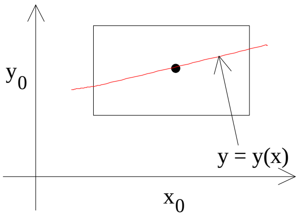
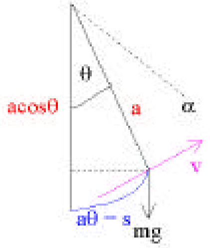
2nd February 2000

Introductory Lecture



(1) **Simplest**: ${}^{dy}/_{dx} = f(x)$. Solution: $y = \int f(x)dx + c$. We assume that we *can integrate* functions or get numerical estimates. (2) **Separable:** ${}^{dy}/_{dx} = f(x)g(y)$. Solution: $\int {}^{1}/_{g(y)}dy = \int f(x)dx + c$. (3) **Existence** of solution Picard's Theorem: If $f(x,y)$ and ${}^{\partial f}/_{\partial y}$ are continuous over a *rectangle* containing point (x_0, y_0), then $\exists$ a **unique** solution curve through (x_0, y_0) o the differential equation (d.e.) ${}^{dy}/_{dx} = f(x,y)$. (4) **Retarded Fall:** If a body falls under gravity, g, with a resistance *proportional* to velocity, then ${}^{d^2y}/_{dt^2} = g - c{}^{dy}/_{dt}$, or ${}^{dv}/_{dt} = g - cv$, where **y** is distance fallen, and **v** is the speed. So $\int dt = \int {}^{dv}/_{g-cv}$, and $g - cv = Ae^{-ct}$. Initially, $t = \underline{0}$ and $v = \underline{0}$, so $A = g$; $v = {}^{g}/_{c}(1-e^{-ct})$. *Terminal velocity* = ${}^{g}/_{c}$.



Pendulum: Mass *m*, length *a*, swing ± α. K.E. = P.E.; $\frac{1}{2}mv^2 = mg(a\cos\theta - a\cos\alpha)$. $v = {}^{ds}/_{dt}$, where $s = a\theta$, arc length. This $\Rightarrow ({}^{d\theta}/_{dt})^2 = 2g^{(\cos\theta-\cos\alpha)}/_{a}$. So $dt = -\sqrt{\frac{a}{2g}}\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}}$ (as θ is ↓ as t is ↑). The **period** T is $T = 4\sqrt{\frac{a}{2g}}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}}$. Put $k = \sin({}^{\alpha}/_{2})$, so $T = 2\sqrt{\frac{a}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{k^2-\sin^2(\theta/2)}}$. Put $\sin\phi = {}^{1}/_{k}\sin({}^{\theta}/_{2})$, so $T = 4\sqrt{\frac{a}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$. These are Elliptic Integrals of the First Kind. THEY ARE **TABULATED**. **Pendulum: Approximation:** $({}^{d\theta}/_{dt})^2 = {}^{2g(\cos\theta-\cos\alpha)}/_{a}$. So $2{}^{d\theta}/_{dt}{}^{d^2\theta}/_{dt^2} = -{}^{2g\sin\theta}/_{a}{}^{d\theta}/_{dt}$; ${}^{d^2\theta}/_{dt^2} = -{}^{g\sin\theta}/_{a}$. *Approximate* sinθ by θ if the amplitude α of the swing is **small**. So ${}^{d^2\theta}/_{dt^2} = -{}^{g\theta}/_{a}$; $\theta = \alpha\cos\sqrt{({}^{g}/_{a})}t$. *The period is* $2\pi\sqrt{({}^{a}/_{g})}$, which is the *limit of the elliptic integral* value as $\alpha \to 0$.

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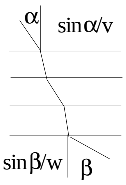
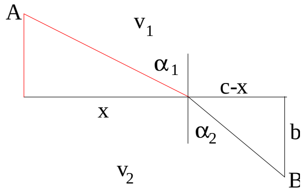
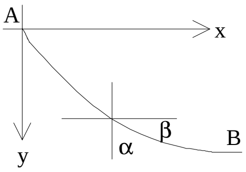
Brachistochrone; The Ring Problem

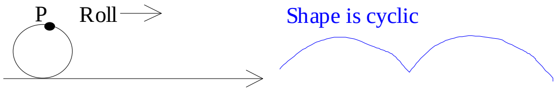




Snell's law of refraction. Light travels *from A to B* with speed v_1 *above* the axis and v_2 *below* the axis. Describe the path, assuming that it takes the **quickest** route. The time is $T = \sqrt{(a^2+x^2)}/v_1 + \sqrt{(b^2+(c-x)^2)}/v_2$. We **want** ${}^{dT}/_{dx} = 0$, $\Rightarrow x/v_1\sqrt{(a^2+x^2)} = c-x/v_2\sqrt{(b^2+(c-x)^2)}$, or $\sin(\alpha_1)/v_1 = \sin(\alpha_2)/v_2$. If light moves through **several** *parallel* layers, then ${}^{\sin\alpha_1}/_{v_1} = {}^{\sin\alpha_2}/_{v_2} = {}^{\sin\alpha_3}/_{v_3} = \ldots$, so ${}^{\sin\alpha}/_{v}$ **= constant**.



Brachistochrone. Consider a curve from A to B in which to *bend wire around*, so that a ring can slide down it with no friction in the shortest time possible. (1693, Bernoulli). Mechanics: $\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{(2gy)}$. **Geometry**: $\sin\alpha = \cos\beta = {}^{1}/_{\sec\beta} = {}^{1}/_{\sqrt{(1+\tan^2\beta)}} \Rightarrow \sin\alpha = {}^{1}/_{\sqrt{(a+(dy/dx)^2)}}$. **Optics**: ${}^{\sin\alpha}/_{v}$ = constant c for the *shortest* speed. Differential *equation:* $y[1+({}^{dy}/_{dx})^2]$ = constant; $({}^{dy}/_{dx})^2 = {}^{c-y}/_{y}$; $x = \int({}^{y}/_{c-y})^{1/2}dy$. Put $y = c\sin^2\phi$, so $x = \int c(1-\cos2\phi)d\phi$, so $x = {}^{c}/_{2}(2\phi - \sin2\phi)$. Or, *putting* $a = {}^{c}/_{2}$ and $\theta = 2\phi$, we get the *standard equation of a cycloid:* $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$.



First Order Linear

Standard Form: ${}^{dy}/_{dx} + P(x)y = Q(x)$.

General Solution: $e^{\int Pdx}y = \int Qe^{\int Pdx}dx + c$, where the *integrating factor is* $e^{\int Pdx}$.

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Tutorial: Linear, Constant Coefficient, Homogeneous

We have an **index** equation. For example, $D^2y-Dy-2y = 0$ *becomes* $m^2-m-2 = 0$ ($y\rightarrow1$, $Dy\rightarrow m$, $D^ry\rightarrow m^r$). *Solving* $m^2-m-2 = 0$, we get $m = 2$ and $m = -1$. (<u>Distinct</u>, <u>Real</u> roots). Solution: $y = Ae^{m_1x}+Be^{m_2x} = Ae^{2x}+Be^{-x}$. If we have a **repeated** root ($m_1 = m_2$), then we have $y = Ae^{m_1x}+xBe^{m_2x}$. If we have **complex** roots ($m_1 = \alpha+i\beta$; $m_2 = \alpha-i\beta$), then we have $y = Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$.

<u>Examples</u>. Q: $D^2y+10y+21y = 0$. A: *Index equation* $m^2+10m+21 = 0$; $(m+3)(m+7) = 0$. So $y = Ae^{-2x}+Be^{-7x}$. Q: $D^2y-6Dy+25y = 0$. A: *Index equation* $m^2-6m+25 = 0$. Using the <u>quadratic equation</u>, $m = {}^{6\pm\surd(36-100)}/_{2} = {}^{6\pm\surd(-64)}/_{2} = {}^{6\pm8i}/_{2} = 3\pm4i$. So $y = Ae^{3x}\cos4x+Be^{3x}\sin4x$. Q: $D^3y-6D^2y+11Dy-6y = 0$. A: *Index equation* $m^3-6m^2+2m+36 = 0$. <u>Factor</u> $m = -2$. Now use *long division* to get $m^3-6m^2+2m+36 = (m+2)(m^2-8m+18)$. We have a **solution** $m = -2$, and use the *quadratic equation* to get the two other solutions: $m = 4\pm\surd(2)i$. So $y = Ae^{-2x}+Be^{4x}\cos\surd(2)x+Ce^{4x}\sin\surd(2)x$.

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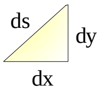
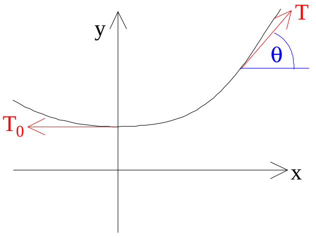
Hanging Chain

Find the *shape of the chain of* variable density hanging from its ends. Take the y-axis through its **lowest** point. Let s be the arc length *from* here; let w(s) be the density at s; and let T be the **tension**. <u>Horizontally</u>, $T\cos\theta = T_0$. <u>Vertically</u>, $T\sin\theta = \int^s_0 w(s)ds$. So $\tan\theta = {}^{dy}/_{dx} \Rightarrow T_0\tan\theta = \int^s_0 w(s)ds = T_0{}^{dy}/_{dx} \Rightarrow T_0{}^{d^2y}/_{dx^2} = {}^{d}/_{dx}[\int^s_0 w(s)ds] = {}^{d}/_{ds}[\int^s_0 w(s)ds]{}^{ds}/_{dx}$. **So** $T_0{}^{d^2y}/_{dx^2} = w(s){}^{ds}/_{dx} = w(s)\surd(1+({}^{dy}/_{dx})^2)$.

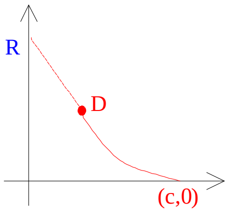




Hanging Chain: we have *uniform density w*. So ${}^{d^2y}/_{dx^2} = a\surd(1+({}^{dy}/_{dx})^2)$, where $a = w/T_0$. *Substitute* $p = {}^{dy}/_{dx}$, so ${}^{dp}/_{dx} = a\surd(1+p^2) \Rightarrow ax = \int{}^{dp}/_{\surd(1+p^2)} \Rightarrow ax = \log[p+\surd(1+p^2)] \Rightarrow {}^{dy}/_{dx} = p = \frac{1}{2}(e^{ax}-e^{-ax})$. Place on the x-axis *so that* $y = {}^{1}/_{a}$ when $x = 0$, so we get $y = {}^{1}/_{2a}[e^{ax}+e^{-ax}] = {}^{1}/_{a}\cosh(x)$. This is the **<u>Catenary</u>**.

Dog & Rabbit

A **Rabbit** starts at (0,0) with speed a. A **Dog** starts at (c,0) with speed b. Let $R = (0, at)$ and $D = (x,y)$. So ${}^{dy}/_{dx} = {}^{y-at}/_{x}$, ${}^{ds}/_{dt} = b$. And $at = y-x{}^{dy}/_{dx}$, so $a{}^{dt}/_{dx} = {}^{dy}/_{dx}-[{}^{dy}/_{dx}+x{}^{d^2y}/_{dx^2}]$; $a{}^{dt}/_{dx} = -x{}^{d^2y}/_{dx^2}$. **Also**, ${}^{dt}/_{dx} = {}^{dt}/_{ds}{}^{ds}/_{dx} = -{}^{1}/_{b}\surd(1+({}^{dy}/_{dx})^2)$. *We have the Differential Equation* $x{}^{d^2y}/_{dx^2} = {}^{a}/_{b}\surd(1+({}^{dy}/_{dx})^2)$, which *can be solved.*



Second Order Linear

Consider $D^2y+P(x)Dy+Q(x)y = R(x)$ (---(1)). **<u>Existence & Uniqueness Theorem</u>**: If P, Q and R are continuous on [a,b]; if $x_0 \in [a,b]$; and if y_0 and y_0' are any two numbers, then *there exists an <u>unique</u>* solution of the Differential Equation (D.E.) over all of the interval [a,b], <u>with</u> $y(x_0) = y_0$ and $Dy(x_0) = y_0'$. Now $R(x) \equiv 0 \Leftrightarrow$ *Homogenous*. **<u>Theorem</u>**: If y is a solution of the D.E. (1), and z is *another* solution, then (y-z) is a solution of the *homogenous* D.E.

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Theorem: The general solution of (1) is the sum of a *particular* solution y_p of (1) and the *general* solution of the *homogenous* D.E. $D^2y+P(x)Dy+Q(x)y = 0$. (---(2)). **Theorem**: The *general solution* of (2) is of form $c_1y_1(x)+c_2y_2(x)$ for *arbitrary* constants c_1 and c_2, *where* y_1 and y_2 are *two independent* solutions of (2).

Proof. Suppose that y, y_1 and y_2 are *solutions* of (2). Then y *and* $c_1y_1+c_2y_2$ are solutions of (2). By the *uniqueness* theorem, we want an x_0 with $y(x_0) = c_1y_1(x_0)+c_2y_2(x_0)$ and $Dy(x_0) = c_1Dy_1(x_0)+c_2Dy_2(x_0)$. **Any** x_0 will do if we can *choose* or *solve equations* for c_1 and c_2, i.e. if

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ Dy_1(x_0) & Dy_2(x_0) \end{vmatrix} \neq 0$$

Definition: The *Wronskian* is $W(y_1, y_2) = y_1Dy_2-y_2Dy_1$. **Theorem**: Either $W(y_1, y_2)$ is always zero or never zero. **Proof:** $DW = y_1D^2y_2-y_2D^2y_1$; $D^2y_1+PDy_1+Qy_1 = 0$; $y_2D^2y_1+y_2PDy_1+y_2Qy_1 = 0$; $y_1D^2y_2+y_1PDy_2+y_1Qy_2 = 0$ (*subtract*). Now $DW+PW = 0 \Rightarrow W = ke^{-\int P(x)dx}$. So either $k = 0$ and $W = 0$ **always,** or $k \neq 0$ and $W \neq 0$ **always**.

Second Solution of the Homogenous D.E.

Suppose that y_1 (= z) satisfies (2). Then $D^2z+P(x)Dz+Q(x)z = 0$. Try $y_2 = vy_1 = vz$ as the *second* solution. Put $y = vz$ in $D^2y+P(x)Dy+Q(x)y = 0$ to get $v(D^2z+P(x)Dz+Q(x)z) + D^2v.z+Dv(2Dz+P(x)z) = 0$. **Hence** DV *satisfies* $zD(Dv) + (2Dz+P(x)z)Dv = 0$, *or* $^{D(Dv)}/_{Dv}+{}^{2Dz}/_{z}+P(x) = 0$, so that $\log(Dv)+2\log z+\int P(x)dx = 0$. Now $DV = {}^{1}/_{z^2}e^{-\int P(x)dx}$, and subject to *integration*, DV (and **hence** v) can be *calculated*.

Constant Coefficients

Consider $D^2y+pDy+qy = 0$ (---(2)). The *auxiliary equation is* $m^2+pm+q = 0$ (---(3)). Consider that we have **roots** m_1 and m_2. Solutions: (Distinct, Real): $y = c_1e^{m_1x}+c_2e^{m_2x}$. (Distinct, Complex): $y = e^{ax}(c_1\cos bx+c_2\sin bx)$ ($m = a\pm ib$). (Equal): $y = c_1e^{mx}+c_2xe^{mx}$.

Variation of Parameters

Assume that $c_1y_1(x)+c_2y_2(x)$ is the *general solution* of the homogeneous D.E. $D^2y+P(x)Dy+Q(x)y = 0$. **Try** $y = v_1y_1+v_2y_2$ as the *solution* of $D^2y+P(x)Dy+Q(x)y = R(x)$. **Now** $Dy = (v_1Dy_1+v_2Dy_2)+(y_1Dv_1+y_2Dv_2)$. *Assume* also that $y = v_1y_1+v_2y_2$ and $y_1Dv_1+y_2Dv_2 = 0$. Then $Dy = v_1Dy_1+v_2Dy_2$, *and* $D^2y = v_1D^2y_1+v_2D^2y_2+Dv_1Dy_1+Dv_2Dy_2$.

Substitute in the **D.E.** to get $v_1(D^2y_1+P(x)Dy_1+Q(x)y_1)$ (= 0) + $v_2(D^2y_2+P(x)Dy_2+Q(x)y_2)$ (= 0) + $Dv_1Dy_1+Dv_2Dy_2 = R(x)$. This gives *two conditions* for Dv_1 and Dv_2, namely $y_1Dv_1+y_2Dv_2 = 0$ and $Dy_1Dv_1+Dy_2Dv_2 = R(x)$. **Hence** $Dv_1 = -y_2R(x)/W(y_1, y_2)$, using the *Wronskian*. **Except** for the problems of integration, this gives v_1. Similarly, we can obtain v_2, and hence the *particular solution* of the **D.E.**

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Tutorial

Q: Use the **Wronskian** to verify that the functions $y_1(x)$ and $y_2(x)$ are *linearly independent*. Show that they are **solutions** of the differential equation, and find the solution satisfying the stated initial conditions. (i) $D^2+Dy-2y = 0$; $y_1(x) = e^x$; $y_2(x) = e^{-2x}$; $y(0) = 8$; and $Dy(0) = 2$. (ii) $D^2+5Dy+6y = 0$; $y_1(x) = e^{-2x}$; $y_2(x) = e^{-3x}$; $y(0) = 1$; and $Dy(0) = 1$.

A: (i) The **Wronskian is** $w(y_1, y_2) = e^x(-2e^{-2x})-e^{-2x}e^x = -2e^xe^{-2x}-e^xe^{-2x} = -3e^{-x} < 0 \Rightarrow$ linearly *independent*. Now **substituting** $y = e^x$ in $D^2y+Dy-2y = 0$, we get $e^x+e^x-2e^x = 0$ (*correct*). **Substituting** $y = e^{-2x}$, we get $4e^{-2x}-2e^{-2x}-2e^{-2x} = 0$ (*correct*). Now we use $y(x_0) = c_1y_1(x_0)+c_2y_2(x_0)$ and $Dy(x_0) = c_1Dy_1(x_0)+c_2Dy_2(x_0)$ to get *two* equations: $8 = c_1+c_2$; $2 = c_1-2c_2$. **Solving** these equations gives $c_1 = 6$ and $c_2 = 2$, so the *general* solution is $y = 6e^x-2e^{-2x}$.

(ii) $w(y_1, y_2) = e^{-2x}(-3e^{-3x})-e^{-3x}(-2e^{-2x}) = -e^{-3x}e^{-2x} = -e^{-5x} < 0 \Rightarrow$ *linearly independent*. Solution 1: $4e^{-2x}-10e^{-2x}+6e^{-2x} = 0$, *correct*. **Similarly** for solution 2. Now $1 = A+B$ and $1 = -2A-3B \Rightarrow A = 4$ and $B = -3$, so the *general solution is* $y = 4e^{-2x}-3e^{-3x}$.

Q: $y_1(x) = x$ is a *solution* of $x^2D^2y+xDy-y = 0$. Find a **second** solution $y_2(x) = vx$. A: Check that it *is* a solution: $x^2(0)+x(1)-x = x-x = 0$, *correct*. Now find a **second** solution using $y_2(x) = vx$. We have $D^2y+{}^1/_xDy-{}^y/_{x^2} = 0$, so $Dv = {}^1/_{z^2}e^{-\int P(x)dx} = {}^1/_{z^2}e^{-\int(1/x)dx} = {}^1/_{z^2}e^{-\ln x} = {}^1/_{z^2x}$. *Integrate* to get v, and thus $y_2(x) = vx$ is the **second** solution.

16th February 2000

Laplace Transforms

[**Log Tables**: Problem: *multiply* $x\times y$ (for large x and y). **Process**: convert x to a and y to b; add a and b; and convert *back*. We require the diagram shown to be commutative: either way around gives the **same** answer. This is based on $e^{a+b} = e^ae^b$: If $x = e^a$ and $y = e^b$, then $x\times y = e^{a+b}$. So we find $a = \log(x)$ and $b = \log(y)$, so that $x\times y = e^{a+b}$].

$$\begin{tikzcd}
\begin{matrix}x\\y\end{matrix} \arrow[r, "\text{TRANSFORM}"] \arrow[d, "\text{MULT.}"'] & \begin{matrix}a\\b\end{matrix} \arrow[d, "\text{ADD}"] \\
xy & a+b \arrow[l, "\text{TRANSFORM}"]
\end{tikzcd}$$

The *Laplace* transform converts certain functions f of parameter t into **other** functions $L(f)$ of a *new parameter* s. We want the transform to be **one-one,** i.e. knowing $L(f)$, we can find f. We want it to have good properties: (i) Linear; (ii) Changes differential equations into polynomials. Definition: $L(f)(s) = \int^\infty_0 e^{-st}f(t)dt$.

This **assumes** that f is of an *appropriate* type to do the integral. We usually require f to be *piecewise continuous* (to have a finite number of jumps); require f to be **defined** for $t \geq 0$; and require f to be of **exponential** order, i.e. $|f(t)| \leq Ce^{\alpha t}$ for *constants* C and α. For **such** functions, $\int^\infty_0 e^{-st}f(t)dt = \lim_{N\to\infty} \int^N_0 e^{-st}f(t)dt$ *exists* for $s > \alpha$.

Example: $f(t) = 1$ *for all* $t \geq 0$. So $Lf(s) = \int^\infty_0 e^{-st}.1dt = \lim_{N\to\infty} \int^N_0 e^{-st}dt$. But $\int^N_0 e^{-st} = [-e^{-st}/s]^N_0 = {}^1/_s-(e^{-sN}/s) \to {}^1/_s$ if $s > 0$, **so** $Lf(s) = {}^1/_s$ for $s > 0$. **Now** let $f(t) = t$ *for all* $t \geq 0$. It follows that $Lf(s) = \lim_{N\to\infty} \int^N_0 e^{-st}.tdt = (-e^{-st}/s)t-\int(-e^{-st}/s)1dt = (-e^{-st}/s)-(e^{-st}/s^2)$. **So** $\int^N_0 e^{-st}.tdt = {}^1/_{s^2}-(e^{-sN}N/s)-(e^{-sN}/s^2)$. The *last two terms* $\to 0$ as $N \to \infty$ (if $s > 0$), so the **Laplace transform** of $f(t) = t$ is $L(f)(s) = {}^1/_{s^2}$.

Suppose that $f(t) = e^{kt}$. Then $L(f)(s) = \int^{\infty}_{0} e^{kt}e^{-st}dt = \int^{\infty}_{0} e^{(k-s)t}dt = {}^{1}/_{s-k}$ for $s > k$. **Example**: $f(t) = \cos kt$. Then $Lf(s) = \int^{\infty}_{0} e^{-st}\cos kt\,dt = [e^{-st}({}^{k\sin kt-s\cos kt}/_{k^2+s^2})]^{\infty}_{0}$. So $L(\cos kt)(s) = {}^{s}/_{k^2+s^2}$ for $s > 0$. *Similarly*, $L(\sin kt)(s) = {}^{k}/_{k^2+s^2}$..

The Laplace Transformation is **Linear:** $L(f+g) = L(f)+L(g)$; $L(\lambda f) = \lambda L(f)$. **Proof**: $L(f)+L(g) = \int^{\infty}_{0} e^{-st}f(t)dt + \int^{\infty}_{0} e^{-st}g(t)dt = \int^{\infty}_{0} e^{-st}f(t)+e^{-st}g(t)dt = \int^{\infty}_{0} e^{-st}(f(t)+g(t))dt = L(f+g)$. **Formula**: $L(Df)(s) = sL(f)(s)-f(0)$. **Proof**: $L(Df)(s) = \int^{\infty}_{0} e^{-st}\,{}^{df}/_{dt}dt = [e^{-st}f(t)]^{\infty}_{0} - \int^{\infty}_{0} (-s)e^{-st}f(t)dt = 0-e^{0}f(0)+s\int^{\infty}_{0} e^{-st}f(t)dt = -f(0)+sL(f)$.

Summary so far: $L(f)(s) = \int^{\infty}_{0} e^{-st}f(t)dt$ is *defined* for all $s > \alpha$. $f(t) = 1$ has *transform* $L(f)(s) = {}^{1}/_{s}$. $f(t) = t$ has *transform* $L(f)(s) = {}^{1}/_{s^2}$. $f(t) = e^{at}$ has *transform* (for all $s > \alpha$) $L(f)(s) = {}^{1}/_{s-a}$. $f(t) = \cos(at)$ has *transform* $L(f)(s) = {}^{s}/_{a^2+s^2}$. $f(t) = \sin(at)$ has *transform* $L(f)(s) = {}^{a}/_{a^2+s^2}$. It is a **linear** operator, $L(f+g) = L(f)+L(g)$, and $L(kf) = kL(f)$. $L(Df)(s) = sL(f)(s)-f(0)$, and for *higher* derivatives, $L(D^n(f))(s) = s^nL(f)(s)-s^{n-1}f(0)-s^{n-2}D(f)(0)-s^{n-3}D^2(f)(0)-\ldots-D^{n-1}(f)(0)$. Hence $f(t) = t^n$ has *transform* $L(f)(s) = n!/s^{n+1}$.

Assignment 1: Set 16/2; In 25/2; Back 10/3

Q: Find the *general solution* of the differential equation $D^2y+8Dy+25y = 0$, and the *particular* solution satisfying $y(0) = 4$ and $Dy(0) = -1$. A: The *differential equation* $D^2y+8Dy+25y = 0$ has index equation $m^2+8m+25 = 0 = (m+4)^2+3^2$. Hence the **general** solution is $y = Ae^{-4x}\cos(3x)+Be^{-4x}\sin(3x)$. The initial conditions are $4 = y(0) = A$, and $-1 = Dy(0) = -4A+3B$. Therefore, $A = 4$ and $B = 5$. It follows that the *particular* solution is $y = 4e^{-3x}\cos(3x)+5e^{-4x}\sin(3x)$. (Or we can use the *quadratic equation* to get the roots, which are $-4\pm3i$, and so $m_1 = -4+3i$; and $m_2 = -4-3i$).

Q: **Assume** that $y_1(x) = e^{-x}$ and $y_2(x) = e^{2x}$ are *solutions* of the differential equation $D^2y-Dy-2y = 0$. Calculate the **Wronskian**, and thus deduce that they are *independent* solutions. Use the method of **variation** of parameters to find the solution of the *non homogenous* equation $D^2y-Dy-2y = e^{-2x}$, with *initial conditions* $y(0) = 0$ and $Dy(0) = 0$.

A: $D^2y-Dy-2y = 0$ has two independent solutions ($y_1(x) = e^{-x}$ and $y_2(x) = e^{2x}$) because the Wronskian is $W = ({}^{y_1(x)}{}_{Dy_1(x)}\ {}^{y_2(x)}{}_{Dy_2(x)}) = ({}^{\exp(-x)}{}_{-\exp(-x)}\ {}^{\exp(2x)}{}_{2\exp(2x)}) = e^{3x} \neq 0$. (This comes *from* $e^x > 0$ for all x). Now assume *that* $y = y_1(x)v_1(x)+y_2(x)v_2(x)$ satisfies the non-homogeneous equation $D^2y-Dy-2y=e^{-2x} = R(x)$. It follows that $Dv_1(x) = -{}^{y_2(x)R(x)}/_{W} = -e^{2x}e^{-2x}/3e^{x} = -1/3e^{x}$. *Integrating* gives $v_1(x) = (e^{-x}/3) + C_1$. Note the *constant*!

Similarly, $Dv_2(x) = +{}^{y_1(x)R(x)}/_{W} = e^{-x}e^{-2x}/3e^{x} = +e^{-4x}/3$. So $v_2(x) = -(e^{-4x}/12) + C_2$; and thus the *general* solution of the non-homogeneous equation is $y = e^{-x}(e^{-x}/3)+C_1)+e^{2x}(-(e^{-4x}/12)+C_2) = C_1e^{-x}+C_2e^{2x}+(e^{-2x}/4)$. The *initial conditions* give $0 = y(0) = C_1+C_2+{}^{1}/_{4}$, and $0 = Dy(0) = -C_1+2C_2-{}^{1}/_{2}$. Thus $C_1 = -{}^{1}/_{3}$ and $C_2 = {}^{1}/_{12}$. The *particular solution* satisfying the initial conditions is $y = -{}^{1}/_{3}e^{-x}+{}^{1}/_{12}e^{2x}+(e^{-2x}/4)$.

Note: Be sure that you know whether initial conditions apply to the *overall* solution or to the *homogenous* equation. In this question, the initial conditions *only* apply to the overall solution — and **not** to the homogenous solution. Therefore, you cannot use them when finding the solution to the *associated* homogenous equation.

In my solution, I said that the *general solution* of $D^2y\text{-}Dy\text{-}2y = e^{-2x}$ was the sum of the **general** solution of $D^2y\text{-}Dy\text{-}2y = 0$ and a particular solution of $D^2y\text{-}Dy\text{-}2y = e^{-2x}$. But we cannot use the *initial* conditions in $y(x_0) = c_1y_1(x_0)+c_2y_2(x_0)$ and $Dy(x_0) = c_1Dy_1(x_0)+c_2Dy_2(x_0)$, the *uniqueness* theorem.

Q: **Assume** that $y_1(x) = e^x\cos(^x/_2)$ and $y_2(x) = e^x\sin(^x/_2)$ are *solutions* of the differential equation $4D^2y\text{-}8Dy+5y = 0$. Calculate the **Wronskian** and deduce that they are *independent* solutions. Convert the equation to **standard** form, and using the method of *variation* of parameters, find the solution of the non homogenous equation $4D^2y\text{-}8Dy+5y = e^x$.

A: The *differential* equation $4D^2y\text{-}8Dy+5y = 0$ has **two** solutions: $y_1(x) = e^x\cos(^x/_2)$, $y_2(x) = e^x\sin(^x/_2)$. These are *independent* because the Wronskian W is $\begin{pmatrix} y_1(x) & y_2(x) \\ Dy_1(x) & Dy_2(x) \end{pmatrix} = \begin{pmatrix} \exp(x)\cos(x/2) & \exp(x)\sin(x/2) \\ \exp(x)[\cos(x/2)\text{-}(1/2)\sin(x/2)] & \exp(x)[\sin(x/2)+(1/2)\cos(x/2)] \end{pmatrix} = {}^1/_2e^{2x} \neq 0$. The solution, $y = y_1(x)v_1(x)+y_2(x)v_2(x)$, of the *non homogeneous* differential equation, in **normal** form, $D^2y\text{-}2Dy+{}^5/_4y = {}^1/_4e^x = R(x)$, has $Dv_1(x) = \text{-}^{y_2(x)R(x)}/_W = \text{-}^1/_2\sin(^x/_2)$, and **so** $v_1(x) = \cos(^x/_2)+C_1$.

Similarly, $Dv_2(x) = +^{y_1(x)R(x)}/_W = {}^1/_2\cos(^x/_2)$, and so $v_2(x) = \sin(^x/_2)+C_2$. Thus $y = C_1e^x\cos(^x/_2)+C_2e^x\sin(^x/_2)+e^x$. The **initial** conditions give $0 = y(0) = C_1+1$; and $0 = Dy(0) = C_1+(C_2/2) + 1$. Thus $C_1 = \text{-}1$ and $C_2 = 0$; and the *solution* is therefore $y = \text{-}e^x\cos(^x/_2)+e^x$.

Q: Show that $z = e^{2x}$ is *one solution* of the differential equation $(x+2)D^2y\text{-}(2x+5)Dy+2y = 0$. Find a **second** solution of the form zv for some function v. (Hint: having *found* Dv, it may be useful to know the derivative of the **function** $(2x+5)e^{-2x}$).

A: Substituting $z = e^{2x}$, $Dz = 2e^{2x}$, and $D^2z = 4e^{2x}$ shows that $(x+2)D^2y\text{-}(2x+5)Dy+2y = (4(x+2)\text{-}2(2x+5)+2)e^{2x} = 0$, and so z satisfies the *differential equation*. To find the second solution, y = zv, write the **differential equation** in normal form: $D^2y+P(x)Dy+Q(x)y = 0$, so that $D^2y\text{-}^{2x+5}/_{x+2}Dy+{}^2/_{x+2y} = 0$, with $P(x) = \text{-}^{2x+5}/_{x+2} = \text{-}2\text{-}^1/_{x+2}$.

Trying $y_2 = vx = ve^{2x}$ as the *second* solution, v can be calculated using $Dv = {}^1/_{z^2}e^{-\int P(x)dx}$. Hence $\int P = \text{-}2x\text{-}\ln(x+2)$, and $e^{-\int P} = e^{2x+\ln(x+2)} = (x+2)e^{2x}$. Finally, use the *equation* $Dv = e^{-\int P}/z^2 = (x+2)e^{2x}/(e^{2x})^2 = (x+2)e^{-2x}$. **Hence** $v = \text{-}^1/_4(2x+5)e^{-2x}$ (hard integration), and so $y = zv = \text{-}^{2x+5}/_5$. **Note**: Always read the question *carefully*. If it says "assume", do not prove it. If it says "show that", remember to *do* this!

17th February 2000

An Application

Q: **Solve** $D^4y+3D^3y+D^2y\text{-}3Dy\text{-}2y = t$ with *boundary condition* $y(0) = Dy(0) = D^2y(0) = D^3(0) = 0$. Put $L = L(y)$. Then $L(Dy) = sL\text{-}0$, *and* $L(D^2y) = s^2L$, etc. **Also**, $L(t) = {}^1/_{s^2}$. The *transform* is linear, hence $s^4L+3s^3L+s^2L\text{-}3sL\text{-}2L = {}^1/_{s^2}$; $(s^4+3s^3+s^2\text{-}3s\text{-}2)L = {}^1/_{s^2}$. *Factorise* the quartic, giving $(s\text{-}1)(s+1)^2(s+2)L = {}^1/_{s^2}$. **Hence** $L = {}^1/_{(s-1)(s+1)^2s^2(s+2)}$.

Split as a **sum** using *partial fractions*: ${}^{A}/_{s-1}+{}^{B}/_{s+1}+{}^{C}/_{(s+1)^2}+{}^{D}/_{s}+{}^{E}/_{s^2}+{}^{F}/_{(s+2)}$. Multiply *through* by the **denominator**, giving $1 = A(s+1)^2s^2(s+2) + B(s-1)(s+1)s^2(s+2) + C(s-1)s^2(s+2) + D(s-1)(s+1)^2s(s+2) + E(s-1)(s+1)^2(s+2) + F(s-1)(s+1)^2s^2$. **Put s = 1** to get $1 = 12A$ so that $A = {}^{1}/_{12}$; **put s = -1** to get $1 = -2C$ so that $C = -{}^{1}/_{2}$; **put s = -2** to get $1 = -12F$ so that $F = -{}^{1}/_{12}$; and **put s = 0** to get $1 = -2E$ so that $E = -{}^{1}/_{2}$.

Now either compare *coefficients* or try *other values* to find the remaining terms. We get out the following: $L = {}^{3}/_{4s}-{}^{1}/_{2s^2}-{}^{3}/_{4(s+1)}-{}^{1}/_{2(s+1)^2}+{}^{1}/_{12(s-1)}-{}^{1}/_{12(s+2)}$. Now look for *functions* with these **transforms, and obtain** $y = {}^{3}/_{4}-{}^{t}/_{2}-{}^{3}/_{4}e^{-t}+(?)+{}^{1}/_{12}e^{t}-{}^{1}/_{12}e^{-2t}$.

18th February 2000

Tutorial

Q: Calculate the **Laplace** transform of $\sin^2(at)$. A: As $\sin^2(at) = {}^{1}/_{2}(1-\cos(2at)) = {}^{1}/_{2}-{}^{\cos(2at)}/_{2}$, *then* $L(f)(s) = {}^{1}/_{2s}-{}^{s}/_{2(s^2+4a^2)} = \ldots = {}^{4a^2}/_{2s^3+8sa^2}$. Q: Find the **Laplace** transform of $f(t) = 2e^{3t}-\sin(5t)$. A: It is ${}^{2}/_{s-3}-{}^{5}/_{25+s^2} = {}^{2(25+s^2)-5(s-3)}/_{(s-3)(25+s^2)}$.

Q: *Express* $g(s) = 1/s^4+s^2$ as a sum of *partial* fractions. Hence find a function $f(t)$ whose Laplace transform is $g(s)$. A: $g(s) = {}^{1}/_{s^2(s^2+1)} = {}^{A}/_{s}+{}^{B}/_{s^2}+{}^{Cs+D}/_{s^2+1}$. So $1 = As(s^2+1)+B(s^2+1)+(Cs+D)s^2$. **Put** $s = 0$ to get $1 = B$. The *coefficient* of s^2 gives $0 = B+D \Rightarrow D = -1$. The *coefficient* of s gives $0 = A$. And the *coefficient* of s^3 gives $0 = A+C \Rightarrow C = 0$. So $g(s) = {}^{1}/_{s^2}-{}^{1}/_{s^2+1}$; $f(t) = t-\sin(t)$.

23rd February 2000

Examples

Q: Solve $Dy+y = \sin x$ with $y(0) = 1$. A: Put L = *the Laplace transform* of y, so that $L = L(y)$. Then the transform of Dy is $sL(y)-y(0)$, i.e. $sL = 1$. The *transform* of $\sin x$ is ${}^{1}/_{1+s^2}$. Transform the *equation*, giving $(sL-1)+L = {}^{1}/_{1+s^2}$. Solve for L: $(s+1)L = 1+{}^{1}/_{1+s^2}$; $L = {}^{1}/_{1+s} + {}^{1}/_{(1+s^2)(1+s)}$.

Partial fractions gives $L = {}^{1}/_{1+s}+{}^{A}/_{1+s}+{}^{Bs+c}/_{1+s^2}$. So $1 = A(1+s^2)+(Bs+C)(1+s)$. Now $s = -1$ *gives* $A = {}^{1}/_{2}$; the *coefficient* of s^2 gives $A+B = 0 \Rightarrow B = -{}^{1}/_{2}$; and the *constant* term gives $A+C = 1 \Rightarrow C = {}^{1}/_{2}$. So $L = {}^{1}/_{1+s}+{}^{1/2}/_{1+s}-{}^{(1/2)s}/_{1+s^2}+{}^{1/2}/_{1+s^2}$. Now *invert* the transformation, **giving** $y(x) = e^{-x}+{}^{1}/_{2}e^{-x}-{}^{1}/_{2}\cos x+{}^{1}/_{2}\sin x$.

Q: **Solve** $D^2y+4y = 0$ with $y(0) = 2$ and $Dy(0) = 2$. A: Put L = *the Laplace transform of y*. The transform of D^2y is $s^2L-sy(0)-Dy(0)$. Hence $(s^2L-s2-2)+4L = 0$. **Solve for L**, giving $(s^2+4)L = 2(1+s)$; $L = {}^{2(1+s)}/_{s^2+4} = {}^{2}/_{s^2+4}+{}^{2s}/_{s^2+4}$. So $y(x) = \sin 2x+2\cos 2x$.

Q: **Solve** $D^3y+Dy = e^x$ with $y(0) = Dy(0) = D^2y(0) = 0$. A: Let L = *the Laplace transform of y*. The transform of Dy is $sL-y(0) = sL$. The transform of D^3y is $s^3L-s^2y(0)-sDy(0)-D^2y(0) = s^3L$. And the *transform* of e^x is ${}^{1}/_{s-1}$. So transforming the *equation* gives $s^3L+sL = {}^{1}/_{s-1}$; $L = {}^{1}/_{(s-1)s(s^2+1)}$. Again do *partial fractions*, and get out in $L = {}^{A}/_{s-1}+{}^{B}/_{s}+{}^{Cs+D}/_{s^2+1}$ that $B = -1$, $A = {}^{1}/_{2}$, $C = {}^{1}/_{2}$ and $D = -{}^{1}/_{2}$. Therefore, $y(x) = {}^{1}/_{2}e^{x}-1+{}^{1}/_{2}\cos x-{}^{1}/_{2}\sin x$.

Laplace Handout: Summary

$L(\int^t_a f(x)dx)(s) = {}^1/_s L(f) - {}^1/_s \int^a_0 f(x)dx$. In *particular,* $L(\int^t_0 f(x)dx)(s) = {}^1/_s L(f)$. Denote the Laplace transform of f by ϕ, i.e. $\phi(s) = L(f)(s)$. **Then** $L(t^n f)(s) = (-1)^n(D^n)\phi(s)$. **Further,** $L(e^{at}f)(s) = \phi(s-a)$. **Example**: $L(e^{at}\cos bt) = {}^{(s-a)}/_{b^2+(s-a)^2}$. When partial fractions *doesn't* work, use convolution: the convolution of 2 functions f & g is the *function* f*g defined **by** $(f*g)(t) = \int^t_0 f(t-x)g(x)dx$.

Theorem: $L(f*g)(s) = L(f)(s)L(g)(s)$. The *step function,* u_a, switches on at value $t = a$. It is given by $u_a(t) = 0$ if $t < a$, and $u_a(t) = 1$ if $t \geq a$. For any *function* g, and any *number* a, we can define a **phase shifted switch** on function f, by $f(t) = u_a(t)g(t-a)$. Thus $f(t) = 0$ for $t < a$, and $f(t) = g(t-a)$ for $t \geq a$. **Theorem**: If $a > 0$, then $L(f) = e^{-as}L(g)$.

Systems of Equations

Suppose that y, z and w are all *functions* of x, with **initial** conditions $y(0) = 1$, $z(0) = 1$, and $w(0) = 0$. Let $D = {}^d/_{dx}$. Suppose that we have a system of D.E.'s: $Dw+y = \sin x$; $Dy-z = e^x$; and $Dz+w+y = 1$. *Transform* these equations: **Let** $Y = L(y)$ [$= \int^\infty_0 e^{-sx}y(x)dx$], $W = L(w)$, and $Z = L(z)$. **Then** $L(Dy) = sY-y(0) = sY-1$; $L(Dw) = sW$; *and* $L(Dz) = sZ-1$.

Transform the equations, giving $sW+Y = {}^1/_{1+s^2}$, $sY-1-Z = {}^1/_{s-1}$, and $sZ-1+W+Y = {}^1/_s$. Thus we have a system of *linear equations* in Y, Z and W: $sW+Y = {}^1/_{1+s^2}$, $sY-Z = {}^s/_{s-1}$, and $W+Y+sZ = {}^{1+s}/_s$. *Solve* this set of **algebraic** equations, and get out that $W = -{}^1/_{s(s-1)} = {}^1/_s - {}^1/_{s-1}$; $Y = {}^{s^2+s}/_{(s-1)(s^2+1)} = {}^1/_{s-1} + {}^1/_{s^2+1}$; and $Z = {}^s/_{s^2+1}$. Therefore, $w(x) = 1-e^{-x}$; $y(x) = e^x+\sin x$; and $z(x) = \cos x$.

24th February 2000

Frobenius Method

First, let us *summarise* the Power Series: $\Sigma^\infty_{n=0} a_nx^n = a_0+a_1x+a_2x^2+\ldots$ (---(1)) is a *power series* in x. Series (1) converges at *a point x* if $\lim_{m\to\infty} \Sigma^m_{n=0} a_nx^n = \lim_{m\to\infty} (a_1+a_1x+\ldots+a_mx^m)$ exists. This limit is called the *sum of series* at x. Series (1) converges at $x = 0$. *Three cases:* (a) The series **converges** for all x; (b) the series **diverges** for all x ($x \neq 0$); (c) there exists an $R > 0$, and the series *converges* for $|x| < R$, and *diverges* for $|x| > R$. Note: the behaviour is **unknown** for $|x| = R$. ($R = \lim_{n\to\infty}|a_n/a_{n+1}|$, if the limit *exists*).

If $|x| < R$, define the **function** f by $f(x) = \Sigma^\infty_{n=0} a_nx^n$. Now f is *continuous* for $|x| < R$, and f is infinitely differentiable. Differentiation is term by term, so $Df(x) = a_1+2a_2x+3a_3x^2+\ldots = \Sigma^\infty_{n=1} na_nx^{n-1}$. *Hence* $a_n = D^nf(0)/n!$. (Compare with *Taylor* series). If $g(x) = \Sigma b_nx^n$ is *another* power series, also converging for $|x| < R$, then $f(x)+g(x) = \Sigma(a_n+b_n)x^n$; $f(x)-g(x) = \Sigma(a_n-b_n)x^n$; $f(x)g(x) = \Sigma^\infty_{n=0} c_nx^n$, *where* $x_n = \Sigma^n_{i=0} a_ib_{n-i} = a_0b_n+\ldots+a_nb_0$.

25th February 2000

Frobenius Method. Consider the D.E. $2x^2D^2y+x(2x+1)Dy-y = 0$. (*Second order*). It has 2 **independent** solutions. Suppose that a solution y can be *written as* $y = \Sigma_{n=0}{}^\infty a_nx^n$, and so $Dy = \Sigma a_nnx^{n-1}$ and $D^2y = \Sigma a_nn(n-1)x^{n-2}$. Substitute in the D.E. and get $\Sigma_{n\geq0} a_nn(n-1)2x^n + \Sigma_{n\geq0} a_nn2x^{n+1} + \Sigma_{n\geq0} a_nnx^n - \Sigma_{n\geq0} a_nx^n = 0$.

i.e. $\Sigma_{n\geq 0}\ a_nx^n\{2n(n-1)+n-1\} + \Sigma_{n\geq 0}\ a_n2nx^{n+1} = 0$, i.e. $\Sigma_{n\geq 0}\ a_nx^n(n-1)(2n+1) + \Sigma_{m\geq 1}\ 2(m-1)a_{m-1}x^m = 0$ *(putting $m = n+1$)*; i.e. $\Sigma_{n\geq 0}\ a_nx^n(n-1)(2n+1) + \Sigma_{n\geq 1}\ 2(n-1)a_{n-1}x^n = 0$ *(putting $n = m$)*. **<u>Summary</u>**: The D.E. implies that $\Sigma_{n\geq 0}\ x^n(n-1)[(2n+1)a_n+2a_{n-1}] = 0$. The *numbers* a_0, a_1, etc. in $y = \Sigma_{n\geq 0}\ a_nx^n$ must be chosen so that the *coefficient* of x^n is **zero** for all n, i.e. $(n-1)[(2n+1)a_n+2a_{n-1}] = 0$ $(n \geq 1)$. **Also,** for $n = 0$, $a_n(n-1)(2n+1) = 0$, i.e. $a_0(-1)(1) = 0$, so that $a_0 = 0$. And if $n \neq 1$, we have $(2n+1)a_n+2a_{n-1} = 0$. **Solution:** $a_0 = 0$; $a_1 =$ *anything*; $(n = 2)$ $5a_2+2a_1 = 0$, so that $a_2 = -{}^{2}/_{5}a_1$; $(n = 3)$ $7a_3+2a_2 = 0$, so that $a_3 = -{}^{2}/_{7}a_2 = {}^{4}/_{35}a_1$. *Solution:* $y = a_1[x-{}^{2}/_{5}x^2+{}^{4}/_{35}x^3-{}^{8}/_{315}x^4+\ldots]$. We have found a series solution of the D.E. But there is one *unknown* constant a_1; the D.E. has a **singularity** at $x = 0$; and there is only 1 solution as a *power* series.

To find two **independent** solutions, use the *Frobenius* method. <u>Assume that</u> $y = [\Sigma_{n\geq 0}\ a_nx^n]x^c$ for some *index* $c \in \mathbf{R}$. *Assume also that* $y = \Sigma_{n\geq 0}\ a_nx^{n+c}$, $Dy = \Sigma\ a_n(n+c)x^{n+c-1}$, and $D^2y = \Sigma\ a_n(n+c)(n+c-1)x^{n+c-2}$. **Substitute**: $2x^2D^2y+2x^2Dy+xDy-y = \Sigma\ a_n2(n+c)(n+c-1)x^{n+c} + \Sigma\ a_n2(n+c)x^{n+c+1} + \Sigma\ a_n(n+c)x^{n+c} - \Sigma\ a_nx^{n+c} = 0$. $(n \geq 0)$.

Look at the *lowest power of x*. In this case, it is x^c. Look at the **coefficient** of x^c: $a_02(0+c)(0+c+1)+a_0(0+c)-a_0$, i.e. $[2c(c-1)+c-1]a_0 = 0$. This is called the <u>index equation</u>. Always *assume* that $a_0 \neq 0$, so that we can divide through by it, giving, in this case, $2c(c-1)+(c-1) = 0$. So we have 2 solutions: $c = 1$ and $c = -½$. Now look at a **general** term x^{m+c}, with *coefficient* $a_m2(m+c)(m+c-1)+a_{m-1}2(m+c-1)+a_m(m+c)-a_m = 0$ for $m \geq 1$ *(putting $n = m-1$)*. <u>Hence</u> $a_m[2(m+c)(m+c-1)+(m+c-1)]+a_{m-1}[2(m+c-1)] = 0$, i.e. $(m+c-1)\{a_m(2m+2c+1)+a_{m-1}2\} = 0$. **Either** $m+c-1 = 0$, which is not possible *this* time as $c = 1$ or $c = -{}^{1}/_{2}$ and $m \geq 1$; **or** $a_m = -2a_{m-1}/2m+2c+1$. **<u>Cases</u>**: $c = 1$ *gives* $a_m = -2a_{m-1}/2m+3$; and $c = -{}^{1}/_{2}$ *gives* $a_m = -a_{m-1}/m$. So $y = a_0x^{-1/2}[1-x+{}^{1}/_{2}x^2-{}^{1}/_{6}x^3+\ldots]$.

1st March 2000

<u>Frobenius Method (Examples)</u>

Consider $(2x+x^3)D^2y-Dy-6xy = 0$. *Assume that* $y = \Sigma_{n=0}^{\infty}\ a_nx^{n+c}$, with $a_0 \neq 0$. Then $y = a_1x^c+a_1x^{c+1}+a_2x^{c+2}+\ldots$, $Dy = ca_0x^{c-1}+(c+1)a_1x^c+(c+2)a_2x^{c+1}+\ldots$, and $D^2y = c(c-1)a_0x^{c-2}+c(c+1)a_1x^{c-1}+(c+2)(c+1)a_2x^c+\ldots$ **Now** $6xy$ starts at x^{c+1}; Dy at x^{c-1}; x^3D^2y at x^{c+1}; and $2xD^2y$ at x^{c-1}. The <u>lowest</u> power of x is x^{c-1}, with coefficient $2c(c-1)a_0-ca_0 = 0$. Now $a_0 \neq 0$, so we have $2c(c+1)-c = 0$. *Solve the index equation* to get $c = 0$ or $2c = 3$. There are <u>two</u> possible solutions: one beginning at $c = 0$, and the other beginning at $c = {}^{3}/_{2}$.

Coefficient of x^c: $2(c+1)ca_1-(c+1)a_1 = 0$, so either $c+1 = 0$ *(not true)*, or $2c-1 = 0$ *(not true)*, or $a_1 = 0$. The x^{c+1} term **occurs as** $2x(c+2)(c+1)a_2x^c$ (<u>From</u> $D^2a_2x^{c+2}$) $+ x^3c(c-1)a_0x^{c-2}$ (<u>From</u> $D^2a_0x^c$) $- (c+2)a_2x^{c+1}$ (<u>From</u> Da_2x^{c+2}) $- 6xa_0x^c$ (<u>From</u> a_0x^c). So $2(c+2)(c+1)a_2+c(c-1)a_0-(c+2)a_2-6a_0 = 0$; $(c+2)a_2[2c+2-1]+(c^2-c-6)a_0 = 0$; $(c+2)(2c+1)a_2+(c+2)(c-3)a_0 = 0$. Now $c \neq -2$, so $a_2 = (3-c)a_0/(2c+1)$. There are *similar equations for* the coefficients of x^{c+2}, x^{c+3}, etc.: $(2c+1)a_2+(c-3)a_0 = 0$; $(2c+3)a_3+(c-2)a_1 = 0$; $(2c+5)a_4+(c-1)a_0 = 0$. **Hence** $a_0 \neq 0$ by definition; $a_1 = 0$; $a_2 = (3-c)a_0/(2c+1)$; $a_3 = ({}^{2-c}/_{2c+3})a_1$ (so $a_3 = 0$); $a_4 = ({}^{1-c}/_{2c+5})a_2 = {}^{(1-c)(3-c)}/_{(2c+1)(2c+5)}a_0$; etc. In *general*, $a_{2n}(2c+4n-3) = a_{2n-2}(5-c-2n)$. **<u>Summary</u>**: $a_0 \neq 0$, $a_1 = 0$. *Hence* $a_3 = a_5 = a_7 = \ldots = 0$. Now $c = 0$ or $c = {}^{3}/_{2}$. If $c = 0$, then $y = a_0[1+3x^2+{}^{3}/_{5}x^4-{}^{1}/_{15}x^6+\ldots]$. If $c = {}^{3}/_{2}$, then $y = a_0x^{3/2}[1+{}^{3}/_{8}x^2-{}^{3}/_{8\times 16}x^4+\ldots]$.

Example: Solve $4xD^2y+2Dy+y = 0$ by the *Frobenius method*. A: Put $y = a_0x^c+a_1x^{c+1}+\ldots$ The *lowest* power in the D.E. is min $\{1+c-2, c-1, c\}$. *The coefficient* of x^{c-1} is $4a_0c(c-1)+2a_0c = 0$. Assuming that $a_0 \neq 0$, the index equation is $2c[2(c+1)+1] = 0$. So $c = 0$ or $c = {}^1/_2$. Now consider the *general term* x^{c+n} in the D.E. The term is $a_nx^{c+n} + 2a_{n+1}(n+1+c)x^{c+n} +4xa_{n+1}(n+1+c)(n+c)x^{c+n-1} = 0$. (The terms *come from* y, Dy and $4xD^2y$ respectively).

2nd March 2000

Solve $2x^2D^2y+(7x^2+7x)Dy-3y = 0$. A: Assume that $y = \Sigma_{n\geq 0}\, a_nx^{n+c}$, with $a_0 \neq 0$. The *lowest* term is x^c. *Substitute*, and get $x^c[2c(c-1)a_0+7ca_0-3a_0] + x^{c+1}[2(c+1)ca_1+7ca_0+7(c+1)a_1-3a_1] + \ldots$ We have *general* term $x^{c+n}[2(c+n)(c+n-1)a_n+7(c+n-1)a_{n-1}+7(c+n)a_n-3a_n] = 0$. Look at the lowest term, x^c. *The coefficient* of x^c must be **zero**. But $a_0 \neq 0$, *hence* $2c(c-1)+7c-3 = 0$, i.e. $2c^2+5c-3 = 0$; $(2c-1)(c+3) = 0$; $c = -3$ or $c = ½$. **General** case: *coefficient* x^{c+n}, $n > 0$. This must be zero, so that $a_n[2(c+n)(c+n-1)+7(c+n)-3] + 7a_{n-1}[c+n-1] = 0$. Note: $2(c+n)(c+n-1)+7(c+n)-3$ is a *quadratic* in $(c+n)$: $2x(x-1)+7x-3$, so factorise to get $(c+n+3)(2(c+n)-1)$, so that $a_n = -{}^{7(c+n-1)}/_{(c+n+3)(2(c+n)-1)}a_{n-1}$. Therefore, all the a_n are determined by the a_0.

Case 1: $c = -3$: $a_n = -{}^{7(n-4)}/_{n(2n-7)}a_{n-1}$, so that $a_1 = -{}^{21}/_5a_0$, $a_2 = -{}^{14}/_6a_1 = {}^{49}/_5a_0$, etc. **Case 2:** $c = ½$: $a_n = -{}^{7(2n-1)}/_{2n(2n+7)}a_{n-1}$, so that $a_1 = -{}^7/_{18}a_0$, $a_2 = {}^{147}/_{492}a_0$, etc. *General solution*: $y = A_0x^{-3}(1-{}^{21}/_5x+{}^{49}/_5x^2-\ldots) + B_0x^{1/2}(1-{}^7/_{18}x+{}^{147}/_{492}x^2-\ldots)$. The *Frobenius* method works for certain D.E.'s, namely those of the **form** $D^2y+P(x)Dy+Q(x)y = 0$ which have a Regular Singular Point at $x = 0$.

For example, the equation *above* has form $D^2y+{}^{(7x^2+7x)}/_{2x^2}Dy-{}^3/_{2x^2} = 0$, so that $P(x) = {}^{7(x+1)}/_{2x}$ and $Q(x) = -{}^3/_{2x^2}$. If P and Q are *defined* and *continuous* at $x = 0$, we have a **regular** point, and the solution is an ordinary series. (Taylor). If P and Q are not defined, we have a **singularity**. A D.E. is said to be *regular singular* at $x = 0$ if $xP(x)$ and $x^2Q(x)$ are defined and continuous at $x = 0$. Example: $xP(x) = {}^{7(x+1)}/_2$; $x^2Q(x) = -{}^3/_2$.

Assignment 2: Set 2/3; In 13/3; Back 24/3

Q: Using *Laplace transforms* and *partial fractions*, solve the differential equation $D^2y+2Dy+y = e^t$ with initial conditions $y(0) = 0$ and $Dy(0) = 0$. A: Let Y denote the *Laplace transform* of the function. Then the Laplace transform satisfies the equation $s^2Y+2sY+Y = {}^1/_{s-1}$. Hence $Y = {}^1/_{(s-1)(s+1)^2}$, and use partial fractions to give ${}^1/_{(s-1)(s+1)^2} = {}^A/_{s-1}+{}^B/_{s+1}+{}^C/_{(s+1)^2}$, so that $y(t) = Ae^t+Be^{-t}+Cte^{-t}$. Multiplying out the partial *fraction* gives $1 = A(s+1)^2+B(s-1)(s+1)+C(s-1)$, so that $A = ¼$, $C = -½$, and $B = -¼$, and therefore $y(t) = ¼e^t-{}^1/_4e^{-t}-{}^1/_2te^{-t}$.

Q: Let Y(s) denote the *Laplace transform* of the function y(t). Find an expression for Y if y satisfies the differential equation $D^2y-Dy-2y = 0$, with **initial** conditions $y(0) = 4$ and $Dy(0) = 4$. Hence find the *solution* y(t). A: Let Y denote the *Laplace transform* of the function. The *transform* of D^2y is $s^2Y-sy(0)-Dy(0) = s^2Y-s(4)-4 = s^2Y-4s-4$.

The *transform* of Dy is $sY-y(0) = sY-4$. Therefore, the Laplace transform satisfies the equation $s^2Y-4s-4-sY+4-sY = 0$. Hence $(s^2-s-2)Y = 4s$, **or** $Y = {}^{4s}/_{(s-2)(s+1)}$. Use partial fractions to give ${}^{4s}/_{(s-2)(s+1)} = {}^A/_{s-2}+{}^B/_{s+1}$, so that $y(t) = Ae^{2t}+Be^{-t}$. Multiplying out the partial fraction *gives* $4s = A(s+1)+B(s-2)$, so that $A = {}^8/_3$ and $B = {}^4/_3$. Therefore, $y(t) = {}^8/_3e^{2t}+{}^4/_3e^{-t}$. We can *check* this by **substitution**.

Q: Use the Frobenius method to solve the D.E. $2xD^2y-Dy+4x^3y = 0$. A: To solve using the **Frobenius** method, substitute $y = \Sigma_{n\geq0}\, a_nx^{n+c}$, so $Dy = \Sigma_{n\geq0}\, a_n(n+c)x^{n+c-1}$ and $D^2y = \Sigma_{n\geq0}\, a_n(n+c)(n+c-1)x^{n+c-2}$. *Hence* $\Sigma_{n\geq0}\, 2xa_n(n+c)(n+c-1)x^{n+c-2} - \Sigma_{n\geq0}\, a_n(n+c)x^{n+c-1} + \Sigma_{n\geq0}\, 4x^3a_nx^{n+c} = 0$. The *lowest* power of x is x^{c-1}, and the **term** is $2xa_0c(c-1)x^{n-2}-a_0cx^{c-1}$. Hence, as we *assume* that $a_0 \neq 0$, we have the *index* equation $2c(c-1)-c = 0$, so $c = 0$ or $2c = 3$.

The x^c term has *coefficient* $2a_1(1+c)(1+c-1)-a_1(1+c) = 0$, and *substituting* the possible values of c gives $a_1 = 0$. Similarly, the x^{c+1} term has *coefficient* $2a_2(2+c)(2+c-1)-a_2(2+c) = 0$, and the x^{c+2} term has *coefficient* $2a_3(3+c)(3+c-1)-a_3(3+c) = 0$, so that *again* we deduce that $a_2 = a_3 = 0$. For $n \geq 4$, we have the x^{c+n-1} term, with *coefficient* $2a_n(n+c)(n+c-1)-a_n(n+c)+4a_{n-4} = 0$. So $a_n = -(4a_{n-4})/(n+c)(2n+2c-3)$; and **thus** $a_4 = -{}^{4}/_{(4+c)(2c+5)}a_0$ and $a_8 = -{}^{4}/_{(8+c)(2c+13)}a_4$. **For** $c = 0$, we have $y = a_0\{1-{}^{1}/_{5}x^4+{}^{1}/_{130}x^8-...\}$. **For** $c = {}^{3}/_{2}$, we have $y = a_0x^{3/2}\{1-{}^{1}/_{11}x^4+{}^{1}/_{418}x^8-...\}$.

Q: Use the Frobenius method to solve the D.E. $2x^2D^2y-xDy+y-xy = 0$, giving the first **3** terms of each series. A: In *standard form*, the D.E. is $D^2y-{}^{Dy}/_{2x}+{}^{y(1-x)}/_{2x^2} = 0$, i.e. $P(x) = -{}^{1}/_{2x}$ and $Q(x) = {}^{1-x}/_{2x^2}$. **As** $xP(x) = -{}^{1}/_{2}$ and $x^2Q(x) = 1-x$ are *defined* and *continuous* at $x = 0$, the differential equation is **regular singular**.

Applying the *substitutions* as in the previous question gives $\Sigma_{n\geq0}\, 2x^2a_n(n+c)(n+c-1)x^{n+c-2} - \Sigma_{n\geq0}\, xa_n(n+c)x^{n+c-1} + \Sigma_{n\geq0}\, a_nx^{n+c} - \Sigma_{n\geq0}\, xa_nx^{n+c} = 0$. Writing out the *first few coefficients* of x enables us to get the **general** term in the sequence as $x^{c+n}[2(x+n)(c+n-1)a_n-(c+n)a_n+a_n-a_{n-1}]$. The **lowest** power of x is x^c, and the term is $2x^2a_0c(c-1)x^{n-2}-xa_0cx^{c-1}+a_0x^c$. Hence, as we assume that $a_0 \neq 0$, we have the *index equation* $2c(c-1)-c+1= 0$ — so $c = 1$ or $2c = 1$.

The x^{c+1} term has **coefficient** $2a_1(1+c)(1+c-1)-a_1(1+c)+a_1-a_0 = 0$, so $a_1 = {}^{1}/_{(2c-1)(1+c)+1}a_0$. For $n \geq 1$, the x^{c+n} term has *coefficient* $2a_n(n+c)(n+c-1)-a_n(n+c)+a_n-a_{n-1} = 0$, so $a_n = {}^{1}/_{(n+c)(2n+2c-3)+1}a_{n-1} = {}^{1}/_{(2n+2c-1)(n+c-1)}a_{n-1}$. **For** $c = 1$, we have $a_n = {}^{1}/_{(n+1)(2n-1)+1}a_{n-1} = {}^{1}/_{(2n+1)n}a_{n-1}$; $y = a_0x\{1+{}^{1}/_{3}x+{}^{1}/_{30}x^2+...\}$. For $2c = 1$, we have $a_n = {}^{1}/_{(2n+1)(n-1)+1}a_{n-1} = {}^{1}/_{n(2n-1)}a_{n-1}$; $y = a_0x^{1/2}\{1+x+{}^{1}/_{6}x^2+...\}$.We can check these by *(complicated)* substitutions.

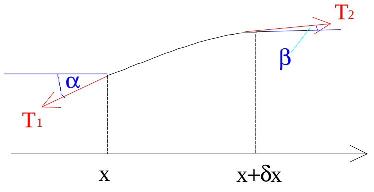
3rd March 2000

Tutorial

To solve e.g. $3x^2D^2y-xDy+y = 0$ using the *Frobenius* method, (1) Check Regular Singular; (2) Find the *index equation;* (3) Find the values of c; (4) Find **solutions;** (5) **Check the** solutions. The coefficient of the lowest term, e.g. x^c, is the index equation, from which you get your values of c. If you get a *general* term e.g. $x^{c+n}[3a_n(c+n)(c+n-1)-a_n(c+n)+a_n]$, its coefficient must be zero. Here, $(3a_n(c+n)(c+n-1)-a_n(c+n)+a_n) = 0$, so $a_n[3(c+n)(c+n-1)-(c+n)+1] = 0$. Substitute your c values in, e.g. for $c = 1$, we get $a_n(n(3n+2)) = 0$. As $n > 0$ and as $n \neq -{}^{2}/_{3}$, then a_n must be zero, i.e. for the *first* part of the solution, $y = Ax^1[1+0+0+...]$.

8th March 2000

Take a position on a *plucked string*, x. Move to $y = y(x,t)$, where the displacement y depends on the distance along the string, x, and the time, t. **Model the** movement of the string: take a small section. Tensions are along the *tangents* at angles α and β to the horizontal. **Resolve** the forces. There is no movement in the x-direction, so $T_1\cos\alpha = T_2\cos\beta$. **Let the constant** along the string be T, say.



The *force* in the y-direction is $T_2\sin\beta - T_1\sin\alpha$. But $T_1 = {}^{T}/_{\cos\alpha}$, so the force in the *y-direction* is $T\tan\beta - T\sin\alpha$. Now $F = ma$, where $a = {}^{d^2y}/_{dt^2}$ and m = density×length ($\rho\delta x$); so $T\tan\beta - T\tan\alpha = \rho\delta x\,{}^{d^2y}/_{dt^2}$; ($\tan\alpha$ = gradient at x, so...); $T\{({}^{dy}/_{dx})_{x+\delta x} - ({}^{dy}/_{dx})_x\} - \rho\delta x\,{}^{d^2y}/_{dt^2}$. So $[({}^{dy}/_{dx})_{x+\delta x} - ({}^{dy}/_{dx})_x]/\delta x = {}^{\rho}/_{T}\,{}^{d^2y}/_{dt^2}$, i.e. ${}^{d^2y}/_{dx^2} = {}^{\rho}/_{T}\,{}^{d^2y}/_{dt^2}$. Note that the derivatives are **partial**. This gives the 1-D *wave* equation, ${}^{\partial^2y}/_{\partial t^2} = c^2\,{}^{\partial^2y}/_{\partial x^2}$, for some constant c. (Note that ${}^{\rho}/_{T} > 0$). How do we solve this?

Separation of Variables

To find *possible functions* $y(x,t)$, try y as a product of a function of x by a function of t. We have *boundary* conditions $y(0,t) = 0$; $y(A\ t) = 0$; and $y(x,0)$ = some **initial** function. Write $y = y(x,t) = f(x)g(t)$. Then ${}^{\partial^2y}/_{\partial t^2} = f(x)\,{}^{d^2g}/_{dt^2}$, and ${}^{\partial^2y}/_{\partial x^2} = {}^{d^2f}/_{dx^2}\,g(t)$. So $f(x)\,{}^{d^2g}/_{dt^2} = c^2g(t)\,{}^{d^2f}/_{dx^2}$, i.e. ${}^{1}/_{c^2g(t)}\,{}^{d^2g}/_{dt^2} = {}^{1}/_{f(x)}\,{}^{d^2f}/_{dx^2} = k$. The LHS is a function of *t only*; and the RHS is a function of *x only*. So both sides must be constant.

9th March 2000

We have a **pair** of ordinary *differential equations:* ${}^{d^2f}/_{dx^2} - kf = 0$ and ${}^{d^2g}/_{dt^2} - c^2kg = 0$. The boundary conditions are $0 = y(0,t) = f(0)g(t)$ and $0 = y(A\ t) = f(A)g(t)$. So *either* $g(t) = 0$ always (ignore this: $y \equiv 0$), or $f(0) = f(A) = 0$. **Solve** ${}^{d^2f}/_{dx^2} - kf = 0$ with $f(0) = f(A) = 0$. What is the *constant* k?

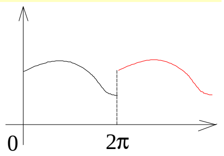
If $k = 0$, then ${}^{d^2f}/_{dx^2} = 0 \Rightarrow f = \alpha x + \beta$. *Boundary conditions:* $f(0) = f(A) = 0 \Rightarrow \alpha = \beta = 0$ (**ignore**!). If $k > 0$, $k = p^2$, then ${}^{d^2f}/_{dx^2} = p^2f$ has *solution* $f(x) = \alpha e^{px} + \beta e^{-px}$. But $f(0) = f(A) = 0 \Rightarrow \alpha = \beta = 0$. Finally, *try* $k = -p^2$, then ${}^{d^2f}/_{dx^2} + p^2f = 0$. **Solution:** $f(x) = \alpha\sin px + \beta\cos px$. But $f(0) = 0$, so $\beta = 0$. And $f(A) = 0$, so $\alpha\sin pA = 0$. So p only has *possible* values $p = {}^{n\pi}/_{A}$.

Summary: If $y = f(x)g(t)$, then ${}^{d^2f}/_{dx^2} + p^2f = 0$, with $p = {}^{n\pi}/_{A}$ for some *integer n*, and $f(x) = \alpha\sin({}^{n\pi x}/_{A})$. Further, g(t) *satisfies* ${}^{d^2g}/_{dt^2} + p^2c^2g = 0$. So $g(t) = \gamma\sin(pct) + \gamma\cos(pct)$. The **only** solutions of ${}^{\partial^2y}/_{\partial t^2} = c^2\,{}^{\partial^2y}/_{\partial x^2}$ are of the form $y = f(x)g(t)$, *with* $y(0,t) = y(A,t) = 0$ as **sums** of $\sin({}^{n\pi x}/_{A})\sin({}^{n\pi ct}/_{A})$ or $\sin({}^{n\pi x}/_{A})\cos({}^{n\pi ct}/_{A})$.

We can therefore have a **solution** of the form $y = \Sigma_{n\geq1}\, b_n.\sin({}^{n\pi x}/_{A})\sin({}^{n\pi ct}/_{A}) + \Sigma_{n\geq1}\, a_n.\sin({}^{n\pi x}/_{A})\cos({}^{n\pi ct}/_{A})$. The *final* boundary condition concerns the behaviour at $t = 0$, i.e. we require $\Sigma\, a_n\sin({}^{n\pi x}/_{A})$ to be the *initial* shape. This leads to **Fourier Series.**

15th March 2000

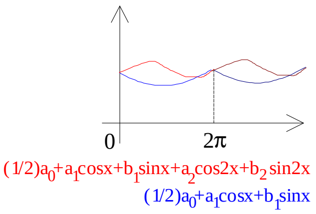
Problem: Write a function given over a given *interval* from $x = 0$ to $x = A$ as a sum of sine and cosine terms. Suppose that a function f(x) is defined fo $0 \leq x \leq 2\pi$. Also assume that f is **periodic**, of period 2π, i.e. $f(x+2\pi) = f(x)$.



Fourier Series of f(x)

Series: ${}^{1}/_{2}a_0 + \Sigma_{n\geq1}\, a_n\cos nx + b_n\sin nx$. **Coefficients:** $a_0 = {}^{1}/_{\pi}\int_0^{2\pi} f(x)dx$; $a_n = {}^{1}/_{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$; and $b_n = {}^{1}/_{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$. This assumes that f is **periodic:** $f(x+2\pi) = f(x)$. **Definition**: f is Bounded if $|f(x)| \leq$ a constant M. f has *jump discontinuity* at x_0 if $f(x_{0-}) = \lim_{x\to x_0, x<x_0} f(x)$ and $\lim_{x\to x_0, x>x_0} f(x) = f(x_{0+})$ both exist, so that **left** and **right** limits exist.

Dirichlet Theorem



If f is *bounded* on $[0, 2\pi]$, and continuous except for a *finite* numbe of jump discontinuities, then the **Fourier Series** converges point wise tc the average of the left and right hand limits, ${}^{1}/_{2}[f(x_{0-})+f(x_{0+})]$. *Convergenc* in the mean. A sequence of functions $S_n(x)$ converges in MEAN to f(x) i $E_n = \int^{2\pi}_{0} (f(x)-S_n(x))^2dx \to 0$. *We write* $f(x) = {}^{\text{L.I.M.}}{}_{n\to\infty} S_n(x)$. (L.I.M. = *Limi in Mean*).

Theorem: The *Fourier* Series of f converges in mean to f. Orthonormal functions. The sequence of functions $\phi_1, \phi_2, \ldots, \phi_k, \ldots$ is *orthonormal* if $\int^{2\pi}_{0} \phi_k(x)\phi_l(x)dx = 0$ if $k \neq l$, and 1 if $k = l$. **Theorem**: The *sequence* ${}^{1}/_{\sqrt{2\pi}}, {}^{\cos x}/_{\sqrt{\pi}}, {}^{\sin x}/_{\sqrt{\pi}}, {}^{\cos 2x}/_{\sqrt{\pi}}, {}^{\sin 2x}/_{\sqrt{\pi}}, \ldots$ is *orthonormal*. **Theorem**: If $S_N = \Sigma_{i=n}^{N} c_i\phi_i(x)$, where the c_i are *Fourier coefficients* of f, then $E_n = \int^{2\pi}_{0} f(x)^2dx - \Sigma_{i=1}^{N} c_i^2$.

16th March 2000

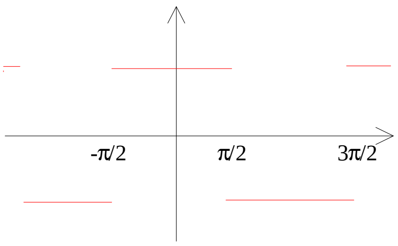
Bessel's Inequality. If the C_i are the *Fourier coefficients* of f w.r.t. orthonormal functions ϕ_i ($C_i = \int^{2\pi}_{0} f(x)\phi_i(x)dx$), then $\Sigma^{N}_{i=1} C_i^2 \leq \int^{2\pi}_{0} f(x)^2dx$. **Parsaval's Identity**. If ϕ_i is the *orthonormal* sequence of ordinary Fourier series, $\ldots, {}^{\cos ix}/_{\sqrt{\pi}}, {}^{\sin ix}/_{\sqrt{\pi}}, \ldots$, *then* $\Sigma^{\infty}_{i=1} C_i^2 = \int^{2\pi}_{0} f(x)^2dx$. So the series *converges in mean* to f.

Even & Odd Functions

Let f be periodic: $f(x+2\pi) = f(x)$. We can **consider** f over an interval $[-\pi,\pi]$, with $a_n = {}^{1}/_{\pi}\int^{2\pi}_{0} f(x)\cos nx dx = {}^{1}/_{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$. *Even* function: $f(-x) = f(x) \Rightarrow b_n = 0$. Only *cosine* series. *Odd* function: $f(-x) = -f(x) \Rightarrow a_n = 0$. Only *sine* series. Consider $a_n = {}^{1}/_{\pi}\int^{2\pi}_{0} (\ldots)dx = {}^{1}/_{\pi}\int_{-\pi}^{\pi} (\ldots)dx = {}^{1}/_{\pi}\int_{-\pi}^{0} f(x)\cos nx dx + {}^{1}/_{\pi}\int^{\pi}_{0} f(x)\cos nx dx$.

Substitute $u = -x$ in $\int_{-\pi}^{0}$, then $\int_{-\pi}^{0} f(x)\cos nx dx = \int_{\pi}^{0} f(-u)\cos(-nu)(-du) = -\int_{\pi}^{0} f(-u)\cos(nu)du = \int_{0}^{\pi} f(-u)\cos(nu)du$. So $a_n = {}^{1}/_{\pi}\int_{0}^{\pi} f(-u)\cos(nu)du + {}^{1}/_{\pi}\int_{0}^{\pi} f(x)\cos nx dx$. If f is ODD: $f(-u) = -f(u)$, then $a_n = 0$. If f is EVEN: $f(-u) = f(u)$, then $a_n = {}^{2}/_{\pi}\int_{0}^{\pi} f(u)\cos nu du$.

Change of Period Interval: $f(x+2L) = f(x)$. **Series** ${}^{1}/_{2}a_0+\Sigma_{n\geq 1} (a_n\cos{}^{n\pi x}/_{L}+b_n\sin{}^{x\pi x}/_{L})$; $a_n = {}^{1}/_{L}\int_{0}^{2L} f(x)\cos{}^{n\pi x}/_{L}dx$.



Example. Find the *Fourier series* of $f(x) = 1$ when $-{}^{\pi}/_{2}<x<{}^{\pi}/_{2}$ and $f(x) = -1$ when ${}^{\pi}/_{2}<x<{}^{3\pi}/_{2}$. *Periodic*, $f(x+2\pi) = f(x)$. The functio f is **even,** so $b_n = 0$. To find a_n, use $\pi a_0 = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0}+\int_{0}^{\pi}$. *Better* use $\int_{-\pi/2}^{\pi/2}+\int_{\pi/2}^{3\pi/2} = \int_{-\pi/2}^{\pi/2} 1dx + \int_{\pi/2}^{3\pi/2} -1.dx = \pi-\pi = 0$, so that $a_0 = 0$. **For** $n \geq 1$, $\pi a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx = \int_{-\pi/2}^{\pi/2} \cos nx dx + \int_{\pi/2}^{3\pi/2} -\cos nx dx = [{}^{\sin nx}/_{n}]_{-\pi/2}^{\pi/2} - [{}^{\sin nx}/_{n}]_{\pi/2}^{3\pi/2} = {}^{4\sin(n\pi/2)}/_{n}$. So $a_{2n} = 0$, and $a_{2n+1} = 4(-1)^n/(2n+1)\pi$.

Hence the *Fourier series* is ${}^{4}/_{\pi}[\cos x - {}^{\cos 3x}/_{3} + {}^{\cos 5x}/_{5} - {}^{\cos 7x}/_{7} + \ldots]$. This series converges *point-wise*. At $x = 0$, it converges to $f(0) = 1$. At $x = {}^{\pi}/_{2}$, it converges to 0 ($0 = [f({}^{\pi}/_{2+})+f({}^{\pi}/_{2-})]/2$). Also, $\int_{-\pi}^{\pi} f(x)^2dx = \int_{-\pi}^{\pi} 1dx = 2\pi$. **Hence** we have the following: $\Sigma a_n^2 = {}^{1}/_{\pi}(2\pi)$; $\Sigma {}^{16}/_{(2n+1)^2\pi^2} = 2$; and $\Sigma {}^{1}/_{(2n+1)^2} = {}^{\pi^2}/_{8}$.

Assignment 3: Set 15/3; In 29/3; Back 13/4

Recap: The Fourier series of the *function* $f(x)$ is $f(x) = {}^{1}/_{2}a_0 + \Sigma_{n\geq1} a_n\cos(nx) + \Sigma_{n\geq1} b_n\sin(nx)$, where $a_n = {}^{1}/_{\pi}\int^{\pi}_{-\pi} f(x)\cos(nx)dx$, and $b_n = {}^{1}/_{\pi}\int^{\pi}_{-\pi} f(x)\sin(nx)dx$. *Useful calculations*: $\int^{\pi}_{-\pi}\cos(nx) = \int^{\pi}_{-\pi} \sin(nx)dx = 0$; $\int x\sin(nx)dx = -{}^{x\cos(nx)}/_{n}+{}^{\sin(nx)}/_{n^2}$; $\int x\cos(nx)dx = {}^{x\sin(nx)}/_{n} + {}^{\cos(nx)}/_{n^2}$; $\int^{\pi}_{-\pi} x\cos(nx)dx = 0$; $\int^{\pi}_{-\pi} x^2\sin(nx)dx = 0$; and $\int x^2\cos(nx)dx = {}^{x^2\sin(nx)}/_{n} + {}^{2x\cos(nx)}/_{n^2}-{}^{2\sin(nx)}/_{n^3}$.

Q: Find the **Fourier series** of the function $f(x) = x^2$, for $-\pi\leq x\leq\pi$. Deduce *formulae* for the sum of the two series ${}^{1}/_{1}+{}^{1}/_{4}+{}^{1}/_{9}+\ldots+{}^{1}/_{n^2}+\ldots$ and ${}^{1}/_{1}-{}^{1}/_{4}+{}^{1}/_{9}-.+-..{}^{1}/_{n^2}-+\ldots$ A: The *function* $f(x) = x^2$ is **even,** so the Fourier coefficients b_n are all zero. Now $a_0 = {}^{2}/_{\pi}\int^{\pi}_{0} x^2dx = {}^{2\pi^2}/_{3}$. For $n > 0$, the *coefficients* a_n are given by the **formula** $a_n = {}^{2}/_{\pi}\int^{\pi}_{0} f(x)\cos(nx)dx$. Hence $a_n = {}^{2}/_{\pi}({}^{x^2\sin(nx)}/_{n}+{}^{2x\cos(nx)}/_{n^2}-{}^{2\sin(nx)}/_{n^3})^{\pi}_{0}$. Evaluating, *we have* $a_n = {}^{2}/_{\pi}{}^{2\pi}/_{n^2}((-1)^n-1)$, so $a_{2n} = 0$ and $a_{2n+1} = -{}^{8}/_{(2n+1)^2}$. At each *point* where the Fourier series of $f(x)$ **converges,** we have $f(x) = {}^{1}/_{2}a_0+\Sigma_{n\geq1} a_n\cos(nx)$, so $x^2 = {}^{\pi^2}/_{3}-\Sigma_{n\geq1}{}^{8}/_{(2n+1)^2}\cos(2n+1)x$.

To *deduce* the formula, we must revise the question. **Define** $f(x) = 0$ for $-\pi\leq x\leq0$, and $f(x) = x^2$ for $0\leq x\leq\pi$: neither an *even* nor an *odd* function. Get out at the end that the Fourier series of f is $f(x) = {}^{\pi^2}/_{6} + 2\Sigma_{n\geq1}(-1)^n\ {}^{\cos nx}/_{x^2} + \pi\Sigma_{n\geq1}(-1)^{n+1}\ {}^{\sin nx}/_{n} - {}^{4}/_{\pi}\Sigma_{n\geq1}\ {}^{\sin(2n-1)x}/_{(2n-1)^3}$. *Substituting* $x = \pi$ in the series gives ${}^{1}/_{1}-{}^{1}/_{4}+{}^{1}/_{9}-\ldots+{}^{1}/_{n^2}-.. = {}^{\pi^2}/_{12}$. *Similarly*, $x = 0$ gives ${}^{1}/_{1}+{}^{1}/_{4}+{}^{1}/_{9}+\ldots+{}^{1}/_{n^2}+\ldots = {}^{\pi^2}/_{6}$.

Q: Find the *Fourier serie*s of the function f defined by $f(x) = \pi$ for $-\pi\leq x\leq{}^{\pi}/_{2}$, and $f(x) = 0$ *elsewhere*. A: $a_0 = {}^{1}/_{\pi}\int^{\pi}_{-\pi} f(x)dx = {}^{1}/_{\pi}\int^{\pi/2}_{-\pi} \pi dx = {}^{3\pi}/_{2}$. *Similarly*, $a_n = {}^{1}/_{\pi}\int^{\pi}_{-\pi} f(x)\cos(nx)dx = \int^{\pi/2}_{-\pi}\cos(nx)dx$, and $b_n = \int^{\pi/2}_{-\pi} \sin(nx)dx$. So $a_n = {}^{\sin(n\pi/2)}/_{n}$: $a_{2n} = 0$; and $a_{2n+1} = (-1)^n$. Similarly, $b_n = {}^{1}/_{n}(\cos(n\pi)-\cos(n{}^{\pi}/_{2}))$, so $b_{4n} = 0$; $b_{4n+1} = -{}^{1}/_{4n+1}$; $b_{4n+2} = {}^{2}/_{4n+2}$; and $b_{4n+3} = -{}^{1}/_{4n+3}$.

Q: Which of the **following** functions are (a) *even;* (b) *odd;* (c) *neither*: $x+x^2$, $|x|$, e^{x^2}, $x\cos(x)$, $\sin(x)+\cos(x)$, $\sin^2(x)+\cos^2(x)$ and $x|x|$. A: e^{x^2}, $|x|$ and $\sin^2(x)+\cos^2(x)$ are all *even*; $x\cos(x)$ and $x|x|$ are *odd; and* $x+x^2$, e^{2x} and $\sin(x)+\cos(x)$ are *neither* odd nor even.

Q: Find the Fourier **sine** series of the function f *defined* by $f(x) = 1$ for $0\leq x\leq\pi$. A: Here, $b_n = {}^{2}/_{\pi}\int^{\pi}_{0} f(x)\sin(nx)dx = {}^{2}/_{\pi}\int^{\pi}_{0} \sin(nx)dx = {}^{2}/_{n\pi}(1-(-1)^n)$. *Hence* $b_{2n} = 0$ and $b_{2n+1} = {}^{4}/_{(2n+1)\pi}$. The series is $\Sigma_{n\geq1} b_n\sin(nx)$, or $\Sigma_{n\geq0}{}^{4}/_{(2n+1)\pi}\sin(2n+1)x$.

Q: Find the Fourier half range *sine* and *cosine* series of the function f defined by $f(x) = x$ for $0\leq x\leq\pi$. A: Here, (sine) $b_n = {}^{2}/_{\pi}\int^{\pi}_{0} x\sin(nx)dx = {}^{2}/_{\pi}(-{}^{\pi\cos(nx)}/_{n})$. Hence $b_{2n} = -{}^{2}/_{2n}$ and $b_{2n+1} = {}^{2}/_{2n+1}$. The *series* is $\Sigma_{n\geq1} b_n\sin(nx)$, or $-\Sigma_{n\geq1}{}^{2}/_{n}(-1)^n\sin(nx)$. The half range Fourier *cosine* series has **coefficients** $a_n = {}^{2}/_{\pi}\int^{\pi}_{0} x\cos(nx)dx$. Hence $a_0 = {}^{2}/_{\pi}{}^{\pi^2}/_{2}$ and $a_n = {}^{2}/_{\pi}[(-1)^n-1/n^2]$. *Hence* $a_{2n} = 0$ and $a_{2n+1} = -{}^{4}/_{\pi(2n+1)^2}$. The *series* is ${}^{\pi}/_{2}-\Sigma_{n\geq0}{}^{4}/_{\pi(2n+1)^2}\cos(2n+1)x$.

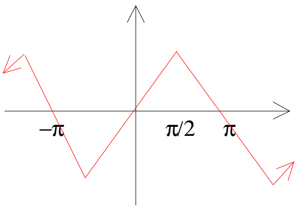
22nd March 2000

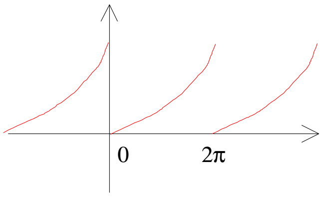
Examples of Fourier Series

Let $f(x) = 1$ if $x > 0$, and $f(x) = -1$ if $x < 0$, defined on $[-\pi,\pi]$. If we have an **odd** function, then $a_n = 0$. Now $b_n = {}^{1}/_{\pi}\int^{\pi}_{-\pi} f(x)\sin nx\,dx = {}^{2}/_{\pi}\int^{\pi}_{0} \sin nx\,dx$ (**odd**) $= -{}^{2}/_{\pi n}[\cos nx]^{\pi}_{0} = {}^{2}/_{\pi n}-{}^{2}/_{\pi n}(-1)^n$. So $b_n = 0$ if n is *even*, and $b_n = {}^{4}/_{\pi n}$ if n is *odd*. So the **Fourier series** of f(x) is $\Sigma_{n\geq 1} b_n\sin nx$, or ${}^{4}/_{\pi}[{}^{\sin x}/_{1}+{}^{\sin 3x}/_{3}+{}^{\sin 5x}/_{5}+\ldots] = {}^{4}/_{\pi}\Sigma_{k=1}^{\infty}\ {}^{\sin(2k-1)x}/_{2k-1}$.

The series converges *point-wise*. If $x > 0$, e.g. $x = {}^{\pi}/_{2}$, the series *converges* to 1. If $x = -{}^{\pi}/_{2}$, the series *converges* to -1. If $x = 0$, the series *converges* to ${}^{1-1}/_{2} = 0$. **So**, with $x = {}^{\pi}/_{2}$ we get $1 = {}^{4}/_{\pi}[\sin(\pi/2)+{}^{\sin(3\pi/2)}/_{3}+\ldots]$; $1 = {}^{4}/_{\pi}[1-{}^{1}/_{3}+{}^{1}/_{5}-{}^{1}/_{7}+\ldots]$; $\pi = 4[1-{}^{1}/_{3}+{}^{1}/_{5}-{}^{1}/_{7}+\ldots]$; and if $x = {}^{\pi}/_{4}$, we get $1 = {}^{4}/_{\pi}[{}^{\sqrt{2}}/_{2}+{}^{\sqrt{2}}/_{2\times 3}-{}^{\sqrt{2}}/_{2\times 5}-{}^{\sqrt{2}}/_{2\times 7}+{}^{\sqrt{2}}/_{2\times 9}+\ldots]$, so that $\pi = 2\sqrt{2}[1+{}^{1}/_{3}-{}^{1}/_{5}-{}^{1}/_{7}+{}^{1}/_{9}+{}^{1}/_{11}-\ldots]$.

Example: $f(x) = x$ for $-{}^{\pi}/_{2}<x<{}^{\pi}/_{2}$, and $f(x) = \pi-x$ for ${}^{\pi}/_{2}<x<{}^{3\pi}/_{2}$. f i **odd,** so that $a_n = 0$. *Now* $b_n = {}^{2}/_{\pi}\int^{\pi}_{0} f(x)\sin nx\,dx = {}^{2}/_{\pi}\int^{\pi/2}_{0} x\sin nx\,dx + {}^{2}/_{\pi}\int^{\pi}_{\pi/}$ $\pi\sin nx\,dx - {}^{2}/_{\pi}\int^{\pi}_{\pi/2} x\sin nx\,dx$. To do this, we need some *facts on integrals*: $\int\sin nx\,dx = -{}^{\cos nx}/_{n}$; $\int x\sin nx\,dx =$ <u>quoted</u> $= -{}^{x\cos nx}/_{n}+{}^{\sin nx}/_{n^2}$. **Hence** $(\pi b_n/2) = [-{}^{x\cos nx}/_{n}+{}^{\sin nx}/_{n^2}]^{\pi/2}_{0} - [-{}^{x\cos nx}/_{n}+{}^{\sin nx}/_{n}]^{\pi}_{\pi/2} + [-{}^{\pi\sin nx}/_{n}]^{\pi}_{\pi/2} \Rightarrow b_n = {}^{4\sin(n\pi/2)}/_{\pi n^2}$. **<u>So</u>** $b_{2n} = 0$, and $b_{2n+1} = 4(-1)^n/\pi(2n+1)^2$. (*Dodgy?*). The series is ${}^{4}/_{\pi}[\sin x-{}^{\sin 3x}/_{9}+{}^{\sin 5x}/_{25}-\ldots]$.

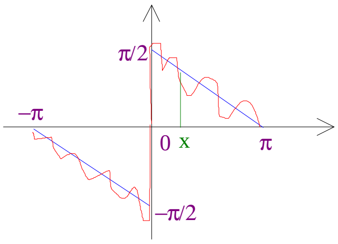




Example: $f(x) = x^2$ for $0\leq x\leq 2\pi$. Here, $a_0 = {}^{1}/_{\pi}\int^{2\pi}_{0} f(x)dx = {}^{1}/_{\pi}\int^{2\pi}_{0} x^2dx = {}^{8\pi^3}/_{3}$. **Now** ${}^{1}/_{\pi}\int^{2\pi}_{0} x^2\cos nx\,dx = [{}^{x^2\sin nx}/_{n}]^{2\pi}_{0} - \int^{2\pi}_{0}\ {}^{2x\sin n\pi}/_{n}dx = 0 - {}^{2}/_{n}[-{}^{x\cos nx}/_{n}+{}^{\sin nx}/_{n^2}]^{2\pi}_{0} = {}^{4\pi}/_{n^2}\cos 2\pi n = {}^{4\pi}/_{n^2}$. *So* $a_n = {}^{4}/_{n^2}$ for $n \geq 1$. Similarly, $b_n = -{}^{4\pi}/_{n}$. The series converges in the *mean:* $\Sigma a_n^2+\Sigma b_n^2 = {}^{1}/_{\pi}\int f(x)^2dx$; $(64\pi^4/18) + \Sigma_{n=1}^{\infty} (16/n^4)+{}^{16\pi^2}/_{n^2} = (32\pi^4/5)$.

23rd March 2000

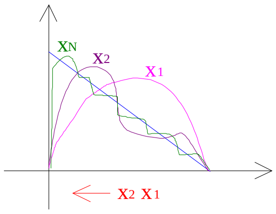
Gibbs Phenomenon



Let $f(x) = {}^{-\pi-x}/_{2}$ for $x < 0$; $f(x) = {}^{\pi-x}/_{2}$ for $x > 0$; and $f(x) = 0$ for $x = 0$. **<u>Fourier Series</u>**: It is odd, so $a_n = 0$. Now $b_n = {}^{2}/_{\pi}\int^{\pi}_{0} ({}^{\pi-x}/_{2})\sin nx\,dx = {}^{1}/_{n}$. Fourier Series: $\Sigma^{\infty}_{n=1}\ {}^{1}/_{n}\sin nx$. Let us draw a curve of the *partial* sum, $S_N(x) = \Sigma^{N}_{n=1}\ {}^{1}/_{n}\sin nx$, and **compare** it with f(x). (Shown red in the diagram). $S_N(x)$ *overshoots* the value ${}^{\pi}/_{2}$ near $x = 0$.

The series **converges** point-wise. If we fit $x \neq 0$, with x *near to* 0, then the values of $S_N(x)$ (for fixed x and varying N) approach a **straight** line. The peak will move *towards* $x = 0$ as N increases, but there will <u>always</u> be an overshoot.

<u>How big is the error?</u> Consider ${}^{x}/_{2}+S_N(x) = {}^{x}/_{2}+\Sigma^{N}_{k=1}\ {}^{1}/_{k}\sin kx = \int^{x}_{0}\ {}^{1}/_{2}+\Sigma^{N}_{k=1}\cos kx\,dx = \int^{x}_{0}\ {}^{\sin(N+½)t}/_{2\sin½t}dt$ (that was *using* $2\sin½t\times\cos kt=\ldots$) $= ½\int^{x}_{0}\ {}^{\sin(N+½)t}/_{½t}dt + ½\int^{x}_{0}({}^{1}/_{\sin½t}-{}^{1}/_{½t})\sin(N+½)t\,dt =$ (using $u = (N+½)t$) $= \int_{0}^{(N+½)x}\ {}^{\sin u}/_{u}du + (\ldots)$. The *2nd* integral tends to **zero** as $x \to 0$ (by l’Hopital’s rule *twice*); so ${}^{1}/_{\sin½t}-{}^{1}/_{½t} \to 0$ as $t \to 0$.

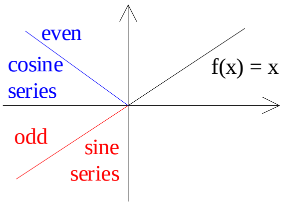


Summary so far: ${}^{x}/_{2}+\Sigma^{N}_{k=1}\,{}^{1}/_{k}\sin kx$ is *approximately* $\int_0^{(N+½)x}\,{}^{\sin u}/_{u}du$. **As** $N\to\infty$, can we *find an* $x_N\to 0$ so that $\int_0^{(N+½)x_N}\,{}^{\sin u}/_{u}du$ takes *substantial* values? Can we also *find* the x_N so that the **value** of ${}^{x}/_{2}+\Sigma^{N}_{k=1}\,{}^{1}/_{k}\sin kx$ is *approximately* $\int^{\infty}_0\,{}^{\sin u}/_{u}du$? Consider the *maximum* value of $\int^{b}_0\,{}^{\sin u}/_{u}du$. Choose x_N with $(N+½)x_N = \pi$, then $({}^{x}/_{2}$ + *series*$)$ **approximates** to $\int^{\pi}_0\,{}^{\sin u}/_{u}du = {}^{\pi}/_{2}(1.089490...)$.

Summary: For all N, we can find an x_N near to zero with the value of the series *overshooting* ${}^{\pi}/_{2}$ by a factor of 1.089490... This is true for **any** function with a discontinuity. The jump in the function is less than the *difference* between the maximum and the minimum of the series near the discontinuity — by the **factor**.

24th March 2000

Half Range Series



Let $f(x) = x$ for $x \geq 0$. Now *extend* $f(x)$ to the values < 0 to make f: (1) Odd; (2) Even. If we put $f(-x) = +f(x)$, we get an *even* function with Fourie series having $b_n = 0$ and $a_n = {}^{2}/_{\pi}\int^{\pi}_0 f(x)\cos nx\,dx$. This is the Fourier half range cosine series. *Similarly* for sine series.

29th March 2000

Legendre Polynomials: Summary Sheet

Let V denote the *vector space* of all Piecewise Continuous functions f defined on the interval [-1,1], with inner product defined *by* $\langle f,g\rangle = \int^{1}_{-1}f(x)g(x)dx$. The Legendre polynomials can be defined by the *Rodrigués* formula $P_n(x) = \frac{1}{2^n n!}D^n((x^2-1)^n)$. **Hence** $P_0(x) = 1$ and $P_1(x) = x$, etc. The functions $P_n(x)$ are *polynomials*. $P_n(x)$ contains only **even** powers of x if n is **even**, and only **odd** powers of x if n is **odd**.

The *polynomial* P_n is a solution of the *differential equation* $(1-x^2)D^2y-2xDy+n(n+1)y = 0$. The functions $P_n(x)$ satisfy the condition $\int^{1}_{-1}P_n(x)P_m(x)dx = 0$ for $m \neq n$. This is *summarised* by the statement that they are **orthogonal**: the polynomial $xP_n(x)$ is orthogonal to $P_k(x)$ for $k < n-1$.

The **Legendre** polynomials satisfy a *recurrence* relation of the form $xP_n(x) = \alpha P_{n+1}(x)+\beta P_{n-1}(x)$. More *specifically*, $(2n+1)xP_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x)$. *Notes*: $P_n(0) = 1$; $P_n(-1) = (-1)^n$; $P_n(x)$ has n distinct **roots** in the interval $-1<x, 1$; and the Legrendre polynomials are the result of applying the *Gram Schmidt process* to the polynomials 1, x, x^2, x^3, ...

Lecture notes on the above. **Inner Products:** f.g or $\langle f,g\rangle = \int^{1}_{-1}f(x)g(x)dx$, where f.g = g.f; $(f_1+f_2).g = f_1.g + f_2.g$; $(\lambda f).g = \lambda(f.g)$ (last two $\Rightarrow$ *linear*); $f.f \geq 0$; and $f.f = 0 \Leftrightarrow f = 0$. Therefore, we have the idea of "**orthogonal**": f.g = 0; and of **"size"**: $\|f\| = \sqrt{(f.f)}$.

In the *equation* $P_n(x) = \frac{1}{2^n n!}D^n((x^2-1)^n)$, for n = 0, we have ${}^{1}/_{2^0 0!}D^0(x^2-1)^0 = {}^{1}/_{1}D^0(1) = 1$; for n =1, we have ${}^{1}/_{2^1 1!}D^1(x^2-1)^1 = {}^{1}/_{2}(2x) = x$; for x = 2, we have ${}^{1}/_{2^2 2!}D^2((x^2-1)^2) = {}^{1}/_{8}D^2(x^4-2x^2+1) = {}^{1}/_{8}(12x^2-4) = {}^{3x^2-1}/_{2}$; and so on. $P_n(x)$ are polynomials of *degree n*. $(x^2-1)^n$ is of degree 2n; we differentiate *n times* to get a polynomial of degree 2n-n. $P_{2n-1}(x)$ has only **odd** powers of x because $(x^2-1)^n$ has only *even* powers of x.

Proposition: $P_n(x)$ satisfies the D.E. $(1-x^2)D^2y-2xDy+n(n+1)y = 0$. **Proof**. Put $w = (x^2-1)^n$. *Differentiating*, $Dw = n(x^2-1)^{n-1}.2x$; so $(x^2-1)Dw = n2xw$. We want **information** on D^nw. *Use Liebnitz's theorem to differentiate a product:* $D(uv) = uDv+vDu$; $D^2(uv) = uD^2v+2DuDv+vD^2u$; $D^n(uv) = uD^nv+(^n{}_1)DuD^{n-v}v+\ldots+(^n{}_r)D^ruD^{n-r}v+\ldots+vD^nu$. Use *Liebnitz* as follows:

Differentiate the equation $[(x^2-1)Dw = 2nxw]$ $(n+1)$ times; so that we have $D^{n+1}[(x^2+1)Dw] = 2nD^{n+1}[xw]$. **RHS** $= 2n[xD^{n+1}w+(n+1)DxD^nw+0]$. **LHS** $= (x^2-1)D^{n+2}w+(n+1)D(x^2-1)D^{n+1}w + (^{n+1}{}_2)D^2(x^2-1)D^nw$, i.e. $2nxD^{n+1}w+2n(n+1)D^nw = (x^2-1)D^{n+2}w+2x(n+1)D^{n+1}w+(n+1)nD^nw$, i.e. $(x^2-1)D^{n+2}w+2xD^{n+1}w-n(n+1)D^nw = 0$.

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Put $P_n(x) = (1/2^nn!)D^nw$ to get $(1-x^2)D^2P-2xDP+n(n+1)P = 0$. (D.E. of *Legendre Polynomials)*. Laplace equation: $^{\partial^2u}/_{\partial x^2}+^{\partial^2u}/_{\partial y^2}+^{\partial^2u}/_{\partial z^2} = 0$. Look for solutions having some *symmetry*. **Spherical** Polars: $z = r\cos\phi$; $x = r\sin\phi\cos\theta$; and $y = r\sin\phi\sin\theta$. This changes the *Laplace* equation to $^{\partial}/_{\partial r}(r^2\,^{\partial u}/_{\partial r}) + ^1/_{\sin\phi}\,^{\partial}/_{\partial\phi}(\sin\phi\,^{\partial u}/_{\partial\phi}) + ^1/_{\sin^2\phi}\,^{\partial^2u}/_{\partial\theta^2} = 0$.

Assume the *independence* from θ, and then assume that $u = F(r)G(\phi)$. The **LHS** depends on r, and the **RHS** on ϕ, so both sides are *constant*. Separate variables and get 2 ordinary D.E.'s: $r^2\,^{d^2F}/_{dr^2}+2r\,^{dF}/_{dr}-\lambda F = 0$ (---(1)), and $^1/_{\sin\phi}\,^{d}/_{d\phi}(\sin\phi\,^{dG}/_{d\phi})+\lambda G = 0$ (---(2)), *depending* on λ. (1) has solutions of the *form* $f = r^{-\frac{1}{2}+\sqrt{\lambda+(1/4)}}$. **Assume** that $-^1/_2+\sqrt{}(\lambda+^1/_4)$ is an *integer*. $n \geq 0 \Rightarrow \lambda = n(n+1)$. This makes the behaviour near $r = 0$ acceptable**.** Substitute $\lambda = n(n+1)$ and $x = \cos\phi$ in (2) to get Legendre's D.E., $(1-x^2)^{d^2G}/_{dx^2}-2x\,^{dG}/_{dx}+n(n+1)G = 0$.

Result: $\int^1_{-1}P_n(x)P_m(x)dx = 0$ if $n \neq m$, and $<P_n, P_m> = 0$. **Proof**: use the D.E.'s $(1-x^2)D^2P_n - 2xDP_n + n(n+1)P_n = 0$ and $(1-x^2)D^2P_m-2xDP_m+m(m+1)P_m = 0$. Multiply the *first* by P_m, and the *second* by P_n; subtract, and obtain $(1-x^2)(P_mD^2P_n-P_nD^2P_m) - 2x(P_mDP_n-P_nDP_m) + (n(n+1)-m(m+1))P_nP_m = 0$, i.e. $D\{(1-x^2)(P_nDP_m-P_mDP_n)\} + (n(n+1)-m(m+1))P_nP_m = 0$. Now *integrate* from -1 to 1 to get $[(1-x^2)(P_nDP_m-P_mDP_n)]^1_{-1} + (n(n+1)-m(m+1))\int^1_{-1}P_m(x)P_n(x)dx = 0$. As $1-x^2 = 0$ at the *limits*, and as $n \neq m$, we have $\int^1_{-1}P_n(x)P_m(x)dx = 0$.

Recurrence Relation

$P_i(x)$ is a polynomial *of degree i*. Hence $P_0, P_1, \ldots, P_n$ form a **basis** for all polynomials of degree $\leq n$. So any polynomial f of *degree* n can be written as $f = \Sigma_{i=0}^n\ \alpha_iP_i$ for some α_i. Further, the P_i are orthogonal, so that $<f, P_i> = \alpha_i<P_i, P_i>$, and $\alpha_i = <f, P_i>/<P_i, P_i>$. Now $f = \Sigma_{i=0}^n\ (<f, P_i>/<P_i, P_i>)P_i$, and $f(x) = \Sigma_{i=0}^n\ (<f, P_i>/<P_i, P_i>)P_i(x)$, where f is of degree $\leq n$.

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We want to *show that* $xP_n = (^{n+1}/_{2n+1})P_{n+1} + (^n/_{2n+1})P_{n-1}$. **Continuation** of Proof: Apply the above to a *polynomial of* degree (n+1), namely $f = xP_n$. **Note**: $<xP_n, P_i> = \int^1_{-1} xP_n(x)P_i(x)dx = <P_n, xP_i>$ by simple *rearrangement* in the integral. xP_i is a polynomial of **degree** (i+1), and so if $i < n-1$, then it is of degree $< n$; and it is *orthogonal* to P_n. **Summary**: $<xP_n, P_i> = <P_n, xP_i> = 0$ *if* $i < n-1$. Hence in the formula for $f = xP_n$, we get only **3** terms. So $f = xP_n = \Sigma_{i=n-1}^{n+1}\ (<xP_n, P_i>/<P_i, P_i>)P_i$; and there *exists* a recurrence relation of the **form** $xP_n = \alpha P_{n+1} + \beta P_n + \gamma P_{n-1}$. *Further*, P_{n+1}, P_{n-1} and xP_n have the **same** parity — and **opposite** to P_n. Remember that P_n involves either **all** *even* powers of x or **all** *odd* powers of x. Hence $\beta = 0$, and *so* $xP_n(x) = \alpha P_{n+1}(x)+\gamma P_{n-1}(x)$.

Stop here — in fact find α by *comparing* the coefficients of x^{n+1}. *Similarly* for γ. Substituting $x = 1$, $P_n(1) = 1$. **Proof:** $P_0(1) = P_1(1) = 1$ by *calculation*. Then use induction on the recurrence relation: $P_{n+1}(x) = ({}^{2n+1}/_{n+1})xP_n(x) - ({}^{n}/_{n+1})P_{n-1}(x)$; so $({}^{2n+1}/_{n+1})-({}^{n}/_{n+1}) = 1$. *Similarly*, $P_n(-1) = (-1)^n$. *Exercise*: Prove that the polynomial $P_n(x)$ does **not** have repeated roots. Method for answering the question: assume that it does have repeated roots, and then use the D.E.

Theorem: $P_n(x) = 0$ has *n roots* in $(-1, 1)$. **Proof**: Obvious for $n = 0$. Assume true for $n \geq 1$. (A) $P_n(x) = 0$ has a *root* in $(-1, 1)$. $\int^1_{-1} P_n(x)dx = \int^1_{-1} P_n(x)P_0(x)dx = \langle P_n, P_0\rangle = 0$, by *orthogonality*. Hence $P_n(x)$ cannot *always be* positive, or *always be* negative, in $[-1,1]$. So we **must** change sign; and we must therefore be a root in $[-1, 1]$. (B) $P_n(x) = 0$ has n roots in $(-1, 1)$. Suppose in fact that it has *m* roots in $(-1, 1)$. Now we must have $m \leq n$ as P_n is of degree n; and $1 \leq m$ by the above. We also know that the roots are *distinct*. Form $Q(x) = (x-x_1)(x-x_2)...(x-x_m)$, where the x_i are these m roots. The **function** $Q(x)P_n(x)$ is of the same sign *throughout* $(-1, 1)$. When $P_n(x)$ changes sign at a distinct root, so does $Q(x)$. Hence $\int^1_{-1} Q(x)P_n(x)dx \neq 0$, i.e. $\langle P_n, Q\rangle \neq 0$. However, if $m < n$, then Q is a *polynomial of degree m*, so $Q = \Sigma_{i=1}^m \alpha_i P_i$; and $\langle Q, P_n\rangle = 0$ by *orthogonality*. We have a **contradiction**, and hence $m = n$.

Gram Schmidt Process

When applied to *polynomials* $1, x, x^2, ..., x^n$, the Gram Schmidt process produces an *orthogonal set of polynomials* $R_0, R_1, ..., R_n$, with R_i being of degree i. **Theorem**: $R_n(x) = \frac{2^n n! n!}{(2n)!}P_n(x)$. **Proof**: Show that R_n is a *multiple* of P_n. Then compare the **coefficients** of x^n.

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Theorem: Let $R_i(x)$ be a sequence of *polynomials of degree i* which are orthogonal, then $R_i(x)$ is a multiple of $P_i(x)$. **Proof**: The polynomials R_i form an *orthogonal* basis for a set of all polynomials of degree $\leq n$. Hence any polynomial of **degree** n can be written as $f(x) = \Sigma_{i=0}^n \alpha_i R_i(x)$, where $\alpha_i = {}^{\langle f(x), R_i(x)\rangle}/_{\langle R_i(x), R_i(x)\rangle}$.

Take f(x) to be $P_n(x)$: $P_n(x) = \Sigma_{i=0}^n {}^{\langle P_n(x), R_i(x)\rangle}/_{\langle R_i(x), R_i(x)\rangle} R_i(x) = {}^{\langle P_n, R_n\rangle}/_{\langle R_n, R_n\rangle} R_n$ (because $\langle P_n(x), R_i(x)\rangle = 0$ for $i < n$). This *follows* from the following: the set of functions P_j is also a **basis**. Hence any function can be written in *terms* of the P_j. So $R_i(x) = \Sigma_{j=0}^i \beta_j P_j(x)$ (summing to i), as $R_i(x)$ is of *degree* i. **Therefore**, $\langle P_n, R_i\rangle = \langle P_n, \Sigma_{j=0}^i \beta_j P_j\rangle = \Sigma_{j=0}^i \beta_j \langle P_n, P_j\rangle = 0$, as $\langle P_n, P_j\rangle = 0$ for $j \leq i < n$.

Assignment 4: Set 5/4; In 12/5

Q: Calculate the *Legendre Polynomial* $P_6(x)$. A: $P_n(x) = (1/2^n n!)D^n((x^2-1)^n)$; $P_6(x) = (1/2^6 6!)D^6((x^2-1)^6)$. Work out that $(x^2-1)^6 = x^{12}-6x^{10}+15x^8-20x^6+15x^4-6x^2+1$. *Differentiate* this 6 times to get $D^6((x^2-1)^6) = 665280x^6-907200x^4+302400x^2-14400$. Dividing *through* by $2^6 6!$, we obtain $P_6(x) = {}^1/_{16}(231x^6-315x^4+105x^2-5)$.

Q: Let $w = (x^2-1)^n$. Prove that w(x) and its *derivatives* $Dw(x), D^2w(x), ..., D^{n-1}w(x)$ (up to order n-1) vanish when $|x| = 1$. Use *integration* by *parts* to show that $\int^1_{-1} x^m P_n(x)dx = 0$ for $0 < m < n$. Use the *above result* to deduce that the Legendre polynomials are mutually **orthogonal**. A: (see over).

A: Let $w(x) = (x^2-1)^n$; $Dw(x) = n(x^2-1)^{n-1}.2x$; $D^2w(x) = n(n-1)(x^2-1)^{n-2}2x+2n(x^2-1)^{n-1}$; and so on. In order for $D^mw(x)$ to vanish, $D^mw(x)$ must have a **factor** of (x^2-1). If it does, then $D^mw(x)$ vanishes as (x^2-1) *vanishes* when $|x| = 1$. However, this factor must be in *all* of the terms. The minimum order of the term (x^2-1) in $D^mw(x)$ is $n-m$. Therefore, for $0 < m < n-1$, $D^mw(x)$ will contain a *factor* of (x^2-1) and will vanish. For $m \geq n$ however, $D^mw(x)$ does not vanish, as each part of the sequence does not **contain** the factor (x^2-1).

Now we **know** that $P_n(x) = (1/2^nn!)D^n((x^2-1)^n) = (1/2^nn!)D^nw(x)$; so $\int^1_{-1} x^mP_n(x)dx = (1/2^nn!)\int^1_{-1} x^mP_nw(x)dx$. Integration *by parts:* Let $u = x^m$ so that ${}^{du}/_{dx} = mx^{m-1}$; and let ${}^{dv}/_{dx} = D^nw(x)$ so that $v = D^{n-1}w(x)$. Therefore, $(1/2^nn!)\int^1_{-1}x^mD^nw(x)dx = (1/2^nn!)\{[x^mD^{n-1}w(x)]^1_{-1} - \int^1_{-1}mx^{m-1}D^{n-1}w(x)dx\}$. The first part is *zero* because $D^{n-1}w(1) = D^{n-1}w(-1) = 0$. So $(1/2^nn!)\int^1_{-1}x^mD^nw(x)dx = (1/2^nn!)[-\int^1_{-1}mx^{m-1}D^{n-1}w(x)dx]$. The result now follows from *induction:* repeat using the same method on the **new** integral. Eventually, the integral will disappear, and all evaluation points disappear *anyway*. Note that x^m will disappear before $D^nw(x)$, as $0 < m < n$. So we can **say** that $\int^1_{-1}x^mP_n(x)dx = 0$ for $0 < m < n$.

Now we must **show** that $\int^1_{-1} P_n(x)P_m(x)dx = 0$. (For $m \neq n$). Take $n < m$. Now $P_n(x)$ is a polynomial which can be **expressed** as $\Sigma_{i=0}^n a_ix^i$. (Some of the a_i (*even* or *odd*) will be zero). So $\int^1_{-1} P_n(x)P_m(x)dx = \int^1_{-1}(\Sigma_{i=0}^n a_ix^i)P_m(x)dx = \Sigma_{i=0}^n\int^1_{-1} a_ix^iP_m(x)dx$. But from the **above,** we know that $\int_1^{-1}a_ix^iP_m(x)dx = 0$ for $i < m$. So we have our **conclusion**.

Q: Write $P_{n+1}(x) = ({}^1/_{2(n+1)})(\frac{1}{2^nn!})D^{n+1}((x^2-1)(x^2-1)^n)$, and apply *Leibnitz* to deduce that $DP_{n+1} = ({}^1/_{2(n+1)})(x^2-1)D^2P_n + ({}^{n+2}/_{n+1})xDP_n + ({}^{n+2}/_2)P_n$. Use the *knowledge* of the differential equation satisfied by P_n to deduce that $DP_{n+1} = xDP_n + (n+1)P_n$. A: Let $P_{n+1}(x) = uv$, where $u = {}^1/_{2(n+1)}[1/(2^nn!)]$, and $v = D^{n+1}((x^2-1)(x^2-1)^n)$. Then, using *Liebnitz,* $DP_{n+1}(x) = uDv+vDu = {}^1/_{2(n+1)}[1/2^nn!]D^{n+2}((x^2-1)(x^2-1)^n) + 0$. (u is a *constant,* so $Du = 0$).

Let us now apply **Liebnitz** to $D^{n+2}((x^2-1)(x^2-1)^n)$, with $u = (x^2-1)$ and $v = (x^2-1)^n$. So $D^{n+2}(uv) = uD^{n+2}(v)+DuD^{n+1}(v)({}^{n+2}{}_1) + ... + ({}^{n+2}{}_r)Du^rD^{(n+2)-r}(v)+...+vD^{n+2}u$. But we *know* that $D^n(u) = 0$ for $n \geq 3$, [$u = (x^2-1)$; $Du = 2x$; $D^2u = 2$; $D^3u = 0$; ...], so we only need the first **three** terms of the series: $D^{n+2}(uv) = uD^{n+2}(v)+({}^{n+2}{}_1)DuD^{n+1}(v)+({}^{n+2}{}_2)D^2uDv = (x^2-1)D^{n+2}((x^2-1)^n) + (n+2)(2x)D^{n+1}((x^2-1)^n) + {}^{(n+2)(n+1)}/_2(2)D^n((x^2-1)^n)$.

So we can *deduce* that $DP_{n+1}(x) = {}^1/_{2(n+1)}[1/2^nn!]\{(x^2-1)D^{n+2}((x^2-1)^n) + (n+2)(2x)D^{n+1}((x^2-1)^n) + (n+2)(n+1)D^n((x^2-1)^n)\}$; $DP_{n+1}(x) = {}^{(x^2-1)}/_{2(n+1)}[1/2^nn!]D^{n+2}((x^2-1)^n) + {}^{(n+2)(2x)}/_{2(n+1)}[1/2^nn!]D^{n+1}((x^2-1)^n) + {}^{(n+2)(n+1)}/_{2(n+1)}[1/2^nn!]D^n((x^2-1)^n)$. Now *as* $P_n(x) = [1/2^nn!]D^n((x^2-1)^n)$, and using *Liebnitz*, like earlier, we obtain $DP_n(x) = [1/2^nn!]D^{n+1}((x^2-1)^n)$; $D^2P_n(x) = [1/2^nn!]D^{n+2}((x^2-1)^n)$. Therefore, *we can deduce that* $DP_{n+1}(x) = {}^{(x^2-1)}/_{2(n+1)}D^2P_n(x) + {}^{(n+2)x}/_{(n+1)}DP_n + {}^{(n+2)}/_2P_n$. **QED**.

Now the D.E. *satisfied by* P_n is $(1-x^2)D^2P_n - 2xDP_n + n(n+1)P_n = 0$; $-2xDP_n + n(n+1)P_n = (x^2-1)D^2P_n$. **Substituting** for $(x^2-1)D^2P_n$ in $DP_{n+1} = {}^1/_{2(n+1)}(x^2-1)D^2P_n+({}^{n+2}/_{n+1})xDP_n + ({}^{n+2}/_2)P_n$, we obtain $DP_{n+1} = {}^1/_{2(n+1)}[-2xDP_n+n(n+1)P_n) + ({}^{n+2}/_{n+1})xDP_n + ({}^{n+2}/_2)P_n = -xDP_n/(n+1) + nP_n/2 + ({}^{n+2}/_{n+1})xDP_n + {}^{n+2}/_2P_n = xDP_n(-{}^1/_{n+1}+{}^{n+2}/_{n+1}) + P_n({}^n/_2+{}^{n+2}/_2) = xDP_n({}^{-1+n+2}/_{n+1}) + P_n({}^{n+n+2}/_2) = xDP_n({}^{n+1}/_{n+1}) + P_n({}^{2n+2}/_2) = xDP_n(1) + P_n({}^{2(n+1)}/_2) = xDP_n+(n+1)P_n$. QED.

Q: Prove that the *Legendre polynomials* do not have **repeated** roots. A: Consider that $P_n(x)$ has a **double** root at $x = a$. Therefore, $(x-a)^2$ divides $P_n(x)$, and $P_n(x)$ can be written in the form $P_n(x) = (x-a)^2Q_n(x)$, where $Q_n(x)$ is a *polynomial* of degree n-2. So $DP_n(x) = 2(x-a)Q_n(x) + (x-a)^2DQ_n(x)$, **and** $D^2P_n(x) = 2Q_n(x) + 2(x-a)DQ_n(x) + 2(x-a)DQ_n(x) + (x-a)^2D^2Q_n(x)$. From the *above*, $P_n(a) = 0$, $DP_n(a) = 0$, and $D^2P_n(a) \neq 0$, as $Q_n(x)$ does **not** have the factor $(x-a)$.

Now $P_n(x)$ satisfies the D.E $(1-x^2)D^2P_n(x) - 2xDP_n(x) + n(n+1)P_n(x) = 0$. Substituting in $x = a$, we *obtain* $(1-a^2)D^2P_n(a) - 2aDP_n(a) + n(n+1)P_n(a) = 0$, where the red segments are zero. *Therefore,* $(1-a^2)D^2P_n(a) = 0$. As $D^2P_n(a) \neq 0$, we can **divide** through by it, giving $(1-a^2) = 0$, or $|a| = 1$. But $P_n(x)$ only has roots in the range $(-1, 1)$. This *contradicts* the above statement, and the root 'a' cannot be -1 or 1 as suggested. **Conclusion:** Legendre polynomials cannot have double roots.

Now *consider* that $P_n(x)$ has a **repeated** root of order r at $x = a$, $(1 < r \leq n)$, so that $P_n(x) = (x-a)^rQ_n(x)$, where $Q_n(x)$ is a polynomial of **degree** n-r. In the same way as before, we can say that $P_n(a) = 0$; $DP_n(a) = 0$; ...; $D^{r-1}P_n(a) = 0$, where $D^rP_n(a) \neq 0$. Now, if $P_n(x)$ satisfies the D.E. $(1-x^2)D^2P_n(x) - 2xDP_n(x) + n(n+1)P_n(x) = 0$, then it must *satisfy* all of its derivatives as well.

For **example**, differentiating once, $P_n(x)$ must *satisfy* $(1-x^2)D^3P_n(x) - 4xD^2P_n(x) + [n(n+1)-2]DP_n(x) = 0$. Differentiating r-2 times, we obtain an *expression* of the form $(1-x^2)D^rP_n(x) + (\alpha(x))D^{r-1}P_n(x) + (\beta(x))D^{r-2}P_n(x) = 0$. Substituting $x = a$ into the *above* equation, we obtain $(1-a^2)D^rP_n(a) + (\alpha(a))D^{r-1}P_n(a) + (\beta(a))D^{r-2}P_n(a) = 0$, i.e. $(1-a^2)D^rP_n(a) = 0$.

We are now in the **same** situation as before, in that we must have $|a| = 1$, because $D^rP_n(a) \neq 0$. But $|a|$ cannot be 1, so we have a *contradiction,* and we therefore **conclude** that there is no such root 'a' such that $(x-a)^r$ is a **repeated** root of $P_n(x)$.

Exam Paper: May 2000

SECTION 1 (Compulsory)

(1) (a) Define the Wronskian W of two independent solutions $y_1(x)$ and $y_2(x)$ of the differential equation $D^2y+P(x)Dy+Q(x)y = 0$. Show that $y = v_1y_1+v_2y_2$ is a particular solution of $D^2y+P(x)Dy+Q(x)y=R(x)$ where $Dv_1 = -y_2R/W$ and $Dv_2 = y_1R/W$.
[10 marks]

(b) Show that $y_1(x) = x$ and $y_2(x) = e^x$ are solutions of the differential equation $(x-1)D^2y-xDy+y=0$. By calculating the Wronskian $W = W(y_1, y_2)$ and by converting the non-homogeneous differential equation $(x-1)D^2y-xDy+y = (x-1)^2e^x$ into standard form, find its general solution. **[10 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Explain what is meant by the term "regular singular point" and show that the origin is a regular singular point of the differential equation $3xD^2y+4Dy+y=0$. Use the method of Frobenius to find the indical equation and a recurrence relation for the coefficients of the series solutions of this differential equation. Hence find the general solution giving the first 3 terms of each series. **[15 marks]**

(3) Let the Legendre polynomials be defined by the Rodrigués formula
$P_n(x) = \frac{1}{2^n n!} D^n((x^2-1)^n)$.
Show that P_n satisfies the differential equation $(1-x^2)D^2y-2xDy+n(n+1)y=0$. **[5 marks]**

Deduce that the functions $P_n(x)$ satisfy $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ for $m \neq n$. **[4 marks]**

Show that the polynomial $xP_n(x)$ is orthogonal to $P_k(x)$ for $k < n-1$. Deduce that there is a recurrence relation of the form $xP_n(x) = \alpha P_{n+1}(x) + \beta P_{n-1}(x)$. **[6 marks]**

(4) Define the Laplace transform Y(s) of y(t). Let D denote differentiation with respect to the dependent variable, $D = {}^{d}/_{dt}$.

(a) Use the method of Laplace transforms to solve the differential equation
$D^2x+4x = \sin t$, where $x(0) = Dx(0) = 0$. **[6 marks]**

(b) Use the method of Laplace transforms to solve the simultaneous differential equations $D^2y+z+y=0$ and $Dy+Dz = 0$, where $y(0) = Dy(0) = 0$ and where $z(0) = 1$.
[7 marks]

(5) Find the Fourier half-range sine and half-range cosine series for the function f defined by: $f(x) = \pi-x$, $0 \leq x \leq \pi$. Show that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$ and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. **[15 marks]**

(Questions done: 1, 2, 4)