for $\int_C Pdx+Qdy$ (---(13)) to be *independent* of the path C joining any two points in R is that in R, ${}^{\partial P}/_{\partial y} = {}^{\partial Q}/_{\partial x}$ (---(14)).

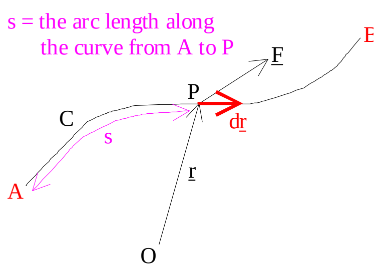


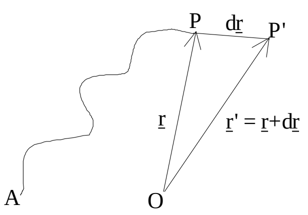
<u>Proof</u>. (i) Do $\Rightarrow$. If $\int_C Pdx+Qdy$ is *independent* of the path C from A to B, then $\int_{c_1} = \int_{c_2} \Leftrightarrow \oint_{ADBEA} = 0$ (clockwise) = (from (9)) = $\oint_{\#}$, (anticlockwise), where # is the path c_2 c_1 (AEB, BDA); $\Leftrightarrow$ (10) ${}^{\partial Q}/_{\partial x} = {}^{\partial P}/_{\partial y}$, by Green's Theorem. (ii) Do $\Leftarrow$. If $\oint_C Pdx+Qdy = 0$ (anticlockwise) for any c, and if ${}^{\partial P}/_{\partial x} \neq {}^{\partial Q}/_{\partial y}$, (a) suppose ${}^{\partial P}/_{\partial y}-{}^{\partial Q}/_{\partial x} > 0$ at some point (x_0,y_0) of R. By **continuity**, there exists a neighbourhood of (x_0,y_0), where ${}^{\partial P}/_{\partial y}-{}^{\partial Q}/_{\partial x} > 0$. Let Γ be the boundary, then (10) $\Rightarrow \oint_\Gamma (Pdx+Qdy) > 0$. (No!). (b) **Since** ${}^{\partial P}/_{\partial y}-{}^{\partial Q}/_{\partial x} \not< 0$, *then* ${}^{\partial P}/_{\partial y}-{}^{\partial Q}/_{\partial x} = 0 \Rightarrow$ <u>necessity</u>. **<u>Note</u>**: $-({}^{\partial P}/_{\partial y}-{}^{\partial Q}/_{\partial x})$ is the z-component of curl(<u>v</u>), where <u>v</u> = (P,Q,0).

2nd March 2000

2.2. Work Done by a Force F moving along a Curve

Let <u>F</u> be defined in a region D of space, and let the curve C joining A and B be in D. The work done by <u>F</u> in the displacement d<u>r</u> is given by dW = <u>F</u>.d<u>r</u>. The total work done (W) when <u>F</u> moves its point of application from A to B *along* C is W = $\int_A^B$ <u>F</u>.d<u>r</u>. In *general*, W depends on the path C as well as on the **end points** A and B. If, however, W *only* depends on A and B, then W is said to be **<u>path independent,</u>** and the force <u>F</u> is said to be **<u>conservative</u>**.





Consider A, a fixed point. Then for a *conservative* force <u>F</u>, the work done by <u>F</u> in moving from A along *any path* to an arbitrary point P, W(P), will only depend on the <u>position</u> of P. W(P) is then a **<u>scalar field</u>** defined in D. Consider a *neighbouring point* P' to P, where $\overrightarrow{PP'}$ = d<u>r</u>. The work done by <u>F</u> in the *displacement* d<u>r</u> is given by dW = <u>F</u>.d<u>r</u>. But dW = W(P')-W(P) = <u>∇</u>(W).d<u>r</u>, where <u>∇</u>(W) is the *gradient* of the scalar field W.

Therefore, <u>F</u>.d<u>r</u> = <u>∇</u>(W).d<u>r</u>, or (<u>F</u>-<u>∇</u>W).d<u>r</u> = 0. But the d<u>r</u> is *arbitrary:* $\Rightarrow$ <u>F</u>-<u>∇</u>(W) = <u>0</u>, or <u>F</u> = <u>∇</u>(W). W is called a *potential function* for the force <u>F</u>. Conversely, if <u>F</u> = <u>∇</u>W, then the work done by <u>F</u> when its *point of application* moves along any path between points A and B **only** depends on the positions of A and B, *since* $\int_A^B$ <u>F</u>.d<u>r</u> = $\int_A^B$ (<u>∇</u>W).d<u>r</u> = $\int_A^B {}^{dW}/_{ds}ds = \int_A^B dW$ = W(B)-W(A). So a force F is *conservative* iff ∃ a potential function W such that <u>F</u> = <u>∇</u>(W).

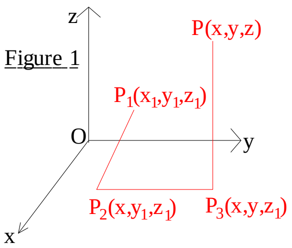
Convenient test for a conservative force. We know that curl(grad(f)) ≡ $\underline{0}$, so if $\underline{F}$ is *conservative*, then curl($\underline{F}$) = curl($\underline{\nabla}$(W)) = curl(grad(W)) ≡ $\underline{0}$. So curl($\underline{F}$) ≡ $\underline{0}$ is a *necessary* condition for $\underline{F}$ to be conservative. Note: we can **also** show that curl($\underline{F}$) ≡ $\underline{0}$ is a *sufficient* condition.

3rd March 2000

Proof of Sufficiency

Given curl($\underline{F}$) ≡ $\underline{0}$, we want to *show that* $\underline{F}$ = $\underline{\nabla}$(W) holds for some **scalar** field W. Now curl$\underline{F}$ ≡ 0 ⇒ the *determinant* shown is ≡ $\underline{0}$, where $\underline{F}$ = $F_1\underline{i}+F_2\underline{j}+F_3\underline{k}$. *Therefore,* $\frac{\partial}{\partial y}F_3 = \frac{\partial}{\partial z}F_2$; $\frac{\partial}{\partial z}F_1 = \frac{\partial}{\partial x}F_3$; and $\frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1$ (---(set A)).

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$



The work done by $\underline{F}$ in *moving* from $P_1(x,y,z_1)$ to P(x,y,z) along <u>any</u> path C is $\int_C \underline{F}.d\underline{r} = \int_C F_1dx+F_2dy+F_3dz$. Now, let C be the <u>particular</u> path shown in Figure 1, which consists of straight line segments from $P_1(x_1,y_1,z_1)$ to $P_2(x,y_1,z_1)$ to $P_3(x,y,z_1)$ to P(x,y,z). Let W(x,y,z) denote the work done along this **particular** path, so that $W = \int_{x_1}^{x} F_1(x_1,y_1,z_1)dx + \int_{y_1}^{y} F_2(x,y,z_1) + \int_{z_1}^{z} F_3(x,y,z)dz$.

It *follows* that $\frac{\partial W}{\partial z} = F_3(x,y,z)$; $\frac{\partial W}{\partial y} = F_2(x,y,z_1) + \int_{z_1}^{z} (\frac{\partial}{\partial y}F_3)(x,y,z)dz$ = (A) = $F_2(x,y,z_1) + \int_{z_1}^{z} (\frac{\partial}{\partial z}F_2)(x,y,z)dz = F_2(x,y,z_1) + [F_2(x,y,z)]_{z_1}^{z} = F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1) = F_2(x,y,z)$; **and** $\frac{\partial W}{\partial x} = F_1(x,y_1,z_1) + \int_{y_1}^{y}(\frac{\partial}{\partial x}F_2)(x,y,z_1)dy + \int_{z_1}^{z} (\frac{\partial}{\partial x}F_3)(x,y,z)dz$ = (A) = $F_1(x,y,z_1) + \int_{y_1}^{y}(\frac{\partial}{\partial y}F_1)(x,y,z_1)dy + \int_{z_1}^{z}(\frac{\partial}{\partial z}F_1)(x,y,z)dz = F_1(x,y_1,z_1) + [F_1(x,y,z_1)]_{y_1}^{y} + [F_1(x,y,z)]_{z_1}^{z} = F_1(x,y_1,z_1) + F_1(x,y,z_1) - F_1(x,y_1,z_1) + F_1(x,y,z) - F_1(x,y,z_1) = F_1(x,y,z)$.

So $\underline{F} = F_1\underline{i}+F_2\underline{j}+F_3\underline{k} = \frac{\partial W}{\partial x}\underline{i}+\frac{\partial W}{\partial y}\underline{j}+\frac{\partial W}{\partial z}\underline{k}$ = $\underline{\nabla}$(W), which proves *sufficiency*. So, a **necessary** and **sufficient** condition for a force $\underline{F}$ to be conservative is curl($\underline{F}$) ≡ $\underline{0}$.

8th March 2000

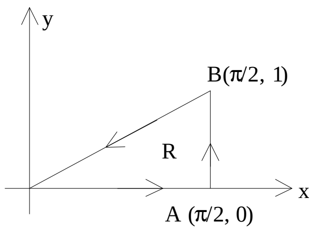
Examples

Q: Show that $\underline{F} = (2xy+z^3)\underline{i}+x^2\underline{j}+3xz^2\underline{k}$ (---(1)) is *conservative*. (b) Find the scalar potential function. (c) Find the **work** done along a path joining (1,-2,1) to (3,1,4). A: (a) A *necessary* and *sufficient* condition for $\underline{F}$ to be conservative is that curl $\underline{F}$ ≡ $\underline{0}$. *Here*, curl $\underline{F}$ = the determinant shown on the **right** = $0\underline{i}-(3z^2-3z^2)\underline{j}+(2x-2x)\underline{k} = \underline{0}$. (b) Part (a) ⇒ $\underline{F}$ = $\underline{\nabla}$(W) for some *scalar* field W. So (1) ⇒ $(2xy+z^3)\underline{i}+x^2\underline{j}+3xz^2\underline{k} = \frac{\partial W}{\partial x}\underline{i}+\frac{\partial W}{\partial y}\underline{j}+\frac{\partial W}{\partial z}\underline{k}$. As $\underline{i}$, $\underline{j}$ and $\underline{k}$ are linearly independent, then $\frac{\partial W}{\partial x} = 2xy+z^3$; $\frac{\partial W}{\partial y} = x^2$; and $\frac{\partial W}{\partial z} = 3xz^2$.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

So $W = x^2y+xz^3+f(y,z)$ (---(2)); $W = x^2y+g(z,x)$ (---(3)); and $W = xz^3+h(x,y)$ (---(4)), with f, g and h *arbitrary*. Choosing f = 0, $g = xz^3$, and $h = x^2y$ gives $W = x^2y+xz^3$, which satisfies <u>all</u> equations (2)-(4) (W is unique up to an *additive* constant). (c) The work done along any path is given by $W(3,1,4)-W(1,-2,1) = (3^2.1+3.4^3)-(1^2.-2+1.1^3) = 202$ **units**.

Q: **Evaluate** $\oint_C$ (y-sin(x))dx+cos(x)dy <u>directly</u>. Now $\int_C = \int_{OA}+\int_{AB}+\int_{BO}$. Therefore, $\int_{OA}$ (with y = 0, so Dy = 0) is $\int^{x=\pi/2}_{x=0}$ (y-sin(x))dx+cos(x)dy = $\int^{x=\pi/2}_{x=0}-\sin(x)dx = [\cos(x)]^{\pi/2}_0 = [0-1] = -1$. And $\int_{AI}$ = $\int_0^1\cos(^{\pi}/_2)dy = \int^1_0 0 = 0$. Consider $\int_{BO}$, with $^y/_x = {}^1/_{\pi/2} = \int^0_{x=\pi/2}$ (y-sin(x))dx+cos(x)dy = $\int^{x=0}_{x=\pi/2}$ $(^2/_\pi x-\sin(x))dx+\cos(x)(^2/_\pi)dx$ = ... = $1-{}^{\pi}/_4-{}^2/_\pi$. So $\int_C = -1+0+1-{}^{\pi}/_4-{}^2/\pi = -{}^{\pi}/_4-{}^2/_\pi$.



Using *Green's* Theorem in the plane, $\oint_C$ (y-sin(x))dx+cos(x)dy = $\iint_R$ (-sin(x)-1)dxdy = $\int^1_0\int^{\pi/2}_{\pi y/2}$ (-sin(x)-1)dxdy = $\int^1_0[\cos(x)-x]^{\pi/2}_{\pi y/2}dy$ = $\int^1_0[(\cos(^{\pi}/_2)-{}^{\pi}/_2)-(\cos(^{\pi y}/_2)-{}^{\pi y}/_2)]$ = $\int^1_0 {}^{\pi y}/_2-{}^{\pi}/_2-\cos(^{\pi y}/_2)dy = [^{\pi y^2}/_4-{}^{\pi y}/_2-{}^2/_\pi\sin(^{\pi y}/_2)]^1_0 = -{}^{\pi}/_4-{}^2/_\pi$. QED.

9th March 2000

<u>Functions of a Complex Variable</u>

<u>1. Introduction, Definitions and Continuity</u>

If w (= u+iv) and z (= x+iy) (u, v, x, y ∈ **R**) are any 2 *complex numbers*, then we might say that w is a function of z, w = f(z), if, to any value of z in a certain **domain** D, there corresponds any one value of w. This is exactly <u>analogous</u> to the definition of a function for real variables. It turns out that this definition is far too <u>*wide*</u>.

It is exactly the same as a function u(x,y)+iv(x,y) of two real variables x and y. For functions defined in this way, the definition of continuity is exactly the same as for **real** functions of real variables: the function f(z) is continuous at the point z_0 if given any ε > 0, there exists a $\delta_\varepsilon(z_0) > 0$ s.t. $|f(z)-f(z_0)| < \varepsilon$ for all z ∈ D satisfying $|z-z_0| < \delta_\varepsilon(z_0)$.

A function is *uniformly continuous* in D if given any ε > 0, there exists a $h_\varepsilon > 0$, independent of z_0, s.t. $|f(z)-f(z_0)| < \varepsilon$ for all z, z_0 ∈ D satisfying $|z-z_0| < h_\varepsilon$. So a function f(z) = u(x,y)+iv(x,y) is a *continuous* function of z iff the u(x,y) and the v(x,y) are <u>continuous</u> functions of x and y. The only class (of functions of z) which is of any *practical* use is that to which the process of <u>differentiation</u> can be applied.

<u>2. Differentiability</u>

The natural definition is: Let f(z) be a **one-valued** function defined in a domain D of the complex plane, then f(z) is differentiable at a point z_0 of D if $f(z)-f(z_0)/z-z_0$ tends to a <u>**unique**</u> limit as $z\to z_0$ through points of D. If the limit exists, it is called the *derivative* of f(z) at z = z_0, and is denoted by f'(z), or by $^{df}/_{dz}|_{z=z_0}$, or by $^{df}/_{dz}(z_0)$.

<u>**Rules of differentiation**</u>. All the rules of *elementary calculus* apply. Let f(z) and g(z) be <u>differentiable</u> functions in D. (1) $^d/_{dz}(f(z)\pm g(z)) = {}^{df}/_{dz}\pm{}^{dg}/_{dz}$. (2) $^d/_{dz}(cf(z)) = c{}^{df}/_{dz}$ (c is an *arbitrary complex constant*). (3) $^d/_{dz}(f(z)g(z)) = f(z){}^{dg}/_{dz}+g(z){}^{df}/_{dz}$. (4) $^d/_{dz}(^{f(z)}/_{g(z)}) = [g(z){}^{df}/_{dz}-f(z){}^{dg}/_{dz}]/[(g(z))^2]$, with g(z) ≠ 0. (5) If w = f(t) and t = g(z), then $^{dw}/_{dz} = {}^{dw}/_{dt}{}^{dt}/_{dz}$. (6) If w = f(z) and z = $f^{-1}(w)$, *then* $^{dw}/_{dz} = {}^1/_{dz/dw}$. (7) If z = f(t) and w = g(t), where t is a *parameter*, then $^{dw}/_{dz} = {}^{dw/dt}/_{dz/dt}$.

3. Analytic Functions

A function which is *one-valued and differentiable* in a domain D is said to be **analytic** in D. [A function is said to be analytic at a point z_0 if $f'(z_0)$ exists, and $f'(z)$ exists at *all* points z of some neighbourhood of the point z_0: $|z-z_0| < \delta > 0$]. A function may be *differentiable* in a domain D except at possibly a **finite** number of points. Such points are called singularities of f(z).

15th March 2000

The Necessary and Sufficient Conditions for $f(z)^n$ to be Analytic

If $f(x) = u(x,y)+iv(x,y)$ is *differentiable* at a given point z, then $^{f(z+\Delta z)-f(z)}/_{\Delta z}$ must tend to a *unique* limit as $\Delta z \to 0$ in any **possible** way. Now $\Delta z = \Delta x+i\Delta y$. Therefore, take Δz to be *wholly* real, so that $\Delta y = 0$, and then, using $z = x+iy$ and $\Delta z = \Delta x+i\Delta y$, $\{^{u(x+\Delta x,\ y)-u(x,y)}/_{\Delta x}\}+i\{^{v(x+\Delta x,\ y)-v(x,y)}/_{\Delta x}\}$ must tend to a *definite* limit as $\Delta x \to 0$. This implies that u_x and v_x must exist at (x,y), and that the *limit* is u_x+iv_x. (Note: $u_x = {}^{\partial u}/_{\partial x}$; $u_y = {}^{\partial u}/_{\partial y}$).

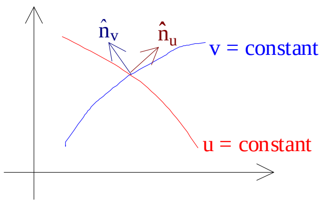
Similarly, if we take Δz to be wholly imaginary, i.e. $\Delta x = 0$, then $\{^{u(x,\ y+\Delta y)-u(x,y)}/_{i\Delta y}\}+i\{^{v(x,\ y+\Delta y)-v(x,y)}/_{i\Delta y}\}$ must tend to a *definite* limit as $\Delta y \to 0$, which is $(^1/_i)u_y+i(^1/_i v_y) = v_y-iu_y$. The **two** limits must be identical, so we must have $u_x = v_y$ and $u_y = -v_x$ (---(1)). (These are the **Cauchy-Riemann Equations, which are** *Partial* Differential Equations).

Note: it can be **shown** that the mere existence of the 4 derivatives (u_x, u_y, v_x and v_y) and the satisfying of the C-R equations is not sufficient to give differentiability. But, if the 4 derivatives are also **continuous**, then sufficiency *follows*. If $w = f(z) = u+iv$, where u and v are *functions* of x and y, then since $x = {}^1/_2(z+\bar{z})$ and $y = {}^1/_{2i}(z-\bar{z})$ (---(2)) ($z = x+iy, \bar{z} = x-iy$), we may **regard** u and v as functions of z and $\bar{z}$.

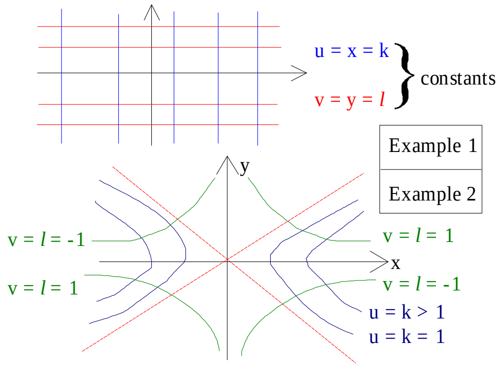
If u_x, u_y, v_x and v_y are continuous, then the *condition* $\partial w/\partial\bar{z} = 0 \Rightarrow (\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}) + i(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}) = 0$. And (2) $\Rightarrow ({}^{\partial u}/_{\partial x}(^1/_2)+{}^{\partial u}/_{\partial y}(-^1/_{2i}))+i({}^{\partial v}/_{\partial x}(^1/_2)+{}^{\partial v}/_{\partial y}(-^1/_{2i})) = 0 \Rightarrow (u_x-v_y)+i(u_y+v_x) = 0 \Rightarrow u_x = v_y$ and $u_y = -v_x$, the *C-R equations*. Hence a formula representing an analytic function of z can **only** have x and y in the combination x+iy. Thus $x+3iy = 2z\bar{z}$ (= 2x+2iy-(x-iy)) **cannot** be analytic. (Check the C-R equations — not satisfied).

Conjugate Functions

(C-R equations) $\Rightarrow (u_x = v_y, v_x = -u_y) \Rightarrow ({}^{\partial^2 u}/_{\partial x^2} = {}^{\partial^2 v}/_{\partial x\partial y}, {}^{\partial^2 v}/_{\partial y\partial x} = -{}^{\partial^2 u}/_{\partial y^2})$. So if the 2nd order *partial derivatives* are **continuous**, then $v_{xy} = v_{yx} \Rightarrow {}^{\partial^2 u}/_{\partial x^2}+{}^{\partial^2 u}/_{\partial y^2} = 0$ (**Laplace's** equations in x and y). Similarly, ${}^{\partial^2 v}/_{\partial x^2} = -{}^{\partial^2 u}/_{\partial x\partial y}$ and ${}^{\partial^2 u}/_{\partial y\partial x} = {}^{\partial^2 v}/_{\partial y^2} \Rightarrow {}^{\partial^2 v}/_{\partial x^2}+{}^{\partial^2 v}/_{\partial y^2} = 0$. So the C-R *equations* $\Rightarrow {}^{\partial^2 u}/_{\partial x^2}+{}^{\partial^2 u}/_{\partial y^2} = 0 = {}^{\partial^2 v}/_{\partial x^2}+{}^{\partial^2 v}/_{\partial y^2}$, Laplace's equations in 2 dimensions. They are called **Harmonic** functions, i.e. Harmonic functions $\phi(x,y)$ satisfy $\nabla^2\phi = {}^{\partial^2\phi}/_{\partial x^2}+{}^{\partial^2\phi}/_{\partial y^2} = 0$.



The C-R equations *imply* $u_x v_x = -v_y u_y \Rightarrow {}^{\partial u}/_{\partial x}{}^{\partial v}/_{\partial x} + {}^{\partial u}/_{\partial y}{}^{\partial v}/_{\partial y} = 0$, $({}^{\partial u}/_{\partial x}, {}^{\partial u}/_{\partial y}).({}^{\partial v}/_{\partial x}, {}^{\partial v}/_{\partial y}) = 0$. **Example 1**: w = z $\Rightarrow$ u = x, v = y. **Example 2**: $w = z^2 = (x+iy)^2 = x^2+2ix-y^2 = (x^2-y^2)+i(2xy)$ [u(x,y)+iv(x,y)]. Now $u_x = 2x$ and $v_y = 2x$, so $u_x = v_y$ (✓). And $u_y = -2y$ and $v_x = 2y$, so $u_y = -v_x$ (✓). **So** $u = k = x^2-y^2$, and $v = l = 2xy$. *Another* example: y = mx, with $x^2+y^2 = r^2$ (A "*Spider's Web*").



It is possible to construct a **function** f(z) which has a given *real* function of x and y for its real or imaginary part, if either of the functions u(x,y) or v(x,y) is a simple combination of elementary functions satisfying *Laplace's* equations.

Milne-Thompson Construction

Let $x = {}^1/_2(z+\bar{z})$, and let $y = {}^1/_{2i}(z-\bar{z})$; so that $f(z) = u(z+\bar{z}/2, z-\bar{z}/2i)+iv(z+\bar{z}/2, z-\bar{z}/2i)$. Now *put* $\bar{z} = z$, i.e. x = z and y = 0. So f(z) = u(z,0)+iv(z,0); $f'(z) = u_x+iv_x = $ (**C-R**) $= u_x+i(-u_y) = u_x-iu_y$. Therefore, if we *write* $\phi_1(x,y)$ and $\phi_2(x,y)$ for u_x and u_y respectively, we have $f'(z) = \phi_1(x,y)-i\phi_2(x,y) = \phi_1(z,0)-i\phi_2(z,0)$. *Integrating*, $f(z) = \int\phi_1(z,0)-i\phi_2(z,0)dz+C$. *Similarly*, if v(x,y) is given, we get $f(z) = \int\varphi_1(z,0)+i\varphi_2(z,0)dz+C$, *where* $\varphi_1(x,y) = v_y$ and $\varphi_2(x,y) = v_x$.

Assignment 3: Set 16/3; In 30/3; Back 14/4

Q: Using the *definition*, find the derivative of $w = f(z) = 3z^{-2}$ at the point where (a) z = 1+i; (b) z = 1-i. A: $f'(z) = \lim_{\Delta z\to 0} {}^{f(z+\Delta z)-f(z)}/_{\Delta z}$. **Here**, $f(z+\Delta z) = {}^3/_{(z+\Delta z)^2} = {}^3/_{z^2+2z\Delta z+\Delta^2 z}$. So $f'(z) = \lim_{\Delta z\to 0} [({}^3/_{z^2+2z\Delta z+\Delta^2 z})-({}^3/_{z^2})]/\Delta z = \lim_{\Delta z\to 0} {}^{3z^2-3(z^2+2z\Delta z+\Delta^2 z)}/_{\Delta z(z^2(z^2+2z\Delta z+\Delta^2 z))} = \lim_{\Delta z\to 0} {}^{3z^2-3z^2-6z\Delta z-3\Delta^2 z}/_{\Delta z(z^2(z^2+2z\Delta z+\Delta^2 z))} = \lim_{\Delta z\to 0} {}^{-6z-3\Delta z}/_{z^2(z^2+2z\Delta z+\Delta^2 z)} = {}^{-6z-3(0)}/_{z^2(z^2+2z(0)+(0)^2)} = -6z/z^4 = {}^{-6}/_{z^3} = -6z^{-3}$.

(a) **When** z = 1+i, $f'(z) = {}^{-6}/_{(1+i)^3}$. Now $(1+i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$. *And* $(1+i)^3 = 2i(1+i) = 2i+2i^2 = 2i-2 = 2(i-1)$. So $f'(z) = {}^6/_{2(i-1)} = -{}^3/_{i-1} = {}^3/_{1-i} = {}^3/_2(1+i)$. (b) *Similarly*, when z = 1-i, $f'(z) = {}^3/_2(1-i)$.

Q: Prove that ${}^d/_{dz}(z^2\bar{z})$ does not *exist anywhere* except at z = 0. A: $f'(z_0) = \lim_{z\to z_0} (z^2\bar{z}-z_0{}^2\bar{z}_0/z-z_0)$. Now *use* $z = z_0+\Delta z$ with $\bar{z} = \bar{z}_0+\Delta\bar{z}$ to *rearrange*, giving $\lim_{\Delta z\to 0}[[2z_0\bar{z}_0 + \bar{z}_0\Delta z + 2z_0\Delta\bar{z} + (\Delta z)(\Delta\bar{z})] + [z_0^2(\frac{\Delta\bar{z}}{\Delta z})]]$. **Letting** $\Delta z = \Delta x\to 0$, we *obtain* $2z_0\bar{z}_0+z_0{}^2(1)$. **Letting** $\Delta z = i\Delta y\to 0$, we *obtain* $2z_0\bar{z}_0+z_0{}^2(-1)$. Thus there is no **unique** limit when $z_0 \neq 0$, but f'(0) = 0!

Q: Verify that the *real* and *imaginary* parts of the function $w = f(z) = e^{z^2}$ satisfies the Cauchy-Riemann equations, and thus deduce the **analyticity** of the function. A: **Using** z = x+iy, $z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$. So $w = e^{(x^2-y^2)+i(2xy)} = e^{(x^2-y^2)}e^{i(2xy)}$. Using Euler's relation, $e^{i\theta} = \cos\theta+i\sin\theta$, we can say that $e^{i(2xy)} = \cos(2xy)+i\sin(2xy)$. Therefore, $w = e^{(x^2-y^2)}[\cos(2xy)+i\sin(2xy)] = (\cos(2xy))^{e(x^2-y^2)}+i[(\sin(2xy))^{e(x^2-y^2)}]$.

Here, $u = (\cos(2xy))e^{(x^2-y^2)}$, and $v = (\sin(2xy))^{e(x^2-y^2)}$. From these, we can *show that the Cauchy-Riemann equations* are satisfied. To deduce the analyticity of the function, we must show that the **four** derivatives u_x, u_y, v_x and v_y are continuous. These derivatives are *mulitplicative* or *additive* combinations of continuous functions, so the four derivatives *are* continuous. Therefore, we can say that w is analytic **everywhere**.

Q: Prove that the *function* $u = x^2-y^2-2xy-2x+3y$ is harmonic. Find a function v such that $f(z) = u+iv$ is **analytic**, i.e. find the conjugate function of u. Express f(z) in *terms* of z. A: By showing that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, we can prove that u is *harmonic*. To find a function v such that u+iv is analytic, let us use the **Cauchy-Riemann** equations. Firstly, $u_x = v_y$, so that $v_y = 2x-2y-2$, and $v = \int 2x-2y-2dy = 2xy-y^2-2y+f(x)$. Secondly, $u_y = -v_x$, so that $v_x = 2y+2x-3$, and $v = \int 2y+2x-3dx = 2yx+x^2-3x+g(y)$. (f(x) and g(y) are *constants*, independent of y and x respectively). Choosing $f(x) = x^2-3x$, and $g(y) = -y^2-2y$, we have $v = 2xy+x^2-y^2-3x-3y+c$. (**Choose** $c = 0$). Now *check* that $v_y = u_x$, and that $v_x = -u_y$, so we have $f(z) = u+iv = x^2-y^2-2xy-2x+3y+i(2xy+x^2-y^2-3x-2y)$. *Now* $z = x+iy$; $-2z = -2x-2iy$; $-3iz = -3ix+3y$; $z^2 = x^2+2ixy-y^2$; and $iz^2 = ix^2-2xy-iy^2$. So $f(z) = (-2x-2iy) + (-3ix+3y) + (x^2+2ixy-y^2) + (ix^2-2xy-iy^2) = (-2x)+(-3iz)+(z^2)+(iz^2) = z^2+iz^2-2z-3iz$.

17th March 2000

Tutorial

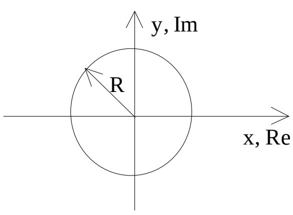
Let $u(x,y) = e^x(x\cos y-y\sin y)$. Verify that u *satisfies* Laplace's equations, $u_{xx}+u_{yy} = 0$. Now $\phi_1(x,y) = u_x = e^x(x\cos(y)-y\sin(y))+e^x(\cos(y)) = e^x(x\cos(y)-y\sin(y)+\cos(y))$; and $\phi_2(x,y) = u_y = e^x(-x\sin(y)-(1.\sin(y)+y\cos(y))) = -e^x(x\sin y+\sin y+y\cos y)$. Therefore, $f'(z) = \phi_1(z,0)-i\phi_2(z,0) = e^z(z+1)+i(e^z(0)) = e^z(1+z) \Rightarrow f(z) = \int e^z(1+z)dz = \int(1+z)d(e^z) = (1+z)e^z-\int e^z dz = (1+z)e^z-e^z+C$; $f(z) = ze^z+C$. *(C is an arbitrary complex constant).*

Q: Using the *definition*, find the derivative of $w = f(z) = z^3-2x$ at the point **where** (a) $z = z_0$; (b) $z = -1$; and (c) $z = i$. A: Use the fact that ${}^{f(z+\Delta z)-f(z)}/_{\Delta z}$ must tend to a *unique* limit as $\Delta z \rightarrow 0$. Now $f(z+\Delta z) = (z+\Delta z)^3-2(z+\Delta z) = (z^2+2z\Delta z+\Delta^2 z)(z+\Delta z)-2(z+\Delta z) = z^3+z^2\Delta z+2z^2\Delta z+2z\Delta^2 z+z\Delta^2 z+\Delta^3 z-2z-2\Delta z$. **And** $f(z+\Delta z)-f(z) = (\ldots)-(z^3-2z) = z^2\Delta z+2z^2\Delta z+2z\Delta^2 z+z\Delta^2 z+\Delta^3 z-2\Delta z$. Therefore, it follows that ${}^{f(z+\Delta z)-f(z)}/_{\Delta z} = z^2+2z^2+2z\Delta z+z\Delta z+\Delta^2 z-2$. As $\Delta z \rightarrow 0$, ${}^{f(z+\Delta z)-f(z)}/_{\Delta z} \rightarrow 3z^2-2$. (a) *When* $z = z_0$, $f'(z) = 3z_0^2-2$. (b) *When* $z = -1$, $f'(z) = 3-2 = 1$. (c) *When* $z = i$, $f'(z) = 3(i)^2-2 = -3-2 = -5$.

22nd March 2000

Section 3: Power Series

Consider $\Sigma_{n=0}^{\infty} a_n z^n$. As far as *absolute convergence* is concerned, everything that has been proved for absolutely convergent series of **real** series extends to complex series, because the series of *moduli* $|a_0| + |a_1||z| + |a_2||z|^2+\ldots$ is a series of +ve terms. By Cauchy's root test, a series of +**ve** terms $\Sigma_n u_n$ is *convergent* or *divergent* according to whether $\lim_{n\rightarrow\infty}((u_n)^{1/n})$ is < 1 or > 1. *Letting* $u_n = |a_n z^n|$, we get $\lim_{n\rightarrow\infty}(u_n)^{1/n} = \lim_{n\rightarrow\infty}|a_n z^n|^{1/n} = |z|\lim_{n\rightarrow\infty}|a_n|^{1/n}$, and if we let $\lim_{n\rightarrow\infty}|a_n|^{1/n} = {}^1/_R$, then we see that the *power* series $\Sigma_{n=0}^{\infty} a_n z^n$ is absolutely *convergent* if $|z| < R$, and *divergent* if $|z| > R$. (If $|z| = R$, **anything** can happen). The number R is called the radius of convergence, and the circle with centre at the origin and *radius* R is called the circle of convergence.



Important Theorem: If $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, then f(z) is *analytic* at every point within the circle of convergence of the power series. This means that a power series may be **differentiated** or **integrated** term-by-term as often as we like (and at any point) within its circle of convergence. The converse of the above is also *true:* if f(z) is analytic in a region, it can be expanded as a power series.

Elementary Functions of a Complex Variable

(I) **Rational Functions**. A polynomial in z: $a_0+a_1z+a_2z^2+...+a_nz^n$ may be regarded as a *power series* which converges for all values of z. As each such function is analytic in the whole complex plane, then rational functions: $f(z) = \frac{a_0+a_1z+...+a_mz^m}{b_0+b_1z+...+b_kz^k}$ are analytic at **all** points of a complex plane, except where the denominator *vanishes*.

(II) **Exponential Functions**. We define exp(z) as the *sum* function of the series of complex terms: $\exp(z) = 1+{}^{z}/_{1!}+{}^{z^2}/_{2!}+{}^{z^3}/_{3!}+...+(z^n/n!)+...$ Since this series converges *for all z* in the complex plane, it defines an **analytic** function in the whole complex plane. Such functions are called *integral* functions.

When z is real ($z = x$, $x \in \mathbf{R}$), then exp(z) is identical to the real exponential function, $\exp(x) = e^x$. When z is complex, it is *convenient* to write e^z for exp(z), since the formula $\exp(z)\exp(\zeta) = \exp(z+\zeta)$ can be proved by **multiplication** of series, where z is *real* or *complex*. The real number e with complex exponent obeys $e^z.e^\zeta = e^{z+\zeta}$.

So, we may define the *power* e^z unambiguously by $e^z = 1+{}^{z}/_{1!}+{}^{z^2}/_{2!}+...+(z^n/n!)+....$ If a is any **positive** number, then a^z is unambiguously *defined* by $a^z = e^{z\log(a)}$. ($\log(a) = \ln(a)$). Now take $z = iy$ ($x = 0$) — z *pure imaginary*. Then $e^z = e^{iy} = 1+{}^{(iy)}/_{1!}+{}^{(iy)^2}/_{2!}+{}^{(iy)^3}/_{3!}+... = (1-{}^{y^2}/_{2!}+(y^4/4!)-...) + i(y-{}^{y^3}/_{3!}+(y^5/5!)-...)$. So $e^{iy} = \cos(y)+i\sin(y)$.

Hence $e^z = e^{(x+iy)} = e^xe^{iy} = e^x(\cos(y)+i\sin(y))$. [**Note**: sometimes, cos(y)+isin(y) is denoted by cis(y)]. *Also,* $|e^{iy}| = |\cos(y)+i\sin(y)| = 1$; $|e^z| = |e^x||e^{iy}| = |e^x| = e^x$ ($e^x > 0$); and $\arg(e^z) = y = \mathrm{Im}(z)$. Further, $e^{(z+2\pi i)} = e^ze^{2\pi i} = e^z$ ($e^{2\pi i} = 1$). So e^z has a ***period*** of $2\pi i$. Finally, *term by term* differentiation of the power series for e^z gives ${}^{d}/_{dz}(e^z) = e^z$.

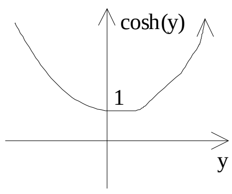
23rd March 2000

(III) **Trigonometric and Hyperbolic functions**. We define the functions sin(z) and cos(z) (when z is complex) as the *sum* functions of power series, just as we do for sin(x) and cos(x) when x is real. So define $\sin(z) = z-{}^{z^3}/_{3!}+(z^5/5!)-... = \Sigma_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$; $\cos(z) = 1-{}^{z^2}/_{2!}+(z^4/4!)-... = \Sigma_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!}$, and as both have *infinite radius of convergence*, both are **integral** functions.

Term-by-term differentiation $\Rightarrow$ ${}^{d}/_{dz}(\sin(z)) = \cos(z)$; ${}^{d}/_{dz}(\cos(z)) = -\sin(z)$. The *other trigonometric* functions are defined as $\tan(z) = {}^{\sin(z)}/_{\cos(z)}$; $\cot(z) = {}^{1}/_{\tan(z)}$; $\sec(z) = {}^{1}/_{\cos(z)}$; and $\mathrm{cosec}(z) = {}^{1}/_{\sin(z)}$. From the **series** definitions of cos(z), sin(z) and e^z, we obtain $\cos(z)+i\sin(z) = e^{iz}$, and $\cos(z)-i\sin(z) = e^{-iz}$.

These *Euler formulae* give rise to $\cos^2(z)+\sin^2(z) = (^1/_2(e^{iz}+e^{-iz}))^2+(^1/_{2i}(e^{iz}-e^{-iz}))^2 = 1$ (using $\cos(z) = {}^1/_2(e^{iz}+e^{-iz})$ and $\sin(z) = {}^1/_{2i}(e^{iz}-e^{-iz})$). Also, the *addition* formulae hold, that is $\sin(z\pm\zeta) = \sin(z)\cos(\zeta)\pm\cos(z)\sin(\zeta)$; and $\cos(z\pm\zeta) = \cos(z)\cos(\zeta)\mp(\sin(z)\sin(\zeta)$.

The *hyperbolic functions* of a complex variable are defined in the **same** way as for real variables: $\sinh(z) = {}^1/_2(e^z-e^{-z})$; $\cosh(z) = {}^1/_2(e^z+e^{-z})$; $\tanh(z) = {}^{\sinh(z)}/_{\cosh(z)}$; etc. **Note**: $\sin(iz) = {}^1/_{2i}(e^{iiz}-e^{-iiz}) = {}^1/_{2i}(e^{-z}-e^z) = -{}^1/_{2i}(e^z-e^{-z}) = {}^i/_2(e^z-e^{-z}) = i\sinh(z)$. **Similarly**, $\cos(iz) = \cosh(z)$, $\sinh(iz) = i\sin(z)$, *and* $\cosh(iz) = \cos(z)$.



Now $\sin(z) = \sin(x+iy) = \sin(x)\cos(iy)+\cos(x)\sin(iy) = \sin(x)\cosh(y)+i\cos(x)\sinh(y)$. Now *if* $\sin(z) = 0$, then $\sin(x)\cosh(y) = 0$ (---(1)) and $\cos(x)\sinh(y) = 0$ (---(2)). But $\cosh(y) \geq 1$ as *shown*, so $\sin(x) = 0$ in (1) $\Rightarrow x = n\pi$ ($n = 0,\pm1,\pm2,\ldots$). So $\cos(x)$ in (2) $= (-1)^n$, and so (2) $\Rightarrow \sinh(y) = 0 \Rightarrow y = 0$. *Therefore*, $\sin(z) = 0 \Rightarrow z = x+iy = n\pi$ ($n = 0,\pm1,\pm2,\ldots$). *Similarly*, $\cos(z) = 0 \Rightarrow z = (n+{}^1/_2)\pi$ ($n = 0,\pm1,\pm2,\ldots$).

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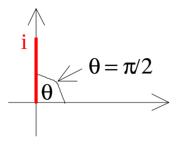
(IV) **The logarithmic function**. When x is *real* and *positive*, $e^u = x$ has 1 real solution, $u = \log_e(x)$. If z is complex, but not *zero*, then $e^w = z$ has an **infinite** number of solutions, each of which is called a *logarithm* of z. Let $w = u+iv$, then $e^w = z \Rightarrow e^{u+iv} = e^ue^{iv} = e^u(\cos u+i\sin v) = z = r(\cos\theta+i\sin\theta) \Rightarrow e^u = r$ (so $u = \log_e(r) = \log_e(|z|)$ *and* $v = \theta = \arg(z)$.

So $w = \mathrm{Log}(z) = \log(|z|)+i\arg(z)$. **Note** that $\arg(z)$ has an *infinite* number of values. The principal value of $\arg(z)$ is $-\pi < \arg(z) \leq \pi$. So the *principal* value of Log(z), which is obtained by giving arg(z) its principal value ($-\pi < \arg(z) \leq \pi$), is denoted by log(z), since it is *identical* to the ordinary logarithm when z is real and positive.

So $\mathrm{Log}(z) = \log(z)+2k\pi i$ ($k = 0,\pm1,\pm2,\ldots$), and $\log(z) = \log_e(|z|)+i\arg(z)$ ($-\pi < \arg(z) \leq \pi$). The *exponential* and *logarithm* functions may be used to define complex **powers** of complex numbers. We define $\zeta^z = e^{z\mathrm{Log}(\zeta)}$, with $\zeta \neq 0$. $\mathrm{Log}(\zeta)$ is *multi-valued*, so ζ^z is multi-valued. The *principal* value of ζ^z occurs when $\zeta^z = e^{z\log(\zeta)}$, i.e. where $\log(\zeta)$ is the principal *value* of $\mathrm{Log}(\zeta)$.

Example: $i^i = e^{i\mathrm{Log}(i)} = e^{i(\log(i)+2k\pi i)}$ ($k = 0,\pm1,\pm2,\ldots$). **But** $\log(i) = \log_e|i|+i\arg(i) =$ (as $|i| = 1$ so that $\log_e|i| = 0$; *and as* $\arg(i) = {}^\pi/_2$) $= 0+i{}^\pi/_2 = i{}^\pi/_2$. Therefore, $i^i = e^{i(i(\pi/2)+2k\pi i)} = e^{-((\pi/2)+2k\pi)}$ ($k = 0,\pm1,\pm2,\ldots$). The *principal value* of i^i is given by $k = 0$, i.e. $i^i = \exp(-{}^\pi/_2)$. *Alternatively*, $i = e^{i(\pi/2)}e^{2k\pi i}$, **where** $e^{2k\pi i} = 1$ and $k = 0, \pm1, \pm2, \ldots$. So $i = e^{i((\pi/2)+2k\pi)}$. Raise *both sides* to the power i, so that (as before) $i^i = e^{i((\pi/2)+2k\pi)i} = e^{-((\pi/2)+2k\pi)}$.



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Complex Integration

The **definite** integral of a complex valued function along a curve in the complex plane can be expressed in terms of two line integrals *as follows*: Let f(z) be analytic in a domain D of the complex plane, and let C denote a **piecewise** smooth curve in D. Then $\int_C f(z)dz = \int_C(u+iv)(dx+idy) = \int_C(udx-vdy) + i\int_C(vdx+udy)$.

If $|f(z)| \le M$ on C, and l is the **length** of C, then $|\int_C f(z)dz| \le Ml$ (---(1)). **Proof:** Let C be given by $x = x(t)$ and $y = y(t)$ for $t_0 \le t \le t_1$, then $|\int_C f(z)dz| = |\int_C (u+iv)(dx+idy)| = \int_C|u+iv||dx+idy| \le \int_C M|dx+idy| = M \int_C|(\dot{x}+i\dot{y})dt|$ = (because $dt \geq 0$) $= M \int_{C}+ \sqrt{(\dot{x}^2+\dot{y}^2)}\, dt = Ml$. *Notes*: $|u+iv| \le M$ by *assumption*; $\int_C \sqrt{(\dot{x}^2+\dot{y}^2)}\, dt$ is l, or $\int_C \sqrt{(\dot{x}^2+\dot{y}^2)}\, dt$ is the **length** of C.

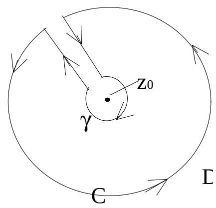
Cauchy's Theorem

Note: all integrals are *anticlockwise* unless otherwise indicated. **Theorem**: If f(z) is analytic, and if f'(z) is *continuous* in a simply connected domain D, then $\oint_C f(z)dz = 0$ (---(2)) on every *closed contour* in D.

The **Proof** uses Green's Theorem. (If P(x,y), Q(x,y), $^{\partial Q}/_{\partial x}$ and $^{\partial P}/_{\partial y}$ are all continuous in D, then we have $\oint_C Pdx+Qdy = \iint_R(^{\partial Q}/_{\partial x}-^{\partial P}/_{\partial y})dxdy$). **Now** $\oint_C f(z)dz = \oint_C(udx-vdy)+i\oint_C(vdx+udy)$ = (by *Green's Theorem*) $= -\iint_R(^{\partial v}/_{\partial x}+^{\partial u}/_{\partial y})dxdy+i\iint_R(^{\partial u}/_{\partial x}-^{\partial v}/_{\partial y})dxdy$. Using *Cauchy-Riemann*: $u_x = v_y$ or $u_x-v_y = 0$; *and* $u_y = -v_x$ or $v_x+u_y = 0$; then we **have** $\oint_C Pdx+Qdy = -\iint_R 0dxdy+i\iint_R 0dxdy = 0$.

Note: Goursat showed that it was *not necessary* to assume the continuity of f'(z) — the theorem also holds provided f'(z) exists everywhere in D.

Cauchy's Integral



If f(z) is **analytic** in a simply connected domain D (and on its *boundary* C), and if z_0 is a point within C, then $f(z_0) = {}^1/_{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$ (---(3)). Describe *about* $z = z_0$ a small circle γ of radius δ lying entirely within C. In the **region** between C and γ, the function $f(z)/z-z_0$ is analytic, but the region is **not** simply connected.

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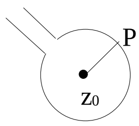
By making a **cross-cut** joining any point of γ to any point of C, we form a closed contour Γ bounding a region which **is** *simply* connected, so by Cauchy's Theorem, $\oint_\Gamma(f(z)/z-z_0)dz = 0$. In traversing Γ, the cross-cut is traversed *twice* in the opposite direction, so the integrals here cancel out. So ${}_C\oint(f(z)/z-z_0)dz + {}_\gamma\oint(f(z)/z-z_0)dz = 0$. (*The second* $\oint$ is **clockwise**). Or, ${}_C\oint(f(z)/z-z_0)dz-{}_\gamma\oint(f(z)/z-z_0) = 0$. (Both $\oint$ are *anticlockwise).*



Now **using** $f(z) = f(z_0)+f(z)-f(z_0)$, ${}^1/_{2\pi i}\,{}_\gamma\oint(f(z)/z-z_0)dz = {}^1/_{2\pi i}\oint(f(z_0)/(z-z_0))dz + {}^1/_{2\pi i}\oint(f(z)-f(z_0)/z-z_0)dz$. Looking at the **diagram**, we know that $z = z_0+\delta e^{i\theta}$, so $dz = \delta ie^{i\theta}d\theta$. But on δ, $z-z_0 = \delta e^{i\theta}$ for $0 \le \theta \le 2\pi$; **so** ${}^1/_{2\pi i}\,{}_\gamma\oint(f(z_0)/z-z_0)dz = {}^1/_{2\pi i}\int_{\theta=0}^{\theta=2\pi}\frac{f(z_0)}{\delta e^{i\theta}}\delta ie^{i\theta}d\theta = {}^{f(z_0)}/_{2\pi}\int_{\theta=0}^{\theta=2\pi}d\theta = {}^{f(z_0)}/_{2\pi}[2\pi] = f(z_0)$.

Also, $|{}^1/_{2\pi i}\,{}_\gamma\oint(f(z)-f(z_0)/z-z_0)dz| \le$ (by (1)) ${}^1/_{(2\pi)}\frac{\max_\gamma|f(z)-f(z_0)|}{\delta}.2\pi\delta \to 0$ as $\delta \to 0$, since f(z) is *continuous* at $z = z_0$ — this **proves** the theorem. Further, $f^{(n)}(z_0) = {}^{n!}/_{2\pi i}\,{}_C\oint(f(z)/(z-z_0)^{n+1})dz$. Using *Cauchy's Integral*, $f(z_0+\delta z_0)-f(z_0)/\delta z_0 = ({}^1/_{2\pi i})_C\oint(f(z)/\delta z_0)\{\frac{1}{z-z_0-\delta z_0} - \frac{1}{z-z_0}\}dz = ({}^1/_{2\pi i})_C\oint(f(z)/\delta z_0)\{\frac{z-z_0-(z-z_0-\delta z_0)}{(z-z_0-\delta z_0)(z-z_0)}\}dz = ({}^1/_{2\pi i})_C\oint\frac{f(z)dz}{(z-z_0-\delta z_0)(z-z_0)} \to ({}^1/_{2\pi i})_C\oint(f(z)/(z-z_0)^2)dz$ as $\delta z_0 \to 0$.

Therefore, $f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$ (---(4)). If the process is *repeated n* times, then $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$ (---(5)). (For the n^{th} derivative of an **analytic** function f(z) at an *interior* point of its domain).

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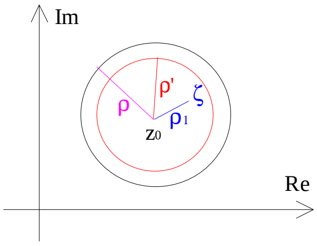
Taylor's Theorem



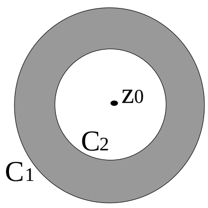
If f(z) is *analytic* within and on a circle $|z-z_0| \le \rho$, and if ζ is a point such that $|\zeta-z_0| = \rho_1$ ($< \rho$), then $f(\zeta) = \Sigma_{n=0}^{\infty} a_n(\zeta-z_0)^n$ (---(6)), where $a_n = f^{(n)}(z_0)/n!$. **Proof**: We consider the *identity* $^1/_{z-\zeta} = \frac{1}{z-z_0} + \frac{\xi-z_0}{(z-z_0)^2} + \ldots + \frac{(\xi-z_0)^{n-1}}{(z-z_0)^n} + \frac{(\xi-z_0)^n}{(z-z_0)^n(z-\xi)}$. Let $S_n = 1+r+r^2+\ldots+r^n$; so that $rS_n = r+r^2+\ldots+r^n+r^{n+1} \Rightarrow S_n(1-r) = 1-r^{n+1}$; $S_n = (1-r^{n+1})/(1-r)$. Therefore, the **RHS** of the identity, *except for the final term*, is $1/(z-z_0)\{1+r+r^2+\ldots+r^{n-1}\}$, where $r = (\zeta-z_0)/(z-z_0)$. So we **have** $\frac{1}{z-z_0}(1-(\frac{\xi-z_0}{z-z_0})^2/1-(\frac{\xi-z_0}{z-z_0})) = (1-(\frac{\xi-z_0}{z-z_0})^n)/\xi - z$, and the *identity* is therefore **true**.

Now *multiply each term* of the identity by $^{f(z)}/_{2\pi i}$, and integrate round the contour C, where C is the *circle* of ρ' with centre $z = z_0$, where $\rho_1 < \rho' < \rho$. **Then, it follows that**

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-\xi)} = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-z_0} + \frac{(\xi-z_0)}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^2} + \ldots + \frac{(\xi-z_0)^{n-1}}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^n} + \frac{(\xi-z_0)^n}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^n(z-\xi)}.$$

Now Cauchy's Integral (3) $\Rightarrow f(\zeta) = f(z_0) + {}^{(\zeta-z_0)}/_{1!}f'(z_0) + {}^{(\zeta-z_0)^2}/_{2!}f''(z_0) + \ldots + \frac{(\xi-z_0)^{n-1}}{(n-1)!}f^{(n-1)}(z_0) + R_n$, *where* $R_n = (\xi-z_0)^n\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^n(z-\xi)}$. If $|f(z)| \le M$ on C, *then* $|R_n| \le \frac{\rho_1^n}{2\pi}\frac{M}{(\rho')^n(\rho'-\rho_1)}2\pi\rho' = k(\rho_1/\rho')^n$, where k is a constant *independent* of n. Since $\rho_1 < \rho'$, then $|R_n| \to 0$ as $n \to \infty$, *proving* our theorem.

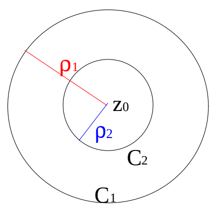
Laurent's Theorem



Let C_1 and C_2 be two circles with centres z_0 and radii ρ_1 and ρ_2 ($\rho_2 < \rho_1$). It follows that f(z) is analytic on the *circles* and within the annulus between C_1 and C_2, and if ζ is any point of the annulus, then $f(\zeta) = \Sigma_{n=0}^{\infty} a_n(\zeta-z_0)^n + \Sigma_{n=1}^{\infty} b_n(\zeta-z_0)^{-n}$ (---(8)), *where* $a_n = \frac{1}{2\pi i}\oint_{C1} \frac{f(z)dz}{(z-z_0)^{n+1}}$ (---(9a)), and $b_n = \frac{1}{2\pi i}\oint_{C2} (z-z_0)^{n-1}f(z)dz$ (---(9b)).

5th April 2000

Zeros, Poles and Residues of an Analytic Function

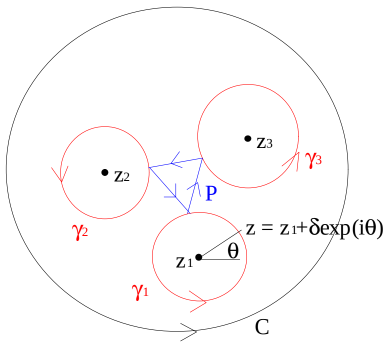


Consider the **Taylor** Expansion (6) $f(z) = \Sigma_0^{\infty} a_n(z-z_0)^n$ (---(15)). If $a_0 = a_1 = \ldots = a_{m-1} = 0$, and if $a_m \neq 0$, then the *first* term in (15) is $a_m(z-z_0)^m$. In **this** case, f(z) is said to have a *zero of order m* at $z = z_0$. A singularity of a function f(z) is a **point** where f(z) is <u>not</u> analytic. If $z = z_0$ is an isolated *singularity* of f(z), then f(z) has a <u>Laurent</u> expansion in the annulus between the two circles C_1 and C_2.

$f(z) = \Sigma_0^\infty a_n(z-z_0)^n + \Sigma_1^\infty b_n(z-z_0)^{-n}$ (---(16)), where the *principal* part of f(z) at $z = z_0$ is the red part. It may *happen* that $b_m \neq 0$, but we could have $b_{m+1}, b_{m+2}, \ldots, b_\infty = 0$; i.e. the *principal* part consists of a finite number of terms: $b_1/(z-z_0) + b_2/(z-z_0)^2 + \ldots + b_m/(z-z_0)^m$ ---(17)), and the *singularity* at $z = z_0$ is called a **pole** of order m of f(z). The coefficient b_1 is called the *residue* of f(z) at the pole $z = z_0$. If the **pole** is of order 1 (called a *simple* pole), then (16) $\Rightarrow b_1 = \lim_{z\to z_0}[(z-z_0)f(z)]$ (---(18)). For a pole of *order n*, $b_1 = \lim_{z\to z_0}[{}^1/_{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]]$. Example: for **m = 2**, $b_1(z-z_0)^2/z-z_0 = b_1(z-z_0) \Rightarrow$ (by d^{2-1}/dz^{2-1}) b_1.

Cauchy's Residue Theorem

$_C\oint f(z)dz = (2\pi i)\Sigma(\text{Residues})$ (---(19)), where f(z) is *analytic* within and on a closed contour C, except at a **finite** number of poles within C, where Σ(Residues) (= ΣR) denotes the sum of the *residues* of f(z) at all of its poles within C. **Proof**. Let $z_1, z_2, \ldots, z_n$ be the *n* poles within C. Draw a set of circles γ_r, of radius δ and centre z_r, which do not **intersect**, and which all lie *within* C. (r = 1,...,n).



Now f(z) is certainly analytic in the region between C and the small circles γ_r. We can therefore *deform* C until it consists of the small circles γ_r and a polygon P which *joins* together the small circles γ_r. It follows that $_C\oint f(z)dz = {}_P\oint f(z)dz + \Sigma_{r=1}^n {}_{\gamma r}\oint f(z)dz$, where the *first term* is 0 because f(z) is **analytic** within P. So $_C\oint f(z)dz = \Sigma_{r=1}^n {}_{\gamma r}\oint f(z)dz$ (---(20)).

If z_r is a **pole** of order m, then $f(z) = \phi(z) + \Sigma_{s=1}^m b_s/(z-z_r)^s$, where φ(z) is *analytic within and on* γ_r. Hence $_{\gamma r}\oint f(z)dz = \Sigma_{s=1}^m {}_{\gamma r}\oint [b_s/(z-z_r)^s]dz$. Now let $z = z_r+\delta e^{i\theta}$ (so $dz = \delta ie^{i\theta}d\theta$), for $0 \le \theta \le 2\pi$, and so $_{\gamma r}\int f(z)dz = \Sigma_{s=1}^m b_s\int_{\theta=0}^{\theta=2\pi}\frac{\delta ie^{i\theta}d\theta}{\delta^s e^{is\theta}} = \Sigma_{s=1}^m b_s\, \delta^{1-s}\int_{\theta=0}^{\theta=2\pi} e^{i(1-s)\theta}id\theta$. *Now* $\int_{\theta=0}^{\theta=2\pi} e^{i(1-s)\theta}id\theta =$ (*when* $s \neq 1$) $= [\frac{e^{i(1-s)\theta}}{i(1-s)}]_0^{2\pi} = 0$. And $\int_{\theta=0}^{\theta=2\pi} e^{i(1-s)\theta}id\theta =$ (*when* s = 1) $= \int_{\theta=0}^{\theta=2\pi} id\theta = [i\theta]_0^{2\pi} = 2\pi i$. So $_{\gamma r}\oint f(z)dz = 2\pi ib_1$ (---(21)), where b_1 is the *residue* at $z = z_r$. Therefore, $_C\oint f(z)dz = \Sigma_{r=1}^n {}_{\gamma r}\oint f(z)dz = 2\pi i\Sigma R$, where ΣR = *the sum of the residues* of f(z) at all its **poles** within C.

6th April 2000

Calculation of Residuals

From equations (16) and (17) (and the *accompanying* discussion), if f(z) has a pole of **order** m at $z = z_0$, then $f(z) = \Sigma_0^\infty a_n(z-z_0)^n + \Sigma_1^m b_n/(z-z_0)^n$. (Blue part = φ(z)). The *residue* of f(z) at $z = z_0$ is b_1. If m = 1, then $f(z) = \phi(z) + (b_1/z-z_0)$. *Multiply* by $z-z_0$ and take **limits** as $z\to z_0$, so that $(z-z_0)f(z) = \phi(z)(z-z_0)+b_1$; $b_1 = \lim_{z\to z_0}[(z-z_0)f(z)]$.

For a pole of **order** m, $b_1 = \lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$. Let f(z) have a *pole of order* 1 at $z = z_0$, with residue b_1. If g(z) is **analytic** at $z = z_0$, then f(z)g(z) has *residue* $b_1g(z_0)$. If f(z) has a *simple* pole at $z = z_0$, then Laurent $\Rightarrow f(z) = \Sigma_0^\infty a_n(z-z_0)^n + (b_1/z-z_0)$. Now *multiply* by g(z) (which is analytic at z_0), so that $f(z)g(z) = \phi(z)g(z) + (b_1g(z)/z-z_0)$. Multiply by $(z-z_0)$, and take the *limit* as $z \to z_0$, so that $\lim_{z\to z_0}[(z-z_0)f(z)g(z)] = b_1g(z_0)$. So the *residue* of f(z)g(z) at z_0 is $b_1g(z_0)$. (b_1 is the residue of f(z) at z_0).

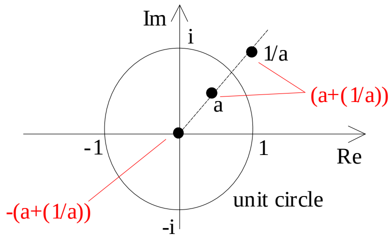
Examples

Determine the *residue* of the following **functions** at their poles. (i) $f(z) = {}^4/_{1-z} = -{}^4/_{z-1}$. A: In $f(z) = \Sigma_0^\infty a_n(z-z_0)^n + \Sigma_1^\infty (b_n/(z-z_0)^n)$, the first *summation* is zero, while in the second, $b_n = -4$, and $z_0 = 1$. Or, we **have** $(z-z_0)f(z)$ = (with $z_0 = 1$) = ${}^{(z-1)4}/_{(1-z)}$ = (take the *limit*) = -4. So $\lim_{z\to 1} (z-1)f(z) = -4$.

(ii) $F(z) = {}^z/_{(z-a)(z-b)(z-c)}$ $(a \neq b \neq c) = ({}^1/_{z-a})({}^z/_{(z-b)(z-c)}) = f(z)g(z)$. A: Now $f(z) = {}^1/_{z-a}$ has a *simple pole* at z = a, with the residue at z = a being 1. g(z) is **analytic** at z = a, so f(z)g(z) also has a *simple* pole at z = a, with the residue of $f(z)\times g(a)$ being $1\times{}^a/_{(a-b)(a-c)}$. Similarly for the *other* two poles (b and c).

7th April 2000

<u>**Example**</u>: $f(z) = z^4-1/z^2(z-a)(z-{}^1/_a)$. There is a pole of *order 2* at z = 0. So $\text{Res}(f(z), 0) = \lim_{z\to z0}[{}^1/_{(2-1)!}(d^{2-1}/dz^{2-1})(z^2f(z))] = \lim_{z\to 0}[{}^d/_{dz}(z^4-1/(z-a)(z-{}^1/_a)) = \ldots = -(a+{}^1/_a)$. To get the *two other residues*, split f(z) up as necessary. From this, we get a **pole** at z = a with residue $a+{}^1/_a$, and a pole at $z = {}^1/_a$ with residue $a+{}^1/_a$. Now *suppose that* $0 < |a| < 1$ — we have ${}_C\oint f(z)dz = 2\pi i\Sigma R$.

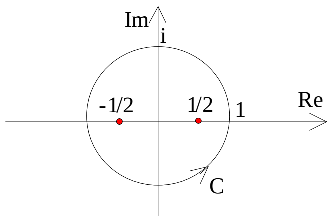


<u>Assignment 4: Set 07/4</u>

Q: Determine the *residues* of the following functions at their poles: (a) $3\exp z/z^4$; (b) ${}^{4z-1}/_{z^2+3z+2}$. A: (a) Let $f(z) = 3\exp z/z^4$, then f(z) has a *pole of order 4* at z = 0. So $b_1 = \text{Res}(f(z), 0) = \lim_{z\to z0}[{}^1/_{(m-1)!}(d^{m-1}/dz^{m-1})((z-z_0)^m f(z))]$ = $(m = 4, z_0 = 0)$ = $\lim_{z\to 0}[{}^1/_{3!}{}^{d^3}/_{dz^3}(z^4f(z))] = \lim_{z\to 0}[{}^1/_6{}^{d^3}/_{dz^3}(3\exp z)] = \lim_{z\to 0}[{}^1/_6(3\exp z)] = {}^1/_6(3\exp(0)) = ½$.

(b) Let $f(z) = {}^{4z-1}/_{z^2+3z+2} = {}^{4z-1}/_{(z+1)(z+2)}$. f(z) has 2 *discontinuities*, at z = -1 and at z = -2. <u>**Pole at z = -1:**</u> Consider that we split f(z) up as $f(z) = ({}^1/_{z+1})({}^{4z-1}/_{z+2}) = (g(z))(h(z))$. Now $g(z) = {}^1/_{z+1}$ has a *simple* pole at z = -1, with $b_1 = \lim_{z\to -1}(z-(-1))g(z)) = \lim_{z\to -1}((z+1){}^1/_{(z+1)}) = \lim_{z\to -1}(1) = 1$. *Therefore*, Res(g(z), -1) = 1. Further, h(z) is **analytic** at z = -1, so g(z)h(z) will also have a *simple* pole at z = -1. Its residue will be given by $\text{Res}(g(z),-1)\times h(-1) = 1\times({}^{4(-1)-1}/_{(-1)+2}) = -5$. Therefore, Res(f(z), -1) = -5. *Similarly*, for the pole at z = -2, we <u>get</u> Res(f(z), -2) = 9.

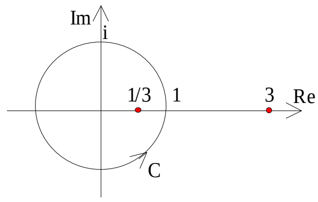
Q: If C is the *unit circle* centred at the origin, evaluate the following integral using the **residue** theorem: ${}_C\oint \tan(\pi z)dz$. A: ${}_C\oint \tan(\pi z)dz = I = {}_C\oint {}^{\sin(\pi z)}/_{\cos(\pi z)}dz$. Now $\cos(\pi z)$ *vanishes* when $\pi z = {}^{n\pi}/_2$ (n = ±1,±2,..), so we will have *discontinuities* when $z = {}^n/_2$. However, we're *only* interested in discontinuities **within** C — at $z = \pm{}^1/_2$.



Pole at z = ${}^1/_2$: $b_1 = \lim_{z\to 1/2} (z-{}^1/_2){}^{\sin(\pi z)}/_{\cos(\pi z)}$. Let $w = z-{}^1/_2$, so the *above* becomes $\lim_{w\to 0} w{}^{\sin(\pi(w+½))}/_{\cos(\pi(w+½))}$. Now apply the *addition* formula, $\cos(z\pm\zeta) = \cos(z)\cos(\zeta)\mp\sin(z)\sin(\zeta)$, to give $\lim_{w\to 0} w{}^{\sin(\pi(w+½))}/_{\cos(\pi w)\cos(\pi/2)-\sin(\pi w)\sin(\pi/2)}$. Because $\cos({}^\pi/_2) = 0$ and because $\sin({}^\pi/_2) = 1$, we *obtain* $\lim_{w\to 0}{}^{w(\sin(\pi(w+½))}/_{-\sin(\pi w)} = \lim_{w\to 0}{}^{\pi w(\sin(\pi(w+½)))}/_{-\pi\sin(\pi w)} = \lim_{w\to 0}(-{}^1/_\pi)(\sin(\pi(w+½)))({}^{\pi w}/_{\sin(\pi w)})$. We can *use* the fact that $\lim_{x\to 0}{}^x/_{\sin(x)} = 1$ to *give* $(-{}^1/_\pi)(\sin(\pi(0+½)))(1) = (-{}^1/_\pi)(\sin({}^\pi/_2)) = (-{}^1/_\pi)(1) = -{}^1/_\pi$. Therefore, if we *let* $f(z) = \tan(\pi z)$, we have $\text{Res}(f(z), ½) = -{}^1/_\pi$.

Similarly (using $w = z+\frac{1}{2}$), we get $\mathrm{Res}(f(z), -\frac{1}{2}) = -\frac{1}{\pi}$. Using the **Residue** Theorem, $I = 2\pi i(\Sigma\text{Residues}) = 2\pi i(-\frac{1}{\pi}+(-\frac{1}{\pi})) = 2\pi i(-\frac{2}{\pi}) = -4i$. *Conclusion*: $\oint_C \tan(\pi z)dz = -4i$, where C is the *unit circle* centred at the origin.

Q: Using *contour* integration, evaluate the following integral: $\int_0^{2\pi} \frac{d\theta}{5-3\cos(\theta)}$. A: To change to a *contour* integral, we use the **substitution** $z = e^{i\theta}$. Our contour using *this* substitution is the unit circle in the **complex** plane. Now $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = izd\theta$. We also *know* that $\cos\theta = e^{i\theta}+e^{-i\theta}/2 = \frac{1}{2}(z+\frac{1}{z})$. So, we *obtain* $I = \oint_C \frac{dz}{iz}\frac{1}{5-3(\frac{1}{2}(z+(1/z)))} = \oint_C \frac{dz}{iz}\frac{2}{10-3(z+(1/z))} = \oint_C \frac{2dz}{10iz-3iz(z+(1/z))} = \oint_C \frac{2dz}{10iz-3iz^2-3i} = \oint_C \frac{2dz}{-3iz^2+10iz-3i} = \oint_C \frac{2dz}{i(-3z^2+10z-3)} = \oint_C \frac{2dz}{i(-z+3)(3z-1)}$. To continue, **let** $f(z) = \frac{2}{i(-z+3)(3z-1)}$.

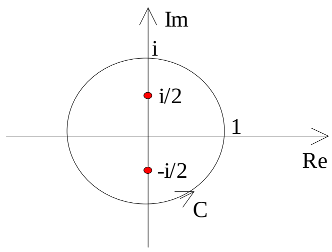


We have **discontinuities** at $z = 3$ and $z = \frac{1}{3}$. But only the *second discontinuity* lies within C, with $b_1 = \mathrm{Res}(f(z), \frac{1}{3}) = \lim_{z\to 1/3}(z-\frac{1}{3})f(z) = \lim_{z\to 1/3}\frac{2}{3i(3-z)} = \frac{2}{3i(3-(1/3))} = \frac{1}{4i}$. So, using the *Residue* Theorem, $I = 2\pi i(\Sigma R) = 2\pi i(\frac{1}{4i}) = \frac{\pi}{2}$. *Conclusion*: $\int_0^{2\pi} \frac{d\theta}{5-3\cos(\theta)} = \frac{\pi}{2}$.

12th April 2000

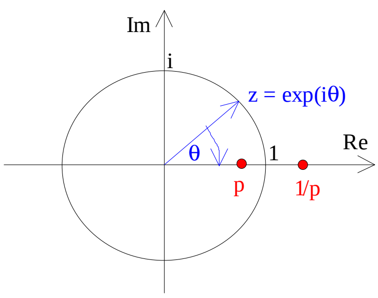
Examples on the Use of the Residue Theorem

Q: $I = \oint_C \frac{dz}{1+4z^2}$, where C is the unit circle *centred at the origin*. A: **Putting** $1+4z^2 = 0$, we get $z^2 = -\frac{1}{4}$, which implies that $z = \pm\frac{i}{2}$. So we have *two simple poles* as shown on the diagram. Now $1+4z^2 = 4(z^2+\frac{1}{4}) = 4(z-\frac{i}{2})(z-(-\frac{i}{2}))$. **Pole at $z = +\frac{i}{2}$:** $b_1 = \lim_{z\to z_0}((z-z_0)f(z)) = \lim_{z\to i/2}((z-(\frac{i}{2}))(\frac{1}{4(z-(i/2))(z-(-i/2))}) = \lim_{z\to i/2}(\frac{1}{4(z+(i/2))}) = \frac{1}{4((i/2)+(i/2))} = -\frac{i}{4}$. Similarly the **pole at $z = -\frac{i}{2}$** is $b_1 = +\frac{i}{4}$. So, by the *residue theorem*, $I = 2\pi i(\Sigma R) = 2\pi i(-\frac{i}{4}+\frac{i}{4}) = 0$.



Q: **Evaluate** $\int_0^{2\pi} \frac{d\theta}{5+3\sin(\theta)}$ (---(1)). A: Let $z = e^{i\theta}$ (---(2)). Now (2) $\Rightarrow dz = e^{i\theta}id\theta$; $dz = izd\theta$ (---(3)). Now, since we *know that* $\sin\theta = e^{i\theta}-e^{-i\theta}/2i = \frac{1}{2i}(z-\frac{1}{z})$, in (1), $I = \oint_C \frac{dz}{iz}\frac{1}{5+\frac{3}{2i}(z-\frac{1}{z})} = \oint_C \frac{2dz}{(3z^2+10iz-3)} = \oint_C \frac{2dz}{(3z+i)(z+3i)}$. *Now let* $f(z) = \frac{2}{(3z+1)(z+3i)}$. We have **two** poles, but only $z = -\frac{i}{3}$ lies *within* C. So $b_1 = \mathrm{Res}(f(z), -\frac{i}{3}) = \lim_{z\to -i/3} (z+\frac{i}{3})\frac{2}{(3z+i)(z+3i)} = \lim_{z\to -i/3} \frac{2}{3(z+3i)} = \frac{1}{4i}$. By the *Residue* Theorem, $I = 2\pi i\frac{1}{4i} = \frac{\pi}{2}$.

13th April 2000



Q: *Evaluate* $I = \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2}$, with $0 < p < 1$. A: Let $z = e^{i\theta}$, so that $\frac{dz}{iz} = d\theta$. By knowing this, and also that $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$, we get $I = \oint_C \frac{dz}{iz}\times\frac{1}{(1+p^2)-p(z+(1/z))} = \frac{i}{p}\oint_C \frac{dz}{(z-p)(z-(1/p))}$. Now **let** $f(z) = \frac{1}{(z-p)(z-(1/p))}$. We have *simple poles* at $z = p$ and at $z = \frac{1}{p}$, but as $0 < p < 1$, *only* $z = p$ lies within C. So $b_1 = \mathrm{Res}(f(z), z=p) = \lim_{z\to p}[(z-p)f(z)] = \lim_{z\to p}[\frac{1}{(z-(1/p))}] = \frac{1}{(p-(1/p))} = \frac{p}{p^2-1}$. So, using the *Residue Theorem*, $I = (\frac{i}{p})2\pi i(\frac{p}{p^2-1}) = -\frac{2\pi}{p^2-1} = \frac{2\pi}{1-p^2}$, with $0 < p < 1$.

Exam Paper: May 2000

SECTION 1 (Compulsory)

(1) (a) Define the gradient of the scalar field f(x, y, z). Find the gradient and the directional derivative in the direction $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ of the scalar field $f = xy^2z^3$ at the point (2, 1, -1). **[5 marks]**

(b) Define the curl of the vector field **v**.
If $\mathbf{v} = (\mathbf{k}\times\mathbf{r})\times\mathbf{r}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, find curl **v**. **[7 marks]**

(c) Write down the Cauchy-Riemann equations satisfied by the real and imaginary parts u, v of the differentiable function f(z) = u(x, y) + iv(x, y). If the real and imaginary parts of a complex valued function satisfy the Cauchy-Riemann equations, is that function differentiable? Explain your answer.

Examine whether or not the real and imaginary parts of the function $f(z) = z^2+5iz+3-i$ satisfy the Cauchy-Riemann equations and state the domain of analyticity of the function. **[8 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Prove that $\text{curl}(\phi\mathbf{A}) = (\text{grad }\phi)\times\mathbf{A} + \phi\,\text{curl}\mathbf{A}$. **[4 marks]**

(b) Simplify curl(grad ϕ). **[2 marks]**

(c) Calculate $\text{curl}(r^2\mathbf{r})$ where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r = |\mathbf{r}|$. **[4 marks]**

(d) Is there a function ϕ such that grad $\phi = r^2\mathbf{r}$? If so, then find such a ϕ. If not, then explain why not. **[5 marks]**

(3) (a) Given a space curve $\mathbf{r} = r(t)$, define the unit tangent vector $\hat{\boldsymbol{T}}$, the unit principal normal vector $\hat{\boldsymbol{N}}$ and the curvature κ. **[3 marks]**

(b) Show that $\kappa = \frac{|\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}|}{|\dot{\boldsymbol{r}}|^3}$ **[4 marks]**

(c) If $\mathbf{r}(t) = t^4\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}$, find $\hat{\boldsymbol{T}}$, κ at the point P = (1, 1, 1). **[4 marks]**

(d) Calculate the work done in moving the force $\mathbf{F} = (3x-2y, y+2z, -x^2)$ from the origin to P along the curve given in part (c). **[4 marks]**

(4) (a) Give the definition of a harmonic function u(x,y). **[2 marks]**

(b) Show that $u = x^3-2xy^2+3x^2-3y^2+1$ is a harmonic function. **[3 marks]**

(c) Find the harmonic conjugate function v(x, y) such that w = u+iv is analytic. **[5 marks]**

(d) Express w in the form f(z) where z = x+iy. **[5 marks]**

(5) Determine the residues at the poles of f(z) where
$f(z) = \frac{z^4+1}{z^2(z-a)(z-a^{-1})}$, $0 < |a| < 1$. **[6 marks]**

Use the substitution $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, and the Residue Theorem to deduce that
$\int_0^{2\pi} \frac{\cos(2\theta)d\theta}{1-2a\cos(\theta)+a^2} = \frac{2\pi a^2}{1-a^2}$, $0 < |a| < 1$. **[9 marks]**

(Questions done: 1, 4, 5)