➢ 30th September 1999

Matrices: Revision

The **zero** matrix is the additive identity. The additive *inverse* of A is (-A). Theorems: (A+B)+C = A+(B+C) is associativity; A+0 = A = 0+A is the additive identity; A+(-A) = 0 is the additive inverse; A+B = B+A is commutativity. k(A+B) = kA+kB. (k+*l*)A = kA+*l*A. (k*l*)A = k(*l*A). 1A = A and 0A = 0 (*field* zero on left, *matrix* zero on right).

Proof of k(A+B) = kA+kB: A = $(a_{i,j})$; B = $(b_{i,j})$. A+B = $(a_{i,j}+b_{i,j})$. k(A+B) = $(k(a_{i,j}+b_{i,j}))$ And kA = $(ka_{i,j})$, kB = $(kb_{i,j})$, kA+kB = $(ka_{i,j}+kb_{i,j})$. Matrix *Multiplication*: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$. *Theorems*: (AB)C = A(BC), A(B+C) = AB+AC, (B+C)A = BA+CA, k(AB) = k(AB) = A(kB), 0A = 0 = 0A. AB is not **necessarily** the same as BA.

Row operations: interchange rows, multiply by *scalars*, add a multiple of one to another. Echelon Form: see the yellow matrix. **Square** Matrix: 1's along diagonal, 0's everywhere else. Inverse: AB = I = BA is *needed* if B is the inverse of A. Determinant: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = ad-bc. In an e.g. 3×3 matrix, *the determinant* is $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$. The *minor* of e.g. c, is $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$. The **trace** of a square matrix = add everything on the *diagonal*. Calculate the inverse using $^{1}/_{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 4 & 6 & 0 & 3 & 6 \\ 0 & 0 & 3 & 4 & 1 & 5 & 6 \\ 0 & 0 & 0 & 4 & 7 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The **Adjunction** of a matrix is the transpose of the matrix of *cofactors*, where $c_{i,j} = (-1)^{i+j}M_{i,j}$ i.e. alternate **signs** as you go along, calculating the minors. Example: $\mathrm{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^t$. The **transpose** of a matrix is when the *rows* become the *columns* and vice-versa. Formula for calculating the inverse: $A^{-1} = {}^{\mathrm{adj}(A)}/_{\det(A)}$. Note: adj(A) is the **transpose** of the minor matrix.

➢ 30th September 1999

Linear Equations

$a_1x_1+a_2x_2+\ldots+a_nx_n = b$ has *solution* ($x_1 = k_1, \ldots, x_n = k_n$) or ($k_1, \ldots, k_n$). So $x_1+2x_2-4x_3+x_4 = 3$ may have *solution* (3,0,0,0) or (3,2,1,0) or (3-2a+4b-c, a, b, c). **Systems** of Linear Equations. Consider m linear equations in n *unknowns* $x_1, \ldots, x_n$. So we have $a_{1,1}x_1+\ldots+a_{1,n}x_n = b_1$ and so on until $a_{m,1}x_1+\ldots+a_{m,n}x_n = b_m$. This system is *homogenous* if $b_1 = \ldots = b_m = 0$.

It has a *particular* solution — an n-tuple satisfying all equations. The solution set is the set of all solutions. It is often specified as the "**general**" solution. We also have a homogenous system associated with a system of *linear* equations: $a_{1,1}x_1+\ldots+a_{1,n}x_n = 0$ until $a_{m,1}x_1+\ldots+a_{m,n}x_n = 0$. This **always** has a solution (the trivial solution $\underline{0}$). So systems of linear equations may be *inconsistent* (no solution) or *consistent* (unique / many solutions).

Theorem: If u is a **particular** solution of the non-homogenous system, and W is the general solution of the *associated* homogenous system, then $u+W = \{u+w\colon w\in W\}$ is the general solution of the original system. Observation: If the homogenous system has only the **trivial** solution, then the solution to the original is unique, and conversely.

Examples: y+4z = 6, y+z = 3, 2x+4y+6z = 20. **Interchange** the rows and form a matrix which we reduce to *echelon* form to get out the answers z = 1, y = 2, x = 3. If we get a last row like 0 0 0 0, for instance, this means that there are infinitely many solutions and we must choose a **parameter**. If we get e.g. 0 0 0 5, this means that the system is *inconsistent* and there are **no** solutions.

➢ 4th October 1999

Tutorial: Remember that **matrices** may only be added if they are *same* size. If det(A) = 0, then A has no inverse. In addition to row operations, remember that you can *rearrange* rows and variables. If you get to a stage when after reduction the last line is e.g. y-3z = 4, then you must assign a **parameter**. So here assign z = a, and then by substitution y-3a = 4 implies y = 4+3a and so on to get the *general* solution.

➢ 7th October 1999

Vector Spaces

A set V is a **Vector** Space over a field k if two operations: *addition* (+) and *scalar multiplication* (.) are defined such that **A1**: (u+v)+w = u+(v+w); **A2**: $\exists\ 0 \in V$ such that u+0 = u $\forall$u; **A3**: $\forall\ u \in V$, $\exists\ -u \in V$ such that u+(-u) = 0; **A4**: u+v = v+u $\forall\ u,v \in V$; **M1**: k(u+v) = ku+kv, $k \in K$; **M2**: (a+b)u = au+bu $(a,b \in K)$; **M3**: (ab)u = a(bu) $(a,b \in K)$; **M4**: $\exists\ 1 \in k$ such that 1u = u $\forall\ u \in V$.

Theorem: Let V be a vector space over a field k. Then (1) k0 = 0 $(0 \in V, k \in K)$; (2) 0u = 0 $(u \in V, 0 \in K)$; (3) ku = 0 if k = 0 or u = 0; (4) (-k)u = k(-u) = -(ku) (for all $k \in K$, $u \in V$). **Proofs**: (1) By A2, $\exists\ 0 \in V$ such that 0+0 = 0 (in V). By M1, k0 = k(0+0) = k0+k0. So k0+(-k0) = k0+k0+(-k0), *therefore* 0 = k0. Proof of (4): u+(-u) = 0 $\Rightarrow$ 0 = k0 = k(u+(-u)) = ku+k(-u). Therefore -ku = ku+k(-u)-ku = k(-u). Similarly, k+(-k) = 0 $\Rightarrow$ 0+0u = (k+(-k))u = ku+(-k)u $\Rightarrow$ -ku = ku+(-k)u = ku = (-k)u.

Example: Vector space K^n; *elements* $(x_1,\dots,x_n)$. Addition: $(x_1,\dots,x_n) + (y_1,\dots,y_n) = (x_1+y_1,\dots.)$. Scalar multiplication: $k(x_1,\dots,x_n) = (kx_1,\dots,kx_n)$. Other examples include *matrices*; *polynomials*; functions from *sets* into *fields*. Use the axioms to prove/disprove that something is a vector space. **Example**: elements (a,b); addition (a,b)+(c,d) = (a,b); scalar multiplication k(a+b) = (ka, kb). This is not a vector space because A4 doesn't hold: If u = (1,2) and v = (2,4), then (1,2)+(2,4) $\neq$ (2,4)+(1,2); (1,2) $\neq$ (2,4).

Subspace: w is a subspace of a vector space v iff (i) w is *non-empty*; $(w \neq \phi)$; (ii) w is closed under scalar multiplication, $kw \in W$; (iii) w is **closed** under vector addition: $w_1+w_2 \in W$. So W is a subspace of v iff $0 \in W$ and $kw_1+lw_2 \in W$ for all $k,l \in K$; $w_1,w_2 \in W$.

Let $\{v_1,\dots,v_n\}$ be a set of vectors in a vector **space** V over a field k. Any vector $v \in V$ of the form $v = k_1v_1+k_2v_2+\dots+k_nv_n$ is called a **Linear** Combination of $v_1,\dots,v_n$. Example: v = (1,-2,5); $\{v_1 = (1,1,1)\ v_2 = (1,2,3)\ v_3 = (2,-1,1)\}$. $v = av_1+bv_2+cv_3$; v = (a+b+2c, a+2b-c, a+3b+c) = (1,-2,5). Solve by **matrix** to give $v = -6v_1+3v_2+2v_3$. Linear independence: A set of *linear vectors* is linearly independent if none is a linear combination of the others. Spanning set: A set of vectors spans a space v if every vector is a linear **combination** of those in the set. So for all $v \in V$ there exists $k_1,\dots,k_n$ such that $V = k_1v_1+\dots+k_nv_n$.

Let us work in $\mathbf{R}^3$: (a,b,c), with a,b,c, $\in$ **R**. Let u = (1,2,3), v = (0,1,2), and w = (0,0,1). And let (a,b,c) = xu+yv+zw. *Solving* gives x = a, 2x+y = b, and 3x+2y+z = c. Therefore, x = a, y = b-2a, and c = 2b+a+5c. Proving that S is a *subspace* of V: (1) Prove that $0 \in$ S, and for all k,*l* $\in$ K; u,v $\in$ S, we have ku+*l*v $\in$ S. (2) Prove that $\{v_1,...,v_n\}$ generates a vector space V — prove that a **general** v $\in$ V can be written as a *linear combination* of the v_i.

Co-ordinates of a Vector Relative to a Basis

If $\{v_1,...,v_n\}$ is a basis for V, then for *all v* $\in$ V, there is a unique set of scalars $k_1,...,k_n$ such that $v = k_1v_1+...+k_nv_n$. (existence $\Leftarrow$ spanning property, uniqueness $\Leftarrow$ linear independence). $(k_1,...,k_n)$ or the ***column*** vector $[k_1,...,k_n]$ is called the co-ordinate vector of v relative to the basis $\{v_1,...,v_n\}$. *Example*: V = (1,-2,3) $\{v_1(1,1,1)\ v_2(1,2,3)\ v_3(2,-1,1)\}$. $V = -6v_1+3v_2+2v_3$. So we have (-6,3,2) or the ***column*** vector $[-6\ 3\ 2]v_i$.

Linear Transformations

A *linear* transformation T: U $\rightarrow$ V is a *mapping* of vector spaces such that $T(u_1+u_2) = T(u_1)+T(u_2)$; $T(ku_1) = k(T(u_1))$, where $u_1,u_2 \in$ U and k $\in$ K (field). **Notes**: T(0) = 0, so the origin is fixed. {T: U$\rightarrow$V is a *linear* transformation} is a vector space. A linear transformation can be specified by its effect on the **basis** elements. Consider an m×n matrix A over K. Let T: $K^n \rightarrow K^m$, where T(u) = Au. So we have an m×n matrix **multiplied** by a n×*l* matrix to **give** an m×*l* matrix. Now T(v+w) = A(v+w) = Av+Aw = T(v)+T(w). And T(kv) = A(kv) = k(Av) = k(T(v)). So $[{}^{2}{}_{1}{}^{0}{}_{3}]$ $\mathbf{R}^2\rightarrow\mathbf{R}$ $[{}^{x}{}_{y}] = [{}^{2x}{}_{x+3y}]$.

Matrix of a Linear Transformation

Let T: U$\rightarrow$V be a *linear* transformation defined on the basis $\{u_1,...,u_n\}$ with respect to the basis $\{v_1,...,v_n\}$. Then $T(u_1),...,T(u_n) \in$ V can be **written** (uniquely) as linear combinations of the $\{u_1,...,u_n\}$. So $T(u_1) = a_{1,1}v_1+...+a_{m,1}v_m$; $T(u_2) = a_{1,2}v_1+...+a_{m,2}v_m$, ..., $T(u_n) = a_{1,n}v_n+...+a_{m,n}v_m$. The *transpose* of the matrix of coefficients, denoted by $[T]^v{}_u$, is the Matrix **Representation** of T relative to the bases $\{u_1\}$ and $\{v_i\}$. $[T]^v{}_u$ is as shown on the right. (Note: an m×n matrix maps vectors of length n to *vectors* of length m).

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & & \dots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix}$$

<u>**Theorem**</u>: For any vector u $\in$ U, $[T]^v{}_u\ [u]_u = [T(u)]_v$. [(1)(2) = (3), where (1) = matrix representation of the linear transformation T w.r.t. u_i, v_i; (2) = U in terms of the basis $\{u_1,...,u_n\}$; (3) = the image of U under T in terms of $\{v_1,...,v_m\}$. **Change** of Basis. Suppose that $\{u_1,...,u_n\}$ and $\{u_1',....,u_n'\}$ are *bases* for a vector space u. <u>Let</u> $\{u_1' = a_{1,1}u_1+...+a_{1,m}u_n$; $u_2' = a_{2,1}u_1+...+a_{2,n}u_n$; ...; $u_m' = a_{m,1}u_1+...+a_{m,n}u_n$. (**the new** basis in terms of the **old** basis). Then the *transition* matrix (from old basis to new) is P = as shown on the right. The *transpose* of the matrix of <u>coefficients</u> from the equations.

$$\begin{bmatrix} a_{1,1} & \dots & a_{n,1} \\ \dots & & \dots \\ a_{1,n} & \dots & a_{m,n} \end{bmatrix}$$

Example: $\{u_1 = (1,0)\ u_2 = (0,1)$; $\{u_1' = (1,1)\ u_2' = (1,0)$. Let T: $\mathbf{R}^2\rightarrow\mathbf{R}^2$ be defined by T(x,y) = (4x-2y, 2x+y). So $T(u_1)$ = T(1,0) = (4,2). And $T(u_2)$ = T(0,1) = (-2,1). And so $[T]^u{}_u = [{}^{4}{}_{2}{}^{-2}{}_{1}]$. Now $[{}^{4}{}_{2}{}^{-2}{}_{1}][{}^{x}{}_{y}] = [{}^{4x-2y}{}_{2x+y}]$, so $u_1' = (1,1) = 1u_1+1u_2$; $u_2' = (1,0) = 1u_1+0u_2$. Let P = the transition matrix = $[{}^{1}{}_{1}{}^{-1}{}_{0}]$. New *basis* $u_1'+u_2'$; old basis u_2. So $[{}^{1}{}_{1}{}^{-1}{}_{0}][{}^{1}{}_{1}]_{u'} = [{}^{0}{}_{1}]_u$.

Theorem: Let P be the *transition* matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space v. Then for any **vector** v, $P[v]_f = [v]_e$. The effect is “from *new* to *old*”. LHS: v as a co-ordinate vector in the new basis $\{f_i\}$. RHS: v as a co-ordinate vector in the old basis $\{e_i\}$. **Theorem**: Let P be the *transition* matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space V. Then for any linear *mapping* T: V → V, we have $[T]_f = P^{-1}[T]_eP$. LHS: matrix representation of T w.r.t. the *new* basis $\{f_i\}$. RHS: matrix representation of T w.r.t. the *old* basis $\{e_i\}$. **Example**: Let $Q = [{}^{0}{}_{-1}{}^{1}{}_{-1}]$ and $P = [{}^{1}{}_{1}{}^{-1}{}_{0}]$. Here, PQ = I or $Q = P^{-1}$. So $[T]_{u'} = Q[T]_uP = [{}^{0}{}_{-1}{}^{1}{}_{1}][{}^{4}{}_{2}{}^{-2}{}_{1}][{}^{1}{}_{1}{}^{-1}{}_{0}]$; $[T]_{u'} = [{}^{3}{}_{1}{}^{-2}{}_{2}]$.

➢ 11th October 1999

Tutorial: Vector Spaces & Linear Transformations

The **standard** basis for $\mathbf{R}^2$ is $E = \{e_1 = (1,0), e_2 = (0,1)\}$. An *alternative* basis for $\mathbf{R}^2$ is $F = \{f_1 = (0,3), f_2 = (-1,2)\}$. Q: Write the following elements of $\mathbf{R}^2$ as *co-ordinate* vectors of E and then F: u = (1,1), v = (1,3), w = (x,y). A: $(1,1) = e_1+e_2 = f_1-f_2$. $(1,3) = e_1+3e_2 = {}^{5}/_{3}f_1-f_2$. $(x,y) = xe_1+ye_2 = -xf_1+{}^{1}/_{3}(y+2x)f_2$. All these coefficients could be expressed as **column** vectors.

Q: Express F in terms of E and E in terms of F to calculate the *transition* matrices P from E to F and Q from F to E. A: $f_1 = (0,3) = 0e_1+3e_2$; $f_2 = (-1,2) = -e_1+2e_2$. And $e_1 = (1,0) = {}^{2}/_{3}f_1-f_2$; $e_2 = (0,1) = {}^{1}/_{3}f_1+0f_2$. So $P = [{}^{0}{}_{3}{}^{-1}{}_{2}]$ and $Q = [{}^{2/3}{}_{-1}{}^{1/3}{}_{0}]$. Check your answers against the <u>vectors</u> u,v,w — i.e. check that QP = PQ = I, and that e.g. $[u]_E = P[u]_F$, $[v]_F = Q[v]_E$. A: Verify: $[{}^{0}{}_{3}{}^{-1}{}_{2}][{}^{2/3}{}_{-1}{}^{1/3}{}_{0}] = [{}^{1}{}_{0}{}^{0}{}_{1}]$, so $Q = P^{-1}$. Then perform the *simple* calculations to confirm that the answers are **correct**.

Q: Let T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the linear transformation defined by T(x,y) = (3x,2x+2y). Prove that Im(T) is a subspace of $\mathbf{R}^2$. Calculate $[T]^E{}_E$, $[T]^E{}_F$, $[T]^F{}_E$, $[T]^F{}_F$. A: $0 = T(0,0) \in$ Im(T). And $k_1(3x_1,2x_1+2y_1)+k_2(3x_2,2x_2+2y_2) = (3k_1x_1+3k_2x_2,\ 2k_1x_1+2k_1y_1+2k_2x_2+2k_2y_2) = (3(k_1x_1+k_2x_2),\ 2(k_1x_1+k_2x_2)+2(k_2x_2+k_2y_2)) = T((k_1x_1+k_2x_2), (k_1y_1+k_2y_2)) \in$ Im(T). So T <u>is</u> a subspace of $\mathbf{R}^2$.

2nd part. $T(e_1) = (3,2) = 3e_1+2e_2$; $T(e_2) = (0,2) = 0e_1+2e_2$. So $[T]^E{}_E = [{}^{3}{}_{2}{}^{0}{}_{2}]$. Now $T(e_1) = (3,2) = {}^{8}/_{3}f_1-3f_2$, and $T(e_2) = (0,2) = {}^{2}/_{3}f_1+0f_2$. So $[T]^E{}_F = [{}^{8/3}{}_{-3}{}^{2/3}{}_{0}]$. Further, $T(f_1) = (0,6) = 0e_1+6e_2$, and $T(f_2) = (-3,2) = -3e_1+2e_2$. So $[T]^E{}_F = [{}^{0}{}_{6}{}^{-3}{}_{2}]$. Finally, $T(f_1) = (0,6) = 2f_1+0f_2$, and $T(f_2) = (-3,2) = -{}^{4}/_{3}f_1+3f_2$. So $[T]^F{}_F = [{}^{2}{}_{0}{}^{-4/3}{}_{3}]$.

Q: Verify that $P^{-1}[T]^E{}_EP = [T]^F{}_F$; $P^{-1}[T]^E{}_E = [T]^E{}_F$, and that $[T]^E{}_EP = [T]^F{}_E$. A: $P^{-1}[T]^E{}_EP = [{}^{2/3}{}_{-1}{}^{1/3}{}_{0}][{}^{3}{}_{2}{}^{0}{}_{2}][{}^{0}{}_{3}{}^{-1}{}_{2}] = \ldots = [{}^{2}{}_{0}{}^{-4/3}{}_{3}] = [T]^F{}_F$. And $P^{-1}[T]^E{}_E = [{}^{2/3}{}_{-1}{}^{1/3}{}_{0}][{}^{3}{}_{2}{}^{0}{}_{2}] = [{}^{8/3}{}_{-3}{}^{2/3}{}_{0}] = [T]^F{}_E$. And $[T]^E{}_EP = [{}^{3}{}_{2}{}^{0}{}_{2}][{}^{0}{}_{3}{}^{-1}{}_{2}] = [{}^{0}{}_{6}{}^{-3}{}_{2}] = [T]^F{}_E$. QED.

➢ 14th October 1999

Subspaces Associated with a Matrix

The **Rowspace/Column space** of A is the space spanned by the *rows/columns* of A. The **Nullspace** of A is the space of all vectors $\underline{x}$ such that $A\underline{x} = 0$ (the *solution* space). **Null** Space of A^T: as above, but for A^T instead of A. Rank and Nullity: The **RANK** is the *dimension* of the rowspace (or column space). The **NULLITY** is the *dimension* of the null space (Solve $A\underline{x} = 0$). So a matrix has **Row** space (Rank) and **Null** Space (Nullity).

Row operations: *Interchange 2 rows, multiply a row by a non-zero scalar; add a multiple of one row to another row.*

Row **Equivalence**: B is row equivalent to A if B can be obtained from A in a finite sequence of *elementary row operations*. Theorem: Row equivalent matrices have the **same** row space, and the **same** null space. Elementary matrices: Row operations can be interpreted as *pre-multiplying* by elementary matrices, e.g. multiplying $[^{a}{}_{b}{}^{c}{}_{d}]$ on the left by $[^{0}{}_{1}{}^{1}{}_{0}]$ or $[^{1}{}_{0}{}^{0}{}_{1}]$.

To calculate a *basis* for the row space/rank, do the following: (1) Reduce the **matrix** to row-reduced echelon form. (2) The *non-zero* rows are a basis for the row space. The number of non-zero rows is the rank. (To calculate row space use **row reduction**). Example: For A = $[^{1}{}_{2}{}^{1}{}_{3}{}^{5}{}_{13}]$, row *reduce* to $[^{1}{}_{0}{}^{1}{}_{1}{}^{5}{}_{3}]$ ~ $[^{1}{}_{0}{}^{0}{}_{1}{}^{2}{}_{3}]$. So A {(1,0,2), (0,1,3)} has rank 2. For B = $[^{1}{}_{3}{}^{-1}{}_{-2}{}^{-2}{}_{-3}]$, reduce to $[^{1}{}_{0}{}^{-1}{}_{1}{}^{2}{}_{3}]$ ~ $[^{1}{}_{0}{}^{0}{}_{1}{}^{1}{}_{3}]$. So B {(1,0,1), (0,1,3)} has rank 2. **And** for C = $[^{1}4_{0}{}^{-1}{-3}_{-1}{}^{-1}{-1}_{3}]$, *reduce* to $[^{1}0_{0}{}^{-1}1_{2}{}^{-1}3_{6}]$ ~ $[^{1}0_{0}{}^{-1}1_{0}{}^{-1}3_{0}]$ (*echelon* form) ~ $[^{1}0_{0}{}^{0}1_{0}{}^{2}3_{0}]$ (reduced *echelon* form). So C is {(1,0,2), (0,1,3)}.

To calculate a **basis** for the *null space/nullity*, do the following: (1) Find the general solution of A$\underline{x}$ = 0. (2) Write down a **basis** of vectors which generates the *general* solution, e.g. general solution (a, 2a, b). So a basis for the null space *might* be {$[^{1}2_{0}]$, $[^{0}0_{1}]$} (a = 1, b = 0; then a = 0, b = 1). (3) The *Nullity* is the number of **free** variables in the general solution.

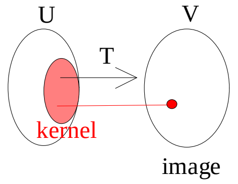
$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 3 & -7 & 2 & 0 & 1 & 4\\ 2 & -5 & 2 & 4 & 6 & 1\\ 4 & -9 & 2 & -4 & -4 & 7\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1 & 0 & -4 & -28 & -37 & 13\\ 0 & 1 & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.\ S=\begin{bmatrix}4a+28b+37c-13d\\ 2a+12b+16c-5d\\ a\\ b\\ c\\ d\end{bmatrix}$$

On the left, we have a matrix A. When we *row reduce* it, we get the second matrix, which has rank 2 and rowspace basis {(1st row), (2nd row)}. So $x_1-4x_3-28x_4-37x_5+13x_6 = 0$ and $x_2-2x_3-12x_4-16x_5+5x_6 = 0$. Set $x_3 = a$, $x_4 = b$, $x_5 = c$, and $x_6 = d$, and we get the *solution* vector (S) as shown above. We can express the vector *separately* as shown to the right as a **basis**. (1st: a = 1 (b = c = d = 0); 2nd: b = 1 (a = c = d = 0); 3rd: c = 1 (a = b = d = 0); 4th: d = 1). So the **nullity** is 4.

$$\left\{\begin{bmatrix}4\\2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}28\\12\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}37\\16\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}13\\-5\\0\\0\\0\\1\end{bmatrix}\right\}$$

Relation to Linear Transformations

Given a *Linear Transformation* T: U→V, $[T]^{v}{}_{u}$ is a matrix *representation* of T. Column space ([T]) = Im(T). Nullspace ([T]) = ker(T). Dimension(u) = Rank + Nullity. This is the **dimension formula**. So, given T, to calculate a basis for the *image*, find a basis for the column space of [T] and calculate a basis for the **kernel**; find a a basis for the nullspace of [T]. Im(T) = {T(u) *such that* u∈ U}. ker(T) = {u *such that* T(u) = 0}.



Relation to Systems of Equations

Let A be a m×n matrix. Then A$\underline{x}$ = 0 has only the *trivial* solution ⇔ A$\underline{x}$ = b has at most **one** solution ⇔ the column vectors of A are *linearly dependent*. So if we have a **system** of linear equations, {**rows** ($a_{1,1}x_1+\ldots+a_{1,n}x_n+k_1$), ($a_{2,1}x_1+\ldots+a_{2,n}x_n+k_2$), ..., {$a_{m,1}x_1+\ldots+a_{m,n}x_n+k_m$)}, then A$\underline{x}$ = $\underline{b}$ is the *matrix* equation. **Consistency** theorem: A$\underline{x}$ = $\underline{b}$ is consistent iff $\underline{b}$ is in the *column space* of A i.e. rank(A) = rank(A|b), the **augmented** matrix.

The Eigenvalues and Eigenvectors of a Square Matrix

An **eigenvalue** of a (square) matrix A is a scalar λ so that for some *non zero vector* v, **Av = λv**. In this case, the vector v is termed an *Eigenvector* of A associated to λ. The set of all such v for a **given** λ is called the *Eigenspace* of λ. Now (A-λI)v = 0 has a *nontrivial* solution iff det(λI-A) ≠ 0. So the **eigenvalues** can be found by solving this *equation* for λ, using the "λ→D→R→E" technique. *Example*: $[^3{}_8{}^0{}_{-1}][^1{}_2] = [^3{}_6]$ *represents* Ax = 3x.

Example: $[^1{}_{-2}{}^4{}_7]$. (λ, calculate A-λI)) $[^{1-\lambda}{}_{-2}{}^4{}_{7-\lambda}]$. (D, find the *determinant* of this matrix) (1-λ)(7-λ)-(-2)(4) = 7-7λ-λ+λ^2+8 = λ^2-8λ+15. (R, find the *roots*) λ^2-8λ+15 = 0; (λ-3)(λ-5) = 0. (E, find the *eigenvalues*) λ = 3 or λ = 5. Now find the **eigenvectors**. Using $[^{1-\lambda}{}_{-2}{}^4{}_{7-\lambda}]$, substitute in our *eigenvalues*.

First substitute λ = 3 to give $[^{1-3}{}_{-2}{}^4{}_{7-3}][^x{}_y]$. **So** (1-3)x+4y = 0 and (-2)x+4y = 0. These give -2x+4y = 0 and -2x+4y = 0. So we *get* x = 2y. **Setting** y = a, *therefore* x = 2a. (2a,a). So we get the eigenspace $\{[^2{}_1]\}$. Now do the *same* for λ = 5: $[^{1-5}{}_{-2}{}^4{}_{7-5}][^x{}_y]$ gives (1-5)x+4y = 0 *and* -2x+(7-5)y = 0. These give -4x+4y = 0 and -2x+2y = 0. **Set** y = a, so that x = a. So we have (a,a) with eigenspace $\{[^1{}_1]\}$. *Summary*: for λ = 3, the eigenspace has **basis** $\{[^2{}_1]\}$. For λ = 5, the eigenspace has **basis** $\{[^1{}_1]\}$. For 3×3 matrices, etc., do **exactly** the same as above. Remember that when calculating the *determinant*, look for zeros to simplify the expression. Important: look for COMMON FACTORS to simplify the **expression** you solve to get λ.

➢ 18th October 1999

Tutorial Questions

Finding Bases. Given a set of vectors that span a space, you can find a basis for the space by writing the vector as rows in a matrix and doing row reduction. Q: Find a basis for the space spanned by {(1,-2,5,-3), (2,3,1,-4), (3,8,-3,-5)}. What is the *dimension* of this space? Then extend the basis to a basis for the whole space $\mathbf{R}^4$. (add *linearly independent* vectors). A: Doing row reduction to eliminate any *linearly dependent vectors*, we get the **final** matrix $[^{1}0_{0}\ ^{-2}7_{0}\ ^{5}{-9}_{0}\ ^{-3}2_{0}]$. So the *basis* is {(1,-2,5,-3), (0,7,-9,2)}. The dimension is 2. Add the (independent) vectors (0,0,1,0) and (0,0,0,1). To check, note that no row of the matrix on the right can be **eliminated**.

$$\begin{bmatrix} 1 & -3 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q: **Rank** and **Nullity**. Given a linear transformation T: U→V, we can use matrices to calculate bases for its *image* (also known as the range) and *kernel*. The image of T is the column space of [T] (the row space of $[T]^t$). The kernel of T is the **null** space of [T]. The dimensions of these spaces are called **rank** and **nullity** respectively. Q: Let T: $\mathbf{R}^3\to\mathbf{R}^3$ be the linear mapping defined by T(x,y,z) = (x+2y-z, y+z, x+y-2z). Find a *basis* and the dimension of the image of T and the kernel of T.

A: The **equations** defining T (w.r.t. the *standard basis* for $\mathbf{R}^3$) are $T(e_1)$ = T(1,0,0) = (1,0,1) = $1e_1+0e_2+1e_3$; $T(e_2)$ = T(0,1,0) = (2,1,1) = $2e_1+1e_2+1e_3$; $T(e_3)$ = T(0,0,1) = (-1,1,-2) = $-e_1+1e_3-2e_3$. So the **matrix** representation of T w.r.t. the *standard* basis is $[T] = [^{1}0_{1}\ ^{2}1_{1}\ ^{-1}1_{-2}]$.

The image is equal to the **column** space of [T]. So find a **basis** for $\{[{}^{1}0_{1}], [{}^{2}1_{1}], [{}^{-1}1_{-2}]\}$. Put vectors as **rows** and do row *reduction*. (i.e. to find the image, do *row reduction* on $[T]^{t}$). $[{}^{1}2_{-1}\ {}^{0}1_{1}\ {}^{1}1_{-2}] \rightarrow [{}^{1}0_{0}\ {}^{0}1_{1}\ {}^{1}{-1}_{-1}] \rightarrow [{}^{1}0_{0}\ {}^{0}1_{0}\ {}^{1}{-1}_{0}]$. A **basis** for Im(T) is $\{[{}^{1}0_{1}], [{}^{0}1_{-1}]\}$. The *rank* of T is 2. The **kernel** is equal to the nullspace of [T], so *solve* $[T]\underline{x} = 0$, i.e. $\{^{x+2y-z=0;}\ y+z=0;\ _{x+y-2x=0;} \rightarrow \{^{x+2y-z=0;}\ y+z=0;\ _{-y-z=0}$. The **free** variable is z, so set z = a and the *general* solution is (3a, -a, a). Therefore, a basis for the **kernel** is $\{[{}^{3}{-1}_{1}]\}$, so the nullity of T is 1. **Check**: rank+nullity = 2+1 = 3 = $\dim(\mathbf{R}^3)$.

Q: Find all **eigenvalues** and a basis for each eigenspace for $A = [{}^{1}3_{6}\ {}^{-3}{-5}_{-6}\ {}^{3}3_{4}]$. A: (λ). Get the *matrix* $[{}^{1-\lambda}3_{6}\ \ {}^{-3}{-5-\lambda}_{-6}\ \ {}^{3}3_{4-\lambda}]$. (minus λ along the *diagonal*). (**D**). Calculate the **determinant**: $(1-\lambda)|{}^{-5-\lambda}{}_{-6}\ {}^{3}{}_{4-\lambda}| + 3|{}^{3}{}_{6}\ {}^{3}{}_{4-\lambda}| + 3|{}^{3}{}_{6}\ {}^{-5-\lambda}{}_{-6}| = \ldots = (-\lambda-2)(-\lambda-2)(-\lambda+4)$. (**R**) $(\lambda+2)^2(\lambda-4) = 0$. Find the *roots* of the polynomial. (**E**) *Eigenvalues* are -2 and 4. *Substitute* λ = -2 in (λ) and find the *nullspace*. $[{}^{3}3_{6}\ \ {}^{-3}{-3}_{-6}\ \ {}^{3}3_{6}] \rightarrow \{^{3x-3y+3z=0;}\ 3x-3y+3z=0;\ _{6x-6y+6z=0;} \rightarrow \{^{x-y+z=0;}\ 0=0;\ _{0=0}$. The free variables are *y and z* so set y = a, z = b to **obtain** the general solution (b-a, a, b) which means that the *eigenspace* has a basis $\{[{}^{-1}1_{0}], [{}^{1}0_{1}]\}$. **Substitute** λ = 4 into (λ) and find the **null** space: $[{}^{-3}3_{6}\ {}^{-3}{-9}_{-6}\ {}^{3}3_{0}] \rightarrow \{^{-3x-3y+3z=0;}\ 3x-9y+3z=0;\ _{6x+6y=0.} \rightarrow \{^{-3x-3y+3z=0;}\ -12y+6z=0;\ _{-12y+6z=0.}\ \{^{x+y-z=0;}\ 2y-z=0$. The **free** variable is z so set z = a. *General* solution: $({}^{1}/_{2}a, {}^{1}/_{2}a, a)$. The **basis** for the *eigenspace* is $\{[{}^{½}\ ½\ {}_{1}]\}$, or *alternatively* $\{[{}^{1}1_{2}]\}$.

➢ 21st October 1999

The Characteristic Polynomial

This is the polynomial we solve to find the **eigenvalues**. Recall: λ is an *eigenvalue* and $\underline{v}$ is an *eigenvector* for the matrix A if $A\underline{v} = \lambda\underline{v}$. (and for the linear **transformation** T if $T(\underline{v}) = \lambda\underline{v}$). Note: If A is **real,** then $\underline{v}$ must be real, but if A is *complex*, then $\underline{v}$ may be complex. It's too hard to solve $A\underline{v} = \lambda\underline{v}$ directly, so we first calculate the **eigenvalues**. To do this, we write the equation in the form $(A-\lambda I)(\underline{v}) = 0$ or $(I\lambda-A)(\underline{v}) = 0$.

This has a **non-zero** solution $\underline{v}$ iff $\det(I\lambda-A) \neq 0$. The *polynomial* $\det(I\lambda-A)$ in powers of λ is called the *characteristic polynomial*. Consider a matrix A, then the matrix A-λI is the characteristic **matrix** (This is the (λ) step: take λ away from the diagonal). Then the characteristic **polynomial** is the *determinant* of the matrix in powers of λ. Summary: In $A\underline{v} = \lambda\underline{v}$, A is the *given* matrix; $\underline{v}$ is the unknown vector; and λ is an *unknown* number. $(A-\lambda I)\underline{v} = 0$ is the characteristic **matrix**.

So an **example** may be where we get in the (R) stage $(1-\lambda)^2(2-\lambda) = 0$. So λ = 1 or λ = 2. We must consider the *algebraic multiplicity* of eigenvalues. Now $\det(I\lambda-A) = 0$ is called the characteristic **equation** of A. The scalars λ that satisfy the equation (the *roots* of the polynomial) are the **Eigenvalues** of A. So (**D**) characteristic polynomial; (**R**) characteristic equation; (**E**) characteristic values (*eigenvalues*).

Theorem: If A is an $n\times n$ *matrix* and λ is a real number, then λ is an *eigenvalue* of A ⇔ $(\lambda I-A)\underline{v} = 0$ has **nontrivial** solutions ⇔ $\exists\ v \in \mathbf{R}^n\ (\underline{v} \neq 0)$ such that $A\underline{v} = \lambda\underline{v}$ ⇔ λ is a solution of the *characteristic equation* of A. The **Eigenvectors** corresponding to an *eigenvalue* λ are the *non-zero vectors* $\underline{v}$ in the **solution** space of $(\lambda I-A)\underline{v} = 0$. For example, λ = 1 gives $\{^{y=0,}\ y=0,\ _{0=0,}$ so we get $\{[{}^{1}0_{0}], [{}^{0}0_{1}]\}$.

To calculate **eigenvectors** for the eigenvalues you have found, substitute in the *characteristic matrix* (A-λI) and solve the **homogenous** system of equations (A-λI)<u>v</u> = 0 for <u>v</u>. The *solution space* is the eigenspace. Example: λ = 2 gives $\{^{-1x+y=0}{}_{0=0}{}_{-12=0}$, so we have $\{[{}^{1}{}_{0}]\}$. The *geometric multiplicity* of eigenvalues = **the dimension** of the corresponding eigenspace.

<u>Relation to Linear Transformations</u>

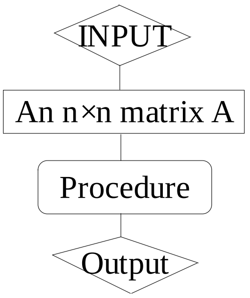
Let T: V→V be a *linear transformation.* Then it can be represented by <u>different</u> matrices $[T]_E$, $[T]_F$, $[T]_G$, ... according to a *choice of bases* E, F, G, ... We can always find a **transition** matrix for the change of basis, and so we write $[T]_F = P^{-1}[T]_E P$. This gives the *notion* of **<u>similarity:</u>** two matrices A & B are defined to be similar if there exists a matrix P so **that** $B = P^{-1}AP$. (Aside: as products of *elementary matrices*, $B = E_r^{-1}....E_1^{-1}AE_1...E_r$, because an *invertible* matrix P can be written as a <u>product</u> of elementary matrices).

<u>Theorem</u>: *similar matrices* have the <u>same</u> eigenvalues. Proof: This follows from the fact that similar matrices represent the **same** linear transformation. **<u>Theorem</u>**: The basis vector v_j is an eigenvector of T with *eigenvalue* λ, iff the j^th column of [T] has the form λv_j. *Corollary*: a matrix [T] is diagonal ⇔ every basis vector v_j of the transformation is an *eigenvector*.

<u>"Nice" matrices for linear transformations.</u> Given a linear transformation T: V→V, can we find a "nice" matrix to represent T? A "nice" matrix would be a *diagonal* one. For this, we would need a basis $\{v_1,...,v_n\}$ for V so that $\{T(v_1) = \lambda_1 v_1, T(v_2) = \lambda_2 v_2, ..., T(v_n) = \lambda_n v_n$. If we **can** find such a basis, then the v_i are *eigenvectors* for the diagonal entries λ_i of the matrix $[T]v_i$.

So **given** a matrix $[T]_E$ for T, we want to find *eigenvectors* and *eigenvalues* in order to construct a diagonal matrix for T — if possible. So, for example, $[T]v_i$ = a matrix with *zeros* everywhere except along the **diagonal,** where we have the λ_i. Given $[T]_E$, we want to find $[T]v_i$ (if possible). $P^{-1}[T]_E P$.

<u>Diagonalisation</u>



The *procedure* in the flow chart is as follows: Calculate **eigenvalues** $\lambda_1,...,\lambda_n$. Calculate *eigenvectors* $v_1,...,v_n$. Define P to be the **matrix** of column v_i's. (Calculate P^{-1}; Calculate $P^{-1}AP$). Therefore $P^{-1}AP$ is A *diagonalised*, a matrix containing zeroes, except on the diagonal, where we have the $\lambda_1,...,\lambda_n$.

<u>Summary</u>: Given a matrix A, perform λ-D-R-E to get the eigenvalues. Then find the *eigenvectors* of the eigenvalues. Using the vectors you get as the basis for the **eigenspace**, put these in a matrix to find P. Invert this, and then calculate $P^{-1}AP$. You should get a diagonalised matrix with the *eigenvalues* on the diagonal. Note that if we get e.g. (3-λ)(3-λ)(2-λ) = 0 in the (R) stage, then we want to find **<u>two</u> vectors** in the eigenspace basis for λ = 3.

Example: Recall that we had in **one** example T: $\mathbf{R}^2 \to \mathbf{R}^2$: (x,y)→(3x, 2x+2y). The bases were E = {(1,0), (0,1)} and F = {(0,3), (-1,2)}; $[T]_E = [{}^{3}{}_{2}{}^{0}{}_{2}]$, $[T]_F = [{}^{2}{}_{0}{}^{-4/3}{}_{3}]$. *Diagonalising* $[T]_E$, we do so as *follows* (see the next page).

(λ) $[^{3-\lambda}_{2}{}^{0}_{2-\lambda}]$. (**D**) $(3-\lambda)(2-\lambda)$. (**R**) $(3-\lambda)(2-\lambda) = 0$. (**E**) $\lambda = 3$ or $\lambda = 2$. For $[T]_F$, we have the *same* eigenvalues. *The eigenvectors* of $[T]_E$: $\lambda = 2 \Rightarrow [^{1}_{2}{}^{0}_{0}][^{x}_{y}] = [^{0}_{0}]$; the two *equations* we get are x = 0 and 2x = 0. So x = 0 and we **parametreise** y. Choosing y = a, and then setting a = 1, we get the *basis* for the eigenspace $\{[^{0}_{1}]\}$.

$\lambda = 3 \Rightarrow [^{0}_{2}{}^{0}_{-1}][^{x}_{y}] = [^{0}_{0}]$. So we get equations 0 = 0 and 2x-y = 0; y = 2x. Set y = a and then let a = 1; we get the *eigenspace* basis $\{[^{1/2}_{1}]\}$. Check that $A\underline{v} = \lambda\underline{v}$: e.g. $[^{3}_{2}{}^{0}_{2}][^{1/2}_{1}] = [^{3/2}_{3}] = 3[^{1/2}_{1}]$. Find **eigenvectors** for the *other* basis using the same method. Doing this, we get out that $\lambda = 3 \Rightarrow \{[^{-4/3}_{1}]\}$ and $\lambda = 2 \Rightarrow \{[^{1}_{0}]\}$. So now $P_E = [^{0}_{1}{}^{1/2}_{1}]$ and $P_F = [^{-4/3}_{1}{}^{1}_{0}]$. All that is left to do is to *invert* the matrices and check that $P^{-1}_n[T]_nP_n$ gives a *diagonalised matrix* with the eigenvalues on the **diagonal**. Example using P_F: *Conclude* that $P_F^{-1} = [^{0}_{1}\ ^{1}_{4/3}]$. **Then** $P_F^{-1}[T]_FP_F = [^{0}_{1}{}^{1}_{4/3}][^{2}_{0}{}^{-4/3}_{3}][^{-4/3}_{1}{}^{1}_{0}] = [^{3}_{0}{}^{0}_{2}]$ as *expected*, because $\lambda = 3$ and $\lambda = 2$.

➢ 25th October 1999

Diagonalisation: What is Going On/What Does it Mean

The eigenvectors must be a *basis* for $\mathbf{R}^n$ — otherwise the matrix would *not be diagonalisable*. To check this, consider the vectors as **rows** of a matrix M (i.e. the matrix P transposed). Reduce this to echelon form. If none of the rows is reduced to zero, then they are linearly independent. Alternatively, continue reducing to **reduced** echelon form, and get the identity matrix.

If the eigenvalues of a **REAL** matrix are complex, then the matrix connected is *not* diagonalisable. In summary, if we have an n×n matrix with entries in **R**, if the eigenvalues are in **C** then it is *not* diagonalisable. If the eigenvalues are in **R**, and we have *n linearly independent eigenvectors*, then the matrix **is** diagonalisable. If we have **less** than n linearly independent vectors, **it is not** diagonalisable. If we have an n×n matrix with entries in **C**, with eigenvalues in **C**, and n linearly independent vectors, then it **is** diagonalisable. If it has *less* than n linearly independent vectors, then it is **not** diagonalisable.

Every Matrix represents a *linear transformation*. If the matrix $A = [^{0}1_{1}\ ^{0}2_{0}\ ^{-2}1_{3}]$ represents a linear transformation $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$, (Using E, the *standard* basis of $\mathbf{R}^3$), then for $(x,y,z) \in \mathbf{R}^3$, find T(x,y,z). A: $T(e_1) = T(1,0,0) = 0e_1+1e_2+1e_3$. $T(e_2) = T(0,1,0) = 0e_1+2e_2+0e_3$. $T(e_3) = T(0,0,1) = -2e_1+1e_2+3e_3$. $[T]_E$ is the *transpose* of matrix of coefficients = A. So $T(x,y,z) = A[^{x}y_{z}] = (-2z, x+2y+z, x+3z)$.

What is the *transition matrix* from the basis E to the basis of eigenvectors? A: Express the **eigenvectors** v_i in terms of the *standard basis* e_i to calculate the transition matrix Q from e_i to v_i. $v_1 = -2e_1+1e_2+e_3$; $v_2 = -1e_1+0e_2+1e_3$; $v_3 = 0e_1+1e_2+0e_3$. The **transition** matrix is the transpose of the *coefficient* matrix, $Q = [^{-2}1_{1}\ ^{-1}0_{1}\ ^{0}1_{0}]$. So $[T]_v = Q^{-1}[T]_EQ$ as expected. Q: Why does putting eigenvectors in **columns** give a matrix which makes A into a diagonal matrix?

Q: What are the *conditions* on a, b, c and d so that the real matrix $[^{a}_{c}{}^{b}_{d}]$ is *diagonalisable*? A: We have the characteristic matrix $[^{a-\lambda}_{c}\ ^{b}_{d-\lambda}]$ and the characteristic equation $(a-\lambda)(d-\lambda)-bc = 0$. Now $ad-d\lambda-a\lambda+\lambda^2-bc = 0$; $\lambda^2+(-1-d)\lambda+(ad-bc) = 0$. We know that $A\lambda^2+B\lambda+C = 0$ has real solutions iff $B^2-4AC > 0$, i.e. $(-a-d)^2-4(ad-bc) > 0$, i.e. $\mathbf{(a+d)^2 > 4(ad-bc)}$.

➢ 28th October 1999

Differential Equations

Note: $\dot{y}$ here is used to *represent* $^{d}/_{dx}(y)$. Do not *confuse* it with $^{d}/_{dt}$, although the notation "dot" represents $^{d}/_{dt}$. Sometimes I will use y' instead of $\dot{y}$. Example: $\dot{y} = ay$ (a = constant). Example: Let $y = f(x)$, a function in x. Then $\dot{y} = {}^{d}/_{dx}f(x)$. Example: Let $\dot{y} = {}^{d}/_{dx}ce^{ax} = cae^{ax} = ay$; so that $y = Ce^{ax}$, the **general** solution. If we have the **initial** condition $y(0) = 5 = ce^{a(0)} = c$, then the *particular* solution is $y = 5e^{ax}$.

Linear system of first order differential equations. Let $y'_1 = a_{1,1}y_1+\ldots+a_{1,n}y_n$; $y'_2 = a_{2,1}y_1+\ldots+a_{2,n}y_n$; …; $y'_n = a_{n,1}y_1+\ldots+a_{n,n}y_n$. Here $y_1 = f_1(x)$, …, $y_n = f_n(x)$. The y_i's are different *functions* of the variable x. ($y'_i = {}^{d}/_{dx}(y_i)$). In matrix *representation*, we have the system as shown on the **right,** i.e. $Y' = AY$.

$$\begin{bmatrix} y'_1 \\ y'_2 \\ \cdots \\ y'_n \end{bmatrix} = \begin{bmatrix} a_{1,1}\ldots\ldots a_{1,n} \\ a_{2,1}\ldots\ldots\ldots a_{2,n} \\ \ldots\ldots\ldots\ldots \\ a_{n,1}\ldots\ldots\ldots a_{n,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_1 \\ \cdots \\ y_n \end{bmatrix}$$

Example: Let $y'_1 = 2y_1$, $y'_2 = 4y_2$, and $y'_3 = -y_3$. The *matrix* representation is $\begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Here, $Y' = AY$. *Solving*, $y_1 = c_1e^{2x}$, $y_2 = c_2e^{4x}$, and $y_3 = c_3e^{-x}$. If we have initial *conditions* $y_1(0) = -1$, $y_2(0) = 4$, and $y_3(0) = 5$, then the *particular solution is* $y_1 = -e^{2x}$, $y_2 = 4e^{4x}$, and $y_3 = 5e^{-x}$.

Solving Systems of First Order Linear Differential Equations

$$\begin{bmatrix} y_1 \\ \cdots \\ y_n \end{bmatrix} \begin{bmatrix} p_{1,1}\ldots\ldots p_{1,n} \\ \ldots\ldots\ldots\ldots\ldots \\ p_{n,1}\ldots\ldots\ldots p_{n,n} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots\cdot \\ u_n \end{bmatrix}$$

This is *easy* if the matrix of coefficients is **diagonal**. If it is **not** diagonal, use diagonalisation. The idea is to substitute the variables y_i by variables that give a *diagonal* system. In the matrix on the left, $Y = PU$. The **constants** p_{ij} are to be determined in such a way that the new system *involving* the variables u_{ij} has a diagonal coefficient matrix. Differentiation gives $Y' = PU'$. **<u>So,</u>** (1) Find P. (2) Substitute $Y = PU$ and $Y' = PU'$. (3) Solve $U' = (P^{-1}AP)U$. (4) Determine Y from $Y = PU$. Summary: **Find** a matrix P that diagonalises A. Set $D = P^{-1}AP$. Substitute $Y = PU$; $Y' = PU'$ to obtain $U' = DU$ (D is *diagonal*). Solve $U' = DU$. Determine Y **from** $Y = PU$.

Example: Let $y'_1 = y_1+y_2$ and $y'_2 = 4y_1-2y_2$. The matrix *representation* is $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$. Find P to *diagonalise* A: the characteristic equation is $\lambda^2+\lambda-6 = 0$; $(\lambda-2)(\lambda+3) = 0$. The *eigenvalues* are $\lambda = 2$ and $\lambda = -3$. The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$. So $P = \begin{bmatrix} 1 & -1/4 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. **Substitute** $Y = PU$ and $Y' = PU'$ to obtain $U' = DU$. So $\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Therefore, $u'_1 = 2u_1$ and $u'_2 = -3u_2$. So $u_1 = c_1e^{2x}$ and $u_2 = c_2e^{-3x}$. Now $Y = PU$; $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1/4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\exp(2x) \\ c_2\exp(-3x) \end{bmatrix} = \begin{bmatrix} c_1\exp(2x)-(1/4)c_2\exp(-3x) \\ c_1\exp(2x)+c_2\exp(-3x) \end{bmatrix}$. So the *general solution* is $y_1 = c_1e^{2x}-{}^{1}/_{4}c_2e^{-3x}$ and $y_2 = c_1e^{2x}+c_2e^{-3x}$.

<u>Example</u>: Let $y'_1 = y_1+3y_2$ and $y'_2 = 4y_1+5y_2$. Solve the system of *linear differential equations* using diagonalisation. Here, $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$, and the *<u>eigenvalues</u>* are $\lambda = -1$ and $\lambda = 7$. The *eigenvectors* are $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $P = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}$. $Y = PU$ and $Y' = PU'$ gives $U' = PU$ as $\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. So $u'_1 = -u_1$, implying that $u_1 = c_1e^{-x}$ and $u'_2 = 7u_2$, implying that $u_2 = c_2e^{7x}$. (The solution of the *diagonalised* system). Now $Y = PU$ gives $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} c_1\exp(-x) \\ c_2\exp(7x) \end{bmatrix}$. So $y_1 = -3c_1e^{-x}+c_2e^{7x}$ and $y_2 = 2c_1e^{-x}+2c_2e^{7x}$ is the *general solution* of the **original** system.

Assignment 1

To find the **inverse** of a matrix, either *row reduce* (A | I), or use the formula (minors, determinant, etc.) Q: Why does the idea of an "inverse" apply only to square matrices? A: B is an **inverse** to A if AB = BA = I. Suppose that AB and BA are defined, and let A be an m×n matrix, and let B be an n×m matrix (for some m, n $\in$ **N**. (*Positive* integers)). So AB would be an m×m **matrix,** and BA would be an n×n matrix. But AB = BA, and so we must have m = n. (We must have the **same** identity matrix). *Therefore*, A & B must be square.

Q: **Linear Equations:** Determine when the following system of *equations* has (i) no solution; (ii) many solutions; (iii) an unique solution: $x+y-z = 1$, $2x+3y+\lambda z = 3$, $x+\lambda y+3z = 2$. What is the **general** solution of the system? A: A **general** solution is a way of representing a whole set of many solutions. So (1, 4, 2) is an *unique* solution, and (a, 2b, b) is a *general* solution, where a and b are variables.

To solve, *row reduce* the associated matrix. We have no solutions when one of the lines is 0 0 0 n ($n \neq 0$). Choose or find the λ that gives any lines like these. We have many solutions when we have a line such that 0 0 0 0 — we then need a parameter. This question appears in the G1M31 Matrix Algebra notes. Look over **those as well!**

Q: Which of the following are subspaces of $\mathbf{R}^3$; which are not, and why? (i) $W_1 = \{(a,b,0): a,b \in \mathbf{R}\}$; (ii) $W_2 = \{(a,b,c): a \geq 0\}$. Can you describe these *geometrically* i.e. as sets of points in 3-dimensional space $\mathbf{R}^3$? A: (i) This is the x-y plane with z co-ordinate. This *is* a **subspace**: (a) $0 = (0,0,0) \in W$; (b) let $k,l \in \mathbf{R}$, and let $u,v \in W_1$; then $u = (a,b,0)$ and $v = (c,d,0)$; (for some a, b, c, d, $\in \mathbf{R}$); then $ku+lv = k(a,b,0) + l(c,d,0) = (ka+lb, kb+ld, 0) \in W_1$. (ii) This is everything on the *positive side of the x-axis*. Now $-5 \in \mathbf{R}$, $(1,1,1) \in W_2$. But $-5(1,1,1) \notin W_2$, so W_2 is not a *subspace*.

Q: Consider the following *bases* of $\mathbf{R}^3$: $E = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$; $F = \{f_1 = (1,1,1), f_2 = (1,1,0), f_3 = (1,0,0)\}$. (i) Find the *transition matrix P* from E to F. (ii) Find the *transition* matrix Q from F to E. (iii) Verify that $Q = P^{-1}$. (iv) Show that $[v]_f = P^{-1}[v]_e$ for any $v \in \mathbf{R}^3$. (v) Show that $[T]_f = P^{-1}[T]_e P$ for T defined by $T(x,y,z) = (2y+z, x-4y, 3x)$.

A: To find the *transition matrix P* from E to F, we must first express the f_i in terms of E and form the matrix of coefficients. The **transpose** of the matrix of coefficients is the transition matrix P. Similarly for (ii). (ii) Get I. (iv) Consider any vector v in $\mathbf{R}^3$, say (a,b,c). This vector can be described using a *basis* for $\mathbf{R}^3$ and a unique set of scalars. So $v = k_1e_1+k_2e_2+k_3e_3$ and similarly for f_i. The set of scalars *associated* with the basis is called the **co-ordinate** vector v relative to the basis.

Now **expand** $v = k_1e_1+k_2e_2+k_3e_3$ and $v = l_1f_1+l_2f_2+l_3f_3$ to get *expressions* for k_1, k_2, and k_3. Now we know that $[v]_e = \begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}$ and that $[v]_f = \begin{bmatrix}l_1\\l_2\\l_3\end{bmatrix}$. As $Q = P^{-1}$, we can modify $[v]_f = P^{-1}[v]_e$ to give $[v]_f = Q[v]_e$. Let us now calculate **both** sides of this equation. It turns out it is the *same* as what we get from the solutions to the k_i i.e. $\begin{bmatrix}l_1\\l_2\\l_3\end{bmatrix} = \begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}$, i.e. $\begin{bmatrix}l_1\\l_2\\l_3\end{bmatrix} = \begin{bmatrix}k_3\\k_2-k_3\\k_1-k_2\end{bmatrix}$. So we *conclude* that $[v]_f = P^{-1}[v]_e$ for all $v \in \mathbf{R}^3$.

(v) First find $[T]_f$ and $[T]_e$, e.g. $T(e_1) = T(1,0,0) = (0,1,3) = 0e_1+1e_2+3e_3$. So ... $[T]_e = [{}^{0}1_{3}\ {}^{-2}{-2}_{4}\ {}^{1}0_{0}]$. Now calculate $Q[T]_eP$. If this is the *same* as $[T]_f$, then we know that the <u>expression</u> $[T]_f = P^{-1}[T]_eP$ is true.

<u>Inner Product Space</u>

It is a **vector** space + *a linear product operation*. [Aside: A vector space consists of: *elements* (vectors) u, v; vector addition u+v; scalar **multiplication** *l*u]. Now $V\times V \rightarrow k$ is represented by $<u,v>$. Axioms: **I1**: $<ku+lw,v> = k<u,v>+l<w,v>$, where k and *l* are *scalars* and u,v,w are *vectors*. **I2**: $<u,v> = \overline{<v,u>}$. **I3**: $<u,u> \geq 0$. It is only 0 if u = 0.

Euclidean Inner Product. Dot product = inner product on $\mathbf{R}^n$. Vector space = $\mathbf{R}^n$. $u = (a_1,...,a_n)$; $v = (b_1,...,b_n)$. $<u,v> = u.v = a_1b_1+...+a_nb_n$. "Euclidean n-space". *Complex* case: vector space $\mathbf{C}^n$. $<u,v> = u.v = a_1\bar{b}_1 + a_2\bar{b}_2 +...+ a_n\bar{b}_n$.

Consider a **vector space** of m×n matrices over **R**. Here, $<A,B> = \text{trace}(B^tA)$. Consider a vector *space* of real continuous *functions* on an interval $a\leq t\leq b$. Here, $<f,g> = \int^b_a f(t)g(t)dt$. For the vector *space* $\mathbf{R}^2$, we get the inner product using $u = (x_1,y_1)$ and $v = (x_2,y_2)$: $<u,v> = x_1x_2-x_1y_2-y_1x_2+3y_1y_2$. Let us try to prove that this satisfies the **axioms**.

I1: Let $w = (x_3,y_3)$. So $ku+lw = (kx_1+lx_3, ky_1+ly_3)$. $<ku+lw,v> = k<u,v>+l<w,v>$. LHS = $(kx_1+lx_3)x_2-(kx_1+lx_3)y_2-(ky_1+ly_3)x_2+3(ky_1+ly_3)y_2$. RHS = $k(x_1x_2-x_1y_2-y_1x_2+3y_1y_2) + l(x_3x_2-x_3y_2-y_3x_2+3y_3y_2)$. As you can see, these match. I2: $<u,v> = x_1x_2-x_1y_2-y_1x_2+3y_1y_2$. $\overline{<v,u>} = x_2x_1-x_2y_1-y_2x_2+3y_2y_1$. These match! I3: $<u,u> = x_1x_2-x_1y_1-y_1x_2+3y_3y_1 = (x_1-y_1)^2+2y_1^2 \geq 0$.

Norm: Define $||u|| = \sqrt{(<u,u>)}$. Let $u = (3,4) \in \mathbf{R}^2$, then $||u||^2 = <u,u> = <(3,4),(3,4)> = 3\times3-3\times4-4\times3+3(4\times4) = 33$. So $||u|| = \sqrt{33}$. Using the *Usual Euclidean inner product space*, $||u||^2 = <u,u> = <(3,4),(3,4)> = 25$. So $||u|| = \sqrt{25} = 5$. And ${}^{u}/_{||u||} = ({}^{3}/_{5}, {}^{4}/_{5})$. *Normalising* vectors: Suppose that $u = (2,1,-1)$. Then $||u||^2 = 2^2+1^2+(-1)^2 = 6$. So $||u|| = \sqrt{6}$. Therefore, u *normalised* is ${}^{u}/_{||u||} = ({}^{2}/_{\sqrt{6}}, {}^{1}/_{\sqrt{6}}, {}^{-1}/_{\sqrt{6}})$. **<u>Distance</u>**: $d(u,v) = ||v-u||$.

Cauchy-Schwarz Inequality: $<u,v> \leq ||u||\,||v||$. *Proof*: Case 1: $v = 0$. $|<u,v>| = |<u,0>| = 0$. So $||u||\ ||v|| = ||u||\sqrt{<v,v>} = ||u||0 = 0$. **So** $0 = |<u,v>| \leq ||u||\ ||v|| = 0$. Case 2: $v \neq 0$. Here $0 \leq ||u-<u,v>tv||^2$ (for *all* $t \in$ **R)** $= <u-<u,v>tv, u-<u,v>tv> = <u,u> - (\overline{<v,u>})t<u,v> - <u,v>t<v,u> + <u,v>(\overline{<v,u>})t^2<v,u> = ||u||^2 - 2t|<u,v>|^2 + |<u,v>|^2t^2||v||^2$. **(Because** $z(\bar{z}) = z$ **for all** $z \in$ **C)**. Now set $t = {}^{1}/_{||v||^2}$ to find that $0 \leq ||u||^2 - {}^{|<u,v>|^2}/_{||v||^2}$. This *implies* that $|<u,v>|^2 \leq ||u||^2||v||^2$. Therefore, $|<u,v>|^2 \leq ||u||\,||v||$. QED.

➢ <u>1st November 1999</u>

<u>Tutorial</u>

Q: A set of vectors $\{v_1,...,v_n\}$ is a basis for a *vector* space V if it is linearly **independent** and **spanning** i.e. linearly independent: $k_1v_1+...+k_nv_n = 0 \Rightarrow k_1 = ... = k_n = 0$. **Span**: for all $v \in V$, $\exists$ scalars $k_1, ..., k_n$ such that $v = k_1v_1+...+k_nv_n$. So prove that $\{v_1 = (1,2,1), v_2 = (2,9,0), v_3 = (3,3,4)\}$ is a basis for $\mathbf{R}^3$. A: (see over).

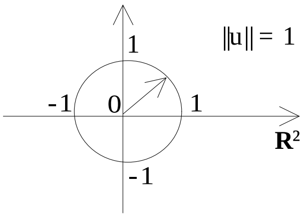
Suppose that $k_1v_1+k_2v_2+k_3v_3 = 0$, i.e. $k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = 0$ so that $(k_1+2k_2+3k_3,\ 2k_1+9k_2+3k_3,\ k_1+4k_3) = (0,0,0)$. Placing these in a *matrix* and reducing, we get the **matrix** $[^{1}0_{0}\ ^{2}5_{0}\ ^{3}\text{-}3_{-1}\ ^{0}0_{0}]$. This matrix *implies that* $k_3 = 0$, $k_2 = 0$, and $k_1 = 0$. So it is **linearly** independent. Span: Let $v = (x,y,z) \in \mathbf{R}^3$. We want to find k_1, k_2, and k_3 such that $v = k_1v_1+k_2v_2+k_3v_3$, i.e. $(x,y,z) = k_1(1,2,1)+k_2(2,9,0)+k_3(3,3,4)$. Placing this in an **augmented** matrix, we obtain $[^{1}2_{1}\ ^{2}9_{0}\ ^{3}3_{4}\ ^{x}y_{z}]$. Reducing this by *row operations*, we obtain $[^{1}0_{0}\ ^{0}1_{0}\ ^{0}0_{1}\ ^{-36x+8y+21z}5x\text{-}y\text{-}3z_{9x-2y-5z}]$, i.e. $\{v_1,v_2,v_3\}$ span $\mathbf{R}^3$ (we can get *expressions* for k_1, k_2 and k_3 in terms of x, y and z.

Q: **Inner product.** Using the definition of *Euclidean inner product,* calculate (i) $\langle v_1,v_2\rangle$; (ii) $\langle v_2,v_1\rangle$; (iii) $\|v_1\|$; (iv) $\|v_2\|$. Hence verify the Cauchy-Scwarz **inequality** for v_1 and v_2. A: (i) $\langle(1,2,1), (2,9,0)\rangle = 1\times2 + 2\times9 + 1\times0 = 20$. (ii) $\langle(2,9,0), (1,2,1)\rangle = 20$. (iii) $\langle(1,2,1), (1,2,1)\rangle = 1^2+2^2+1^2 = 6$. So $\|v_1\| = \sqrt{6}$. (iv) $\langle(2,9,0), (2,9,0)\rangle = 2^2+9^2 = 85$. So $\|v_2\| = \sqrt{85}$. Cauchy Scwarz: $|\langle v_1,v_2\rangle| = 20 = \sqrt{400}$. $\|v_1\|\,\|v_2\| = \sqrt{6}\,\sqrt{85} = \sqrt{510}$. Therefore $|\langle v_1,v_2\rangle| \leq \|v_1\|\,\|v_2\|$.

Q: Now **define** $\langle u,v\rangle = 3x_1x_2+2y_1y_2+z_1z_2$ for $u = (x_1,y_1,z_1)$. Prove that this defines an *inner product*, and recalculate (i), (ii), (iii) and (iv). A: We must verify the *axioms* I_1, I_2 and I_3 for $u, v, w \in \mathbf{R}^3$ and $k,l \in \mathbf{R}$. I_1: $\langle ku+lw,\ v\rangle = \langle(kx_1+lx_3,\ ky_1+ly_3,\ kz_1+lz_3), (x_2,y_2,z_2)\rangle = 3(kx_1+lx_3)x_2 + 2(ky_1+ly_3)y_2 + (kz_1+lz_3)z_1 = 3kx_1x_2+3lx_3x2+2y_1y_2+2ly_3y_2+kz_1z_2+lz_3z_2$. And $k\langle u,v\rangle+l\langle w,v\rangle = k\langle(x_1,y_1,z_1), (x_2,y_2,z_2) + l\langle(x_3,y_3,z_3), (x_2,y_2,z_2)\rangle = k(2x_1x_2+2y_1y_2-z_1z_2) + l(3x_3x_2+2y_3y_2+z_3z_2) = \langle ku,lw, v)$. (From the *above*).

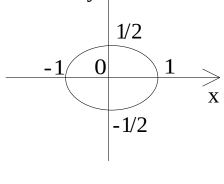
I2: $\langle u,v\rangle = 3x_1x_2+2y_1y_2+z_1z_2$. $\langle v,u\rangle = 3x_2x_1+2y_2y_1+z_2z_1 = \langle u,v\rangle$. I_3: $\langle u,u\rangle = 3(x_1)^2+2(y_1)^2+(z_1)^2 \geq 0$. And $\langle u,u\rangle = 0 \Leftrightarrow 3(x_1)^2+2(y_1)^2+(z_1)^2 = 0 \Leftrightarrow x_1 = y_1 = z_1 = 0$, i.e. $u = 0$. Now (i) $\langle(1,2,1), (2,9,0)\rangle = 3\times1\times2 + 2\times2\times9 + 1\times0 = 42$. (ii) $\langle(2,9,0), (1,2,1)\rangle = 42$. (iii) $\langle(1,2,1), (1,2,1)\rangle = 3\times1\times1 + 2\times2\times2 + 1\times1 = 12$. $\|v_1\| = \sqrt{12}$. (iv) $\langle(2,9,0), (2,9,0)\rangle = 3\times2\times2 + 2\times9\times9 = 174$. $\|v_2\| = \sqrt{174}$.

Q: *Without* the idea of an "inner product", we had no concepts of "**length**" (norm) or "**distance**" in our abstract definition of a vector space. Now we do: the *length* of a vector is $\|v\| = \sqrt{(\langle v,v\rangle)}$; and the distance between vectors is $d(u,v) = \|u-v\|$. So in the vector space $\mathbf{R}^2$, we can define length/distance in *different* ways, depending on the inner product used. The unit circle with the <u>Euclidean</u> inner product consists of all points $u \in \mathbf{R}^2$ s.t. $\|u\| = 1$.



In the diagram, points/vectors are an *abuse* of the definitions, but just relax, and let e.g. (3,4) be a point or a vector depending on the context. What does the **unit** circle with respect to the inner product defined by $\langle u,v\rangle = x_1x_2+4y_1y_2$ for $u = (x_1,y_1)$ and $v = (x_2,y_2)$ in $\mathbf{R}^2$ look like on the *same* scale of graph?

A: With this definition of the inner product, we can see that $\|u\| = 1 \Leftrightarrow \langle u,u\rangle = x_1^2+4y_1^2 = 1$. So if y_1 is 0, then $x_1^2 = 1$, i.e. $x_1 = \pm1$; and if x_1 is 0, then $y_1^2 = ¼$, i.e. $y_1 = \pm ½$. In this way, *we obtain* co-ordinates on the "**unit**" circle in this inner product space.



Q: Use **diagonalisation** to solve the system $y'_1 = -2y_1-y_2-y_3$; $y'_2 = -3y_2+3y_3$, $y'_3 = 4y_2-4y_3$. The **matrix** representation is $A = [{}^{-2}0_0\ {}^{-1}{-3}_4\ {}^{-1}3_{-4}]$. $Y' = AY$. *Eigenvalues* $\lambda = -2$, $\lambda = -7$, $\lambda = 0$. **Eigenvectors** $[{}^{1}0_0]$, $[{}^{1}{-15}_{20}]$, $[{}^{-1}1_1]$. $P = [{}^{1}0_0\ {}^{1}{-15}_{20}\ {}^{-1}1_1]$. $D = [{}^{-2}0_0\ {}^{0}{-7}_0\ {}^{0}0_0]$. We don't need to *calculate* P^{-1} — except if you want to **calculate** $P^{-1}AP$ to check that D is correct. Solve $U' = DU$: $[{}^{u'_1}u'_2{}_{u'_3}] = [{}^{-2}0_0\ {}^{0}{-7}_0\ {}^{0}0_0][{}^{u_1}u_2{}_{u_3}]$, i.e. $\{u'_1 = -2u_1$, $u'_2 = -7u_2$, $u'_3 = 0u_3$. So $u_1 = c_1e^{-2x}$, $u_2 = c_2e^{-7x}$, and $u_3 = c_3$.

Now $Y = PU$ gives $Y' = PU'$. We now do what is *shown in the matrices on the right*. Therefore, the **general** solution to the system is $y_1 = c_1e^{-2x}+c_2e^{-7x}-c_3$, $y_2 = -15c_2e^{-7x}+c_3$, and $y_3=20c_2e^{-7x}+c_3$. **Check**: ${}^{d}/_{dx}(c_1e^{-2x}+c_2{}^{-7x}-c_3) = -2c_1e^{-2x}-7c_2e^{-7x} = -2y_1-y_2-y_3 = y'_1$. (As in the *original* system).

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -15 & 1 \\ 0 & 20 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{-2x} \\ c_2e^{-7x} \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1e^{-2x}+c_2e^{-7x}-c_3 \\ -15c_2e^{-7x}+c_3 \\ 20c_2e^{-7x}+c_3 \end{bmatrix}$$

➢ 4th November 1999

Results on Length and Distance

"Length": $||u|| \geq 0$. Length is zero only if u is zero. $||ku|| = |k|\,||u||$. **Triangle** inequality: $||u+v|| \leq ||u||+||v||$. Proof (by *Cauchy-Schwarz* inequality). $||u+v||^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} + \langle v,v\rangle \leq ||u||^2 + 2||u||\,||v|| + ||v||^2 = (||u|| + ||v||)^2$. *Distance*: $d(u,v) \geq 0$. Distance is only zero iff $u = v$. $d(u,v) = d(v,u)$. $d(u,v) \leq d(u,w) + d(w,v)$.

Angles

Note: $|{}^{(\langle u,v\rangle)}/_{(||u||\ ||v||)}| \leq 1$. θ is the angle *between* u and v. $\cos\theta = {}^{\langle u,v\rangle}/_{||u||\ ||v||}$. Example: $u = (2,-4,6)$, $v = (-6,8,4)$, $w = (3,6,3)$. $\langle u,v\rangle = (2\times-6)+(-4\times8)+(6\times4) = -20$. $||u|| = 4+16+36 = 56$. $||v|| = 36+64+16 = 116$. $\cos\theta = {}^{-20}/_{56\times116}$, an *obtuse* angle, i.e. $\theta > 90°$. $\langle v,w\rangle = 42$, an **acute** angle, i.e. $\theta < 90°$. Note: if $\langle u,w\rangle = 0$, we have *perpendicular* vectors i.e. $\theta = 90^{o}$.

<u>**Orthogonal:**</u> u is orthogonal to v, written $u \perp v$, if $\langle u,v\rangle = 0$. <u>**Othonormal:**</u> vectors are orthonormal if they are *orthogonal* and *normalised*. Basis of a vector space: <u>linearly independent and spanning</u>. Basis of an inner product space = basis for the *vector* space. Other possibilities: (1) Is it an <u>orthogonal</u> basis? i.e. are all vectors *orthogonal* to each other; (2) Is it an <u>orthonormal</u> basis? i.e. *orthogonal* and all vectors are *normalised*.

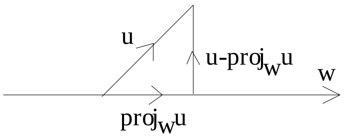
Orthogonal results: Generalised Pythagoras' theorem. Recall that if u and v are orthogonal, then $||u+v||^2 = ||u||^2+||v||^2$. Theorem: If $\{v_1,...,v_n\}$ are orthogonal, then they are *linearly* independent. Proof: $k_1v_1+...+k_nv_n = 0 \Rightarrow k_1 = k_2 = ... = k_n = 0$. Now $\langle v_1, k_1v_1+...+k_nv_n\rangle = \langle v_1,0\rangle$. And $k_1\langle v_1,v_1\rangle + k_2\langle v_1,v_2\rangle+... +k_n\langle v_1,v_n\rangle = \langle v_1,0\rangle \Rightarrow k_1 = 0$.

Example: Prove that the following set of vectors is an orthogonal basis for $\mathbf{R}^3$ (Euclidean inner product): $v_1 = (0,1,0)$, $v_2 = ({}^{1}/_{\sqrt{2}},0,{}^{1}/_{\sqrt{2}})$, $v_3 = ({}^{1}/_{\sqrt{2}},0,-{}^{1}/_{\sqrt{2}})$. Basis for $\mathbf{R}^3$: Linearly *Independent*: we want $k_1v_1+k_2v_2+k_3v_3 = 0$. From these, we get ${}^{k_2}/_{\sqrt{2}}+{}^{k_3}/_{\sqrt{2}} = 0$; $k_1 = 0$, and ${}^{k_2}/_{\sqrt{2}}-{}^{k_3}{}_{\sqrt{2}} = 0$. These imply that $k_1 = k_2 = k_3 = 0$. Span: $(a,b,c) = k_1v_1+k_2v_2+k_3v_3$. From these, we get $k_2+k_3 = a\sqrt{2}$, $k_1 = b$, and $k_2-k_3 = c\sqrt{2}$. We solve the equations to get $k_1 = b$, $k_2 = c\sqrt{2}$, and $k_3 = a\sqrt{2}-c\sqrt{2}$.

Orthogonal: $<v_1,v_2> = (0\times{}^1/_{\sqrt{2}})+(1\times 0)+(0\times{}^1/_{\sqrt{2}}) = 0$. Similarly, $<v_1,v_3> = 0$ and $<v_2,v_3> = 0$. Normal: $||v_1||^2 = <v_1,v_1> = 0^2+1^2+0^2 = 1$. *Similarly,* $||v_2||^2 = 1$ and $||v_3||^2 = 1$.

Orthogonal/Orthonormal Projection

In many applications, it is useful to "*decompose*" a vector u into a sum of *terms* — one parallel to a specified non-zero vector w, and the other perpendicular/orthogonal to w. **Projection of a vector onto a vector:** The projection of a vector u onto the line *containing w* is $\text{proj}_w(u) = {}^{<v,w>}/_{||w||^2}w$. This is "the vector component of **u** along w". It follows that $u\text{-proj}_w(u) = u - {}^{<u,w>}/_{||w||^2}w$ is the "vector component of u **orthogonal** to w".



Examples. In $\mathbf{R}^2$ with the usual Euclidean inner product, find the *projection* of u along w and write u as a sum of two terms — one parallel to w and one perpendicular to w, when w = (4,-2) and u = (9,3). A: $<u,w> = (4\times 9)+(-2\times 3) = 30$. $||w||^2 = 4^2+(-2)^2 = 20$. $\text{proj}_w(u) = {}^{30}/_{20}(4,-2) = (6,-3)$. $u\text{-proj}_w(u) = (9,3)-(6,-3) = (3,6)$. So $u = (9,3) = (3,6)+(6,-3) = (u\text{-proj}_w(u))+\text{proj}_w(u)$.

Projection of a Vector onto a Subspace

Let V be a *vector space* and let W be a subspace **spanned** by $\{w_1,...,w_n\}$. Then $\text{proj}_w(u) = (<u,w_1>/(||w_1||^2))w_1+...+(<u,w_n>/||w_n||^2)w_n$, the "**orthogonal** projection of u onto W". $u\text{-proj}_w(u)$ is the "component of u *orthogonal* to W".

Example: In $\mathbf{R}^3$, with the usual *inner product*, find the projection of u = (1,1,1) on the space W spanned by the set $\{w_1 = (0,1,0), w_2 = (-{}^4/_5, 0, {}^3/_5)\}$. Check that $u\text{-proj}_w(u)$ is **orthogonal** to w_1 and w_2. A: $\text{proj}_w(u) = {}^{<u,v_1>}/_{||v1||^2}+ {}^{<u,v_2>}/_{||v2||^2}+... = (1)(0,1,0) + (-{}^4/_5)({}^4/_5,0,{}^3/_5) = ({}^4/_{25}, 1, -{}^3/_{25})$. Component of u *orthogonal* to W: $(1,1,1)-({}^4/_{25},1,-{}^3/_{25}) = ({}^{21}/_{25},0,{}^{28}/_{25})$.

Theorem: If $\{v_1,...,v_n\}$ is an *orthonormal basis* for an inner product space V, and $u \in V$, then $u = <u,v_1>v_1+...+<u,v_n>v_n$. Theorem: If $\{v_1,...,v_n\}$ is an *orthogonal* basis for an inner product space V, and $u \in V$, then $u = [(<u,v_1>)/(||v_1||)]v_1 + ... + [<u,v_n>/(||v_n||^2)]v_n$.

(**Aside**: $\{v_1,...,v_n\}$ is a basis for V, so if $u \in V$, then there *exists* $k_1,..,k_n$ so that $u = k_1v_1+...+k_nv_n$. We have to **prove that** $k_1 = <u,v_1>, ..., k_n = <u,v_n>$. Now for i = 1,...,n, $<u,v_i> = <k_1v_1+...+k_nv_n, v_i> = k_1<v_1,v_i> + ... + k_n<v_n,v_i>$. So $<v_i,v_1> = ||v_1||^2 = 1$; (*normalised*); $<v_i,v_j> = 0$ (*orthogonal*). Also, $<u,v_i> = k_i<v_i,v_i> = k_i$ (where $<v_i,v_i> = 1$), i.e. $u = <u,v_1>v_1 + ... + <u,v_n>v_n$).

Questions: Working in $\mathbf{R}^3$, with the *Euclidean* inner product, let $w_1 = (0,1,-1)$ and $w_2 = (2,2,-1)$. They are *not* orthogonal, ($w_1.w_2$ is not zero), but they are still a **basis** for a space W. Find the vector (of length 1) *that is orthogonal* to W. A: $<w_1,w_2> = 0\times 2 + 1\times 2 + -1\times -1 = 3 \neq 0$ so w_1 and w_2 are not orthogonal. Let $u = (a,b,c) \in \mathbf{R}^3$. If u has length 1, $||u|| = 1$, i.e. $a^2+b^2+c^2 = 1^2 = 1$ (first equation). If u is orthogonal to W, then u is *orthogonal* to w_1 and w_2, so that $<u,w_1> = 0$ and $<u,w_2> = 0$, i.e. $0a+1b-1c = 0$ and $2a+2a-1c = 0$. Find values for a,b and c for the *equations* $a^2+b^2+c^2 = 1$, $b-c = 0$, and $2a+2-c = 0$. We get $u = (-{}^1/_3,{}^2/_3,{}^2/_3)$ or $({}^1/_3,-{}^2/_3,-{}^2/_3)$.

Q: Let $v_1 = (0,1,0)$, $v_2 = (-^4/_5,0,^3/_5)$, and $v_3 = (^3/_5,0,^4/_5)$. Check that $\{v_1,v_2,v_3\}$ is an *orthogonal* basis for $\mathbf{R}^3$ (with *Euclidean* inner product). Express $u = (1,1,1)$ as a linear combination of the vectors v_1, v_2, and v_3 and find the co-ordinate **vector** $[u]_{v_i}$. A: You have to check that $\{v_1,v_2,v_3\}$ spans $\mathbf{R}^3$, i.e. solve $(a,b,c) = k_1v_1+k_2v_2+k_3v_3$ to show that you can *find* k_1, k_2 and k_3 in terms of a, b and c. You have to **check** that $\{v_1,v_2,v_3\}$ is an orthogonal set (this implies *linear independence*) i.e. check $<v_1,v_2> = 0$, $<v_1,v_2> = 0$ and $<v_2,v_3> = 0$. You have to check that **all** the vectors are normalised, i.e. $\|v_1\| = \|v_2\| = \|v_3\| = 1$. To express $u = (1,1,1)$ as a *linear* combination of the vectors v_1, v_2 and v_3, use the *theorem* $u = {}^{<u,v_1>}/_{\|v_1\|}v_1 + {}^{<u,v_2>}/_{\|v_2\|}v_2 + {}^{<u,v_3>}/_{\|v_3\|}v_3$. $<u,v_1> = 1\times0 + 1\times1 + 1\times0 = 1$. $<u,v_2> = -^1/_5$. And $<u,v_3> = {}^7/_5$. Therefore, $u = v_1-{}^1/_5v_2+{}^7/_5v_3$; $[u]_{v_i} = [^1\ {}_{-1/5}\ {}_{7/5}]$ (*column vector*).

Q: Let $v_1 = (1,-1,2,-1)$, $v_2 = (-2,2,3,2)$, $v_3 = (1,2,0,-1)$, and $v_4 = (1,0,0,1)$. Check that $\{v_1,v_2,v_3,v_4\}$ is an **orthonormal** basis for $\mathbf{R}^4$. (With *Euclidean* inner product). Express the following as a linear combination of the v_i: $u_1 = (1,1,1,1)$, $u_2 = (\sqrt{2}, -3\sqrt{2}, 5\sqrt{2})$, $u_3 = (-^1/_3, {}^2/_3, -^1/_3, {}^4/_3)$. A: Prove that it is spanning and that the vectors are orthogonal as in the above question.

➢ 8th November 1999

Q: *Express* $u = (-1,2,6,0)$ in the form $u = u_1+u_2$, where u_1 is in the *space* W spanned by $w_1 = (-1,0,1,2)$ and $w_2 = (0,1,0,1)$; and u_2 is orthogonal to W. A: $u_1 = (-^5/_4, -^1/_4, {}^5/_4, {}^9/_4)$. $u_2 = (^1/_4, {}^9/_4, {}^{19}/_4, -^9/_4)$. Check that u_1 is orthogonal to u_2: verify that $<u_1,u_2> = \ldots = 0$.

Q: Let W be a **subspace** of an inner product space V. A vector u is orthogonal to W if it is orthogonal to *every* vector in W. The set of vectors in V that are orthogonal to W is called the orthogonal *complement* of W, and denoted by $W^\perp$. Prove that $W^\perp$ is a *subspace* of W. Now let W be the subspace of $\mathbf{R}^5$ spanned by $w_1 = (2,2,-1,0,1)$, $w_2 = (0,0,1,1,1)$, $w_3 = (-1,-1,2,-3,1)$ and $w_4 = (1,1,-2,0,-1)$. Find a *basis* for $W^\perp$.

A: $W^\perp$ is a subspace of W if (i) $0 \in W^\perp$, because for *all* $w \in W$, $<w,0> = 0$. So 0 **is** orthogonal to W. (ii) *Suppose that* $u_1,u_2 \in W^\perp$, then $<u_1,w> = <u_2,w> = 0$ for all $w \in W$ and for all $k_1,k_2 \in K$. Further, $k_1<u_1,w> = k_2<u_2,w> = 0$, so $<k_1u_1+k_2u_2,w> = k_1<u_1,w> + k_2<u_2,w> = 0$. Therefore, $k_1u_1+k_2u_2 \in W^\perp$ for all $k_1,k_2 \in K$ and $u_1,u_2 \in W^\perp$.

To find a *basis* for $W^\perp$, we have to *solve* $<(a,b,c,d,e), w_i> = 0$ for $i = 1\ldots4$. So we have the *equations* $2a+2b-c+e = 0$, $c+d+e = 0$, $-a-b+2c-3d+e = 0$ and $a+b-2c-e = 0$. These imply when solved that $d = 0$, $c+e = 0$, and $a+b+c = 0$. b and e are **free** variables, so set $b = \alpha$ and $e = \beta$ so that the general solution is $(\beta-\alpha, \alpha-\beta, 0, \beta)$. The basis for the *solution* space (and **therefore** for $W^\perp$) is $\{[-1\ -1\ 0\ 0\ 0], [1\ 0\ -1\ 0\ 1]\}$ (These are *COLUMNS*, first comes from $\alpha = 1$, $\beta = 0$; and the second **from** $\alpha = 0$, $\beta = 1$).

Assignment 2

Q: Determine which of the *following* have the same rowspace: $[^1{}_2\ {}^{-1}{}_3\ {}^2{}_{-1}]$, $[^1 2_3\ {}^{-1}{-1}_{-5}\ {}^3 10_1]$, $[^1{}_3\ {}^{-2}{}_{-4}\ {}^{-1}{}_5]$. A: Perform row *operations* on the matrices to get them into row reduced echelon form. The rowspace is the **space** spanned by the rows of the matrix. Example: $A \sim [^1{}_0\ {}^{-1}{}_5\ {}^2{}_{-5}]$ (R_2-2R_1); $\sim[^1{}_0\ {}^{-1}{}_1\ {}^2{}_{-1}]$ $(^1/_5R_2)$; $\sim[^1{}_0\ {}^0{}_1\ {}^1{}_{-1}]$ (R_1+R_2). It turns out that the *rowspaces* of B and C are the same. They each have rank 2 because when **reduced**, one row of B becomes 0 0 0 0.

Q: Find *bases* for the **image** and **kernel** of the linear transformation T: $\mathbf{R}^3 \rightarrow \mathbf{R}^2$ given by T(x,y,z) = (x+y+z, x-z). A: First find the *equations* defining T with respect to the standard basis for $\mathbf{R}^3$. So $T(e_1) = T(1,0,0) = (1,1) = 1(1,0)+1(0,1) = f_1+f_2$, where the f_i make up the *standard* basis for $\mathbf{R}^2$. Similarly, $T(e_2) = f_1$ and $T(e_3) = f_1-f_2$.

The matrix of the **transformation** with respect to the standard basis is the *transpose* of the matrix of the above coefficients. So $[T] = [^{1}1_{1}\ ^{1}0_{-1}]^T = [^{1}{}_{1}\ ^{1}{}_{0}\ ^{1}{}_{-1}]$. The image of the *transformation* is equal to the column space of [T]. So we need to find a **basis** for $\{[^{1}{}_{1}], [^{1}{}_{0}], [^{1}{}_{-1}]\}$ to find a *basis* for the **image** of the transformation T.

To do this, put the vectors as *rows* and do row reduction on $[^{1}1_{1}\ ^{1}0_{-1}]$ to get $[^{1}0_{0}\ ^{0}1_{0}]$. So a *basis* for the image of the linear transformation T is $\{[^{1}{}_{0}], [^{0}{}_{1}]\}$, which is the **standard** basis for $\mathbf{R}^2$. The *rank* of the above basis is 2. The <u>kernel</u> is equal to the nullspace of [T]. We **have** to solve $[T]\underline{x} = 0$. Here, solve $[^{1}{}_{1}\ ^{1}{}_{0}\ ^{1}{}_{-1}][^{x}y_{z}] = [^{0}{}_{0}]$, i.e. x+y+z = 0, x-z = 0. We solve these to get the *general* paramaterised solution (a,2a,a). ($a \in \mathbf{R}$). Setting a = 1, we get a **basis** for the kernel: $\{[^{1}{-2}_{1}]\}$. The *nullity* of [T] is 1. Check: Rank + Nullity = 2 + 1 = 3 = $\dim(\mathbf{R}^3)$.

Q: Find the **dimension** and a **basis** of the solution space W of the system x+2y+2z-s+3t = 0, x+2y+3z+s-t = 0, 3x+6y+8z+s+5t = 0. A: Solve the *augmented* matrix and get solutions out: x = 5a-2b, y = b, z = a, where a and b are parameters. Therefore, the solution space W of the system can be written as W = [5a-2b b -2a a 0] (<u>**Column**</u> vectors) = a[5 0 -2 1 0] + b[-2 1 0 0 0].

Setting a = 1, b = 0; *then* a = 0, b = 1, we see that the vectors v_1 = [5 0 -2 1 0] and v_2 = [-2 1 0 0 0] span the *solution* space. Because v_1 cannot be expressed as a *scalar* multiple of v_2, v_1 and v_2 are linearly independent. So because the **vectors** v_1 and v_2 are linearly independent, then $\{v_1, v_2\}$ is a basis, and the solution space is *two-dimensional*.

Q: Find a *homogenous* system of equations whose solution space W is <u>generated</u> by {(1,-2,0,3), (1,-1,-1,4), (1,0,-2,5)}. A: This is the above question in *reverse*. We know that the vectors came from solving a set of **homogenous** equations that had general solution (column) W = [a+b+b -2a-b -b-2c 3a+4b+5c]. If W represents any *vector* (x y z u), then x = a+b+c, etc.

Writing these equations in **matrix** form, (as shown), we then reduce this matrix as far as it will go. When we do this, we get out <u>that</u> z+y+2x = 0 and that u-5x-y = 0. From these, we see that y = -2x-z and that t = u-5x. So a **homogenous** system of equations whose solution space w is generated by v_1, v_2 and v_3 could be shown to be 2x+y+z = 0, 5x+y-w = 0.

$$\begin{bmatrix} 1 & 1 & 1 & x \\ -2 & -1 & 0 & y \\ 0 & -1 & -2 & z \\ 3 & 4 & 5 & u \end{bmatrix}$$

Q: Find the *eigenvalues* and bases for the eigenspaces of $A = [^{2}0_{0}\ ^{1}1_{2}\ ^{0}{-1}_{4}]$ and $B = [^{3}2_{1}\ ^{1}4_{1}\ ^{1}2_{3}]$. Which matrix can be *diagonalised* and why? <u>Which</u> is the diagonalised matrix? A: Use λ-D-R-E on A to get *eigenvalues* λ = 2, λ = 3; eigenspaces: λ = 2: $x_1 = [^{1}0_{0}]$; λ = 3: $x_2 = [^{-1}{-1}_{2}]$. Since A is a 3×3 matrix, and there are only **two** basis vectors in total, A is *not* diagonalisable. (There are not enough eigenvectors to have a basis of eigenvectors (of A) for $\mathbf{R}^3$).

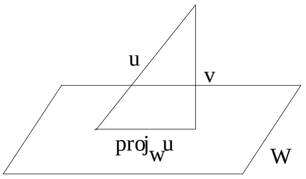
Using λ-D-R-E for B, the *eigenvalues* are λ = 2 and λ = 6. The bases for the eigenspaces are λ = 2: $x_1 = [{}^{-1}1_0]$, $x_2 = [{}^{-1}0_1]$; λ = 6: $x_3 = [{}^{1}2_1]$. Since B is a 3×3 *matrix*, and there are 3 basis vectors in total, B is diagonalisable. Let P be the **matrix** of the x_i's shown above, so that $P = [{}^{-1}1_0\ {}^{-1}0_1\ {}^{1}2_1]$. The *diagonalised* matrix is $[{}^{2}0_0\ {}^{0}2_0\ {}^{0}0_6]$. (*The eigenvalues* are on the diagonal).

(**Note**: we could have (to check) worked out P^{-1}, then $P^{-1}BP$, to make sure that the diagonalised matrix we obtained was the *correct* one. Do not do this in exams: it **wastes** time).

➢ 11th November 1999

Least Squares

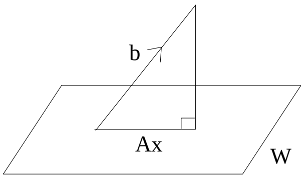
Projections as *Approximations*. $\text{proj}_w u$ is the **closest** vector (or point) in W to u. **Best Approximation Theorem**. Let W be a subspace of V, and let u be a vector of V. $\text{proj}_w u$ is the best approximation to u from W in the *sense* that $||u-\text{proj}_w u|| \le ||u-w||$ for all *other* vectors w ∈ W. Proof: $u-w = (u-\text{proj}_W u) + (\text{proj}_W u-w)$. So by **pythagoras'** theorem, $||u-w||^2 = ||u-\text{proj}_W u||^2 + ||\text{proj}_W u-w||^2$. If $w \neq \text{proj}_w u$, then $||u-w||^2 > ||u-\text{proj}_w u||^2$.



Least Squares Problem. Given a linear system of m equations (Ax = b) in n unknowns, the problem is to find x (if possible) so that ||Ax-b|| is minimised, and then x is called a *least* squares solution. Ax-b is the *error* vector, $\{\varepsilon_1,\ldots,\varepsilon_m\}$. To *minimise* the error, we minimise ||Ax-b|| i.e. $\varepsilon_1^2+..+\varepsilon_m^2$. (Least *squares*).

Geometric view. Solving the least squares problem is like finding a *vector* $x \in \mathbf{R}^n$ so that Ax is the *closest* vector in W to b (W = column space of A). So for x to be a least squares solution of Ax = b, this vector must *satisfy* $Ax = \text{proj}_w b$ (---(*)). So one way to find the least squares solutions of Ax = b would be to *calculate* $\text{proj}_w b$ and then solve (*) directly. However, there is a **better** way...

...Let $Ax = \text{proj}_w b$. Then $b-Ax = b-\text{proj}_w b$ (Where the RHS is *orthogonal* to W). W is the column space (image) of A. So b-Ax is the kernel/null space of A^T. So a *least squares solution* must satisfy $A^T(b-Ax) = 0$. So **$A^TAx = A^Tb$**. This is the "*Normal system associated with Ax = b*".



Theorem: $A^TAx = A^Tb$. This is *consistent*; *all* solutions of the normal system are least squares solutions; if W = column space(A), and x is a *least squares* solution, then $\text{proj}_w b = Ax$. **Theorem**: Let A be an m×n matrix. The column vectors are linearly independent ⇔ A^TA is invertible ⇒ there is a *unique* least squares solution, $x = (A^TA)^{-1}A^Tb$, and $\text{proj}_w b = Ax = A(A^TA)^{-1}A^Tb$.

Example: Find the least squares solution of the *system* x+y = 2, x+4y = 4, and x+5y = 5. (An over *determined* system of equations). In matrix form, Ax = b is $[{}^{1}1_1\ {}^{3}4_5][{}^{x}{}_{y}] = [{}^{2}4_5]$. The *normal* system is $A^TAx = A^Tb$; $A^TA = [{}^{1}{}_{3}\ {}^{1}{}_{4}\ {}^{1}{}_{5}][{}^{1}1_1\ {}^{3}4_5] = [{}^{3}{}_{12}\ {}^{12}{}_{50}]$. So $A^Tb = [{}^{1}{}_{3}\ {}^{1}{}_{4}\ {}^{1}{}_{5}][{}^{2}4_5] = [{}^{11}{}_{47}]$. And $A^TAx = A^Tb$ gives $[{}^{3}{}_{12}\ {}^{12}{}_{50}][{}^{x}{}_{y}] = [{}^{11}{}_{47}]$. We now solve the "*normal equations*" 3x+12y = 11 and 12x+50y = 47 in the usual way, **giving** $x = -{}^{7}/_{3}$ and $y = {}^{3}/_{2}$ as the *least squares* solution.

x	2	4	5	6
y	6.5	8.5	11	12.5

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 8.5 \\ 11 \\ 12.5 \end{bmatrix}$$

Imagine that we perform an *experiment* and get the values shown in the <u>table</u>. It is thought that y is related to x by *a relation* of the form $y = r_0+r_1x$. So $6.5 = r_0+2r_1$, $8.5 = r_0+4r_1$, $11 = r_0+5r_1$, and $12.5 = r_0+6r_1$. What is the least *squares* solution? We get this using the **procedure** on the matrix shown.

Let $Ax = \text{proj}_w b$, where W = *column* space of A. What is the **projection** of b on the column space of A? In the example on the *previous* page, $A = [{}^{1}1_{1}\ {}^{3}4_{5}]$ and $b = [{}^{2}4_{5}]$. So $[{}^{1}1_{1}\ {}^{3}4_{5}][{}^{-7/3}{}_{3/2}] = \text{proj}_w b$. Possible **questions**: Q: Find the least squares solution of $Ax = b$, and hence Q: Find the orthogonal projection of b on the column space of A. Q: Verify that $Ax-b$ is *orthogonal* to W. Q: Calculate the *projection* matrix: $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$.

Projection is a *linear* transformation. Every linear transformation can be **represented** by a matrix: $P: \mathbf{R}^n \rightarrow W$, $x \mapsto \text{proj}_w x$. First choose a *basis* for W, and construct a *matrix* A whose columns are the **basis** vectors $[P] = A(A^TA)^{-1}A^T$. Now consider the projection of $\mathbf{R}^3$ onto the x-y plane (W). **A basis** for $W \in \mathbf{R}^3$ could be $[{}^{1}0_{0}]$, $[{}^{0}1_{0}]$.

Let $A = [{}^{1}0_{0}\ {}^{0}1_{0}]$ so that $A(A^TA)^{-1}A^T = [{}^{1}0_{0}\ {}^{0}1_{0}\ {}^{0}0_{0}]$; $[P][{}^{x}y_{z}] = [{}^{1}0_{0}\ {}^{0}1_{0}\ {}^{0}0_{0}][{}^{x}y_{z}] = [{}^{x}y_{0}]$. Let W be the plane with *equation* $5x-3y+z = 0$. We have basis {(1,0,-5), (0,1,3)}, and the matrix for *orthogonal projection* onto W: $A = [{}^{1}0_{-5}\ {}^{0}1_{3}]$. Now $[P] = A(A^TA)^{-1}A^T = {}^{1}/_{35}[{}^{10}15_{-5}\ {}^{15}26_{3}\ {}^{-5}33_{4}]$. So $[P][{}^{x}y_{z}] = [{}^{10/35x+15/35y-5/35z}\ \ldots]$, the *projection* of $[{}^{x}y_{z}]$ on W.

➢ 15th November 1999

Tutorial

If F = {(0,1,1), (-2,0,0), (3,3,1)} is a *basis* for $\mathbf{R}^3$, and F' = {(3,2), (-1,5)} is a basis for $\mathbf{R}^2$, write down the *matrix* of the linear transformation $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ given by $T(x,y,z) = (x+y, y+z)$, with *respect* to F in the domain and F' in the range. A: $T(f_1) = T(0,1,1) = (1,2) = x(3,2) + y(-1,5)$. From *this*, we get $1 = 3x-y$ and $2 = 2x+5y$, which we solve to get $x = {}^{7}/_{17}$, $y = {}^{4}/_{17}$.

So $T(f_1) = {}^{7}/_{17}(3,2) + {}^{4}/_{17}(-1,5)$. Similarly, $T(f_2) = T(-2,0,0) = (-2,0) = -{}^{10}/_{17}(3,2) + {}^{4}/_{17}(-1,5)$, and $T(f_3) = T(3,3,1) = (6,4) = 2(3,2) + 0(-1,5)$. So $[T]^{F}_{F'}$ is the *transpose* of the matrix of coefficients, so that $[T]^{F}_{F'} = ({}^{7/17}{}_{4/17}\ {}^{-10/17}{}_{4/17}\ {}^{2}{}_{0})$.

Q: Prove that the **subset** S of all vectors of the form (a, b, a+b, a-b) is a *subspace* of $\mathbf{R}^4$. ($a,b \in \mathbf{R}$). A: Let u = (a, b, a+b, a-b); v = (c, d, c+d, c-d) $\in$ S. Then $ku+lv = (ka+lc, kb+ld, (ka+kb)+(lc+ld), (ka-kb)+(lc-ld)) = (ka+lc, kb+ld, (ka+lc)+(kb+ld), (ka+ld)-(kb+ld)) = (e, f, e+f, e-f)$, where $e = ka+lc$ and $f = kb+ld$. So we have $ku+lw \in S$. Also, $0 \in S$, because when $a = b = 0$, we *have* $(0, 0, 0, 0) \in S$.

Q: Show *that* {(3,2,5,1), (0, ½, ½, -½)} is a *basis* for S. A: Linearly **Independent**: prove that $k_1(3,2,5,1) + k_2(0,½,½,-½) = 0$ implies that $k_1 = k_2 = 0$ is the *only* possible solution. Now we get out that $3k_1 = 0$, $2k_1+½k_2 = 0$, $5k_1+½k_2 = 0$, and $k_1-½k_2 = 0$, which **does** imply that $k_1 = k_2 = 0$. Or, we notice *that* $[{}^{3}{}_{0}\ {}^{2}{}_{½}\ {}^{5}{}_{½}\ {}^{1}{}_{-½}]$ is in *echelon* form (so no **rows** can be lost by row reduction).

Span: $k_1(3,2,5,1) + k_2(0,½,½,-½) = (a,b,a+b,a-b)$. So we have $3k_1 = a$, $2k_1+½k_2 = b$, $5k_1+½k_2 = a+b$, and $k_1-½k_2 = a-b$. Reducing this in a *matrix* gives $k_1 = {}^1/_3a$ and $k_2 = 2b-{}^4/_3a$. (Or by *spotting* it directly — notice that the first equation gives $k_1 = {}^1/_3a$). So we *have* the spanning property: every vector of S can be **written** as a linear combination of s_1 and s_2, i.e. $(a,b,a+b,a-b) = ({}^1/_3a)(s_1) + (2b-{}^4/_3a)(s_2)$.

Q: Find a **basis** for the image and a basis for the kernel of the *linear* transformation T: $\mathbf{R}^4\rightarrow\mathbf{R}^2$, defined by $T(x,y,z,w) = (x+y, z+w)$. A: The matrix *representation* of T (with respect to the **standard** bases) is $[T] = [{}^1_0\ {}^1_0\ {}^0_1\ {}^0_1]$. The image is the column space of [T], so write the columns as rows and do row reduction.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we have a **basis** for Im(T): {(1,0), (0,1)}. The *kernel* is the solution space of $[T]\underline{x} = 0$. So solve the expression on the *right*: $x+y = 0$, $z+w = 0$. The **general** solution (with free variables $y = a$, $w = b$) is $x = -a$, $z = -b$. The *basis* for the kernel is {(-1,1,0,0), (0,0,-1,1)}. Check: **Rank** + **Nullity** (dimension of *column* space + dimension of *null space / kernel*) = dimension of $\mathbf{R}^4$. OK.

➢ 18th November 1999

Gram-Schmidt Procedure

Recall: A *basis* of vectors $\{u_1,...,u_n\}$ is orthonormal if it is *orthogonal* ($<u_i, u_j> = 0$) and *normal*. (So we can test if a basis **is** orthonormal). Why are orthonormal bases *useful*? If $\{u_1,...,u_n\}$ is an orthonormal basis for a vector space V, then for *any* $u \in V$, $u = <u,u_1>u_1 + ... <u, u_n>u_n$. Note: $[u]_{u_i}$ = *column* vector $[<u,u_1> ... <u,u_n>]_{u_i}$.

Problem: What if the space is defined using a basis $\{v_1,...,v_n\}$ which is not orthonormal? Can we construct a basis *which is* orthonormal? A: This is always **possible**! How to construct it: the **proof** of the above fact. Theorem: Every *non-zero* finite dimensional space has an orthonormal basis. <u>Proof</u>: (constructive) the Gram-Schmidt Procedure.

<u>**Gram-Schmidt Procedure**</u>. INPUT: $\{v_1,...,v_n\}$ (a *basis* for V). (1) Let $u_1 = v_1$. (2) Let $u_2 = v_2\text{-proj}_{W_1}(v_2)$, where W_1 is the space *spanned* by $u_1 = v_2 - {}^{<v_2,u_1>}/_{||u_1||^2}\,u_1$. (3) Let $u_3 = v_3\text{-proj}_{W_2}(v_3) = v_3-{}^{<v_3,u_1>}/_{||u_1||^2}-{}^{u_1<v_3,u_2>}/_{||u_2||^2}u_2$, where W_2 is the *space spanned* by u_1 and u_2. (4) Let $u_4 = v_4\text{-proj}_{W_3}(v_4) = v_4-{}^{<v_4,u_1>}/_{||u_1||^2}u_1-{}^{<v_4,u_2>}/_{||u_2||^2}u_2-{}^{<v_4,u_3>}/_{||u_3||^2}u_3$. ... (n) Let $u_n = v_n\text{-proj}_{W_{n-1}}(v_n)$. OUTPUT: $\{u_1,...,u_n\}$, *orthogonal*. Then **normalise** $\{{}^{u_1}/_{||u_1||},...,{}^{u_n}/_{||u_n||}\}$ to get an orthonormal basis.

Note that **dividing** or **multiplying** vectors by numbers *does not affect* orthogonality. So we can normalise the vectors at <u>any</u> stage during the algorithm. *Example*: $v_1 = (1,1,1)$, $v_2 = (-1,1,0)$, $v_3 = (1,2,1)$. Q: Find an *orthonormal* basis for the space spanned by $\{v_1,v_2,v_3\}$. (1) $u_1 = v_1 = (1,1,1)$. $(\bar{u}_1)$, i.e. u_1 *normalised*, is ${}^{(1,1,1)}/_{||(1,1,1)||} = ({}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}})$.

(2) $u_2 = v_2\text{-proj}_{W_1}(v_2) = (-1,1,0) - {}^{<(-1,1,0)(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})>}/_{||(\bar{u}_1)||^2}\times({}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}}) = (-1,1,0) - (-{}^1/_{\sqrt{3}}+{}^1/_{\sqrt{3}}+0)({}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}})$. So $u_2 = (-1,1,0)$; $(\bar{u}_2) = (-{}^1/_{\sqrt{2}},{}^1/_{\sqrt{2}},0)$. (3) $u_3 = v_3\text{-proj}_{W_2}(v_3)$, where W_2 is the *subspace spanned by* $(\bar{u}_1)$ and $(\bar{u}_2)$. So $u_3 = (1,2,1)-<(1,2,1)({}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}})>({}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}},{}^1/_{\sqrt{3}}) - <(1,2,1)(-{}^1/_{\sqrt{2}},{}^1/_{\sqrt{2}},0)>(-{}^1/_{\sqrt{2}},{}^1/_{\sqrt{2}},0) = (1,2,1)-({}^4/_3,{}^4/_3,{}^4/_3)-(-{}^1/_2,{}^1/_2,0) = ({}^1/_6,{}^1/_6,-{}^2/_6)$. Therefore, $(\bar{u}_3) = ({}^1/_{\sqrt{6}},{}^1/_{\sqrt{6}},-{}^2/_{\sqrt{6}})$.

Orthogonal Diagonalisation

Matrices: **Symmetric**: A is *symmetric* if $A = A^T$. Example: $[^{10}1/2_{3}\ ^{1/2}0_{6}\ ^{3}6_{5}]$. Results on *symmetric* matrices: If A & C are symmetric, then A+C, A-C, ka (k is a scalar), A^T and A^{-1} (if it *exists*) are ALL symmetric. If A is invertible, then A^TA and AA^T are *invertible*.

Orthogonal. A *matrix* P is orthogonal if $P^{-1} = P^T$. Example: $[^{3/7}-6/7_{2/7}\ ^{2/7}3/7_{6/7}\ ^{6/7}2/7_{-3/7}]$. P is an *orthogonal* n×n matrix ⇔ the row vectors of P are an orthonormal basis for $\mathbf{R}^n$ ⇔ the column vectors are... For example, $PP^T = [P][P^T] = [^{1}0_{0}\ ^{0}1_{0}\ ^{0}0_{1}] = I$.

Proof: $PP^T = [^{<r_1,r_1>}\ldots_{<r_n,r_1>}\ \ldots\ ^{<r_1,r_n>}\ldots_{<r_n,r_n>}] = I$ ⇔ all rows are *normalised*. Now $<r_i,r_j> = 0$ for all $i \neq j$ ⇔ all row vectors are **orthogonal**. Summary: P is "orthogonal" ⇔ $P^{-1} = P$ ⇔ all rows of P are *orthonormal*.

Orthonormal Eigenvector Problem: Given an n×n matrix A, is there an orthonormal basis for $\mathbf{R}^n$ consisting of *eigenvectors* of A?

Matrix *representation* of T: $\mathbf{R}^n \to \mathbf{R}^n$. Let $[T]_u$ be a diagonal matrix, where u = *a basis* of eigenvectors which is orthonormal. This is *equivalent* to the **Orthogonal Diagonalisation Problem**: Given an n×n matrix A, *can we find* an orthogonal matrix P such that $P^{-1}AP = P^TAP$ is diagonal? *Theorem*: A is orthogonally diagonalisable ⇔ A is **symmetric** (easy to test for) ⇔ A has an *orthonormal set* of n eigenvectors.

Orthogonal Diagonalisation Procedure. **INPUT**: A (an n×n matrix). (1) Check that A is *symmetric* (by looking at it). (2) Find a basis for each eigenspace of A (by λ-DRE). (3) Apply *Gram-Schmidt* to each basis to obtain an orthonormal basis for each eigenspace. (4) Form the matrix P whose columns are the *basis* vectors of the previous step. **OUTPUT**: P, which orthogonally diagonalises A.

Example: $A = [^{3}{}_{1}\ ^{1}{}_{3}]$. *Eigenspaces:* (λ) $[^{3-\lambda}{}_{1}\ ^{1}{}_{3-\lambda}]$. (D) $(3-\lambda)(3-\lambda)-1$. (R) $0 = (3-\lambda)^2-1$, $0 = (\lambda-2)(\lambda-4)$. (E) *Eigenvalues* $\lambda = 2$, $\lambda = 4$. For $\lambda = 2$, the basis for the eigenspace is $\{[^{-1}{}_{1}]\}$. $\lambda = 4$: $\{[^{1}{}_{1}]\}$. So $v_1 = (-1,1)$; $v_2 = (1,1)$. Therefore, $<v_1,v_2> = -1+1 = 0$. So the vectors are *orthogonal*. **Normalise**: $(\bar{u}_1) = (-{}^{1}/_{\sqrt{2}}, {}^{1}/_{\sqrt{2}})$, $(\bar{u}_2) = ({}^{1}/_{\sqrt{2}}, {}^{1}/_{\sqrt{2}})$.

These are now *orthonormal* bases for the eigenspaces of A. Use as *columns* for P, so that $P = [^{-1/\sqrt{2}}{}_{1/\sqrt{2}}\ ^{1/\sqrt{2}}{}_{1/\sqrt{2}}]$. It follows that $P^{-1} = P^T = [^{-1/\sqrt{2}}{}_{1/\sqrt{2}}\ ^{1/\sqrt{2}}{}_{1/\sqrt{2}}]$. So $P^{-1}AP = [^{2}{}_{0}\ ^{0}{}_{4}]$. Example: Let T: $\mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation, where $T(x,y) = (3x+y, x+3y)$. Now $[T]_E = [^{3}{}_{1}\ ^{1}{}_{3}] = A$. But $[T]_u = [^{2}{}_{0}\ ^{0}{}_{4}]$, representing the *same* linear transformation, but diagonal — makes calculation **easier**.

Assignment 3

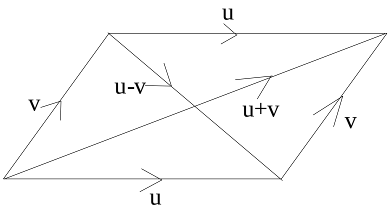
Q: Let $u = (x_1,y_1)$ and let $v = (x_2,y_2)$. (a) **Verify** that $<u,v> = x_1x_2-2x_1y_2-2y_1x_2+5y_1y_2$ is an inner product on the *vector* space $\mathbf{R}^2$. (b) For what values of k is the *following* an inner product on $\mathbf{R}^2$?: $<u,v> = x_1x_2-3x_1y_2-3y_1x_2+ky_1y_2$. (c) For what values of a,b,c,d is the **following** an inner product on $\mathbf{R}^2$?: $<u,v> = ax_1x_2+bx_1y_2+cy_1x_2+dy_1y_2$.

A: First **verify** the *axioms*: <ku+*l*w, v> = k<u,v> + *l*<w,v>. Concentrate on this one! Then verify that <u,v> = <v,u>, followed by checking that <u,u> $\geq$ 0. (b) $x_1^2-6x_1y_1+ky_1^2 \geq 0$ implies k > 9, because $x_1^2-6x_1y_1+ky_1^2 = (x_1-3y_1)^2-9y_1^2+ky_1^2$. (If k = 9, then we could get <u,u> = 0 when u $\neq$ 0, which **isn't** allowed! (c) First of all (by symmetry) <u,v> = <v,u>, so we *need* b = c. Then consider (positivity) <u,u> = $ax_1^2+bx_1y_1+bx_1y_1+dy_1^2$ = ... = $a(x_1+(^{b}/_{a})y_1)^2 + (d-(^{b^2}/_{a}))y_1^2$. So we need a > 0 (or d > 0) *and* ad-b² = ab-bc > 0.

Q: Find the *norm* of $(^{1}/_{2}, -^{1}/_{4}, ^{1}/_{3}, ^{1}/_{6}) \in$ **R**^{4} and (1-2i, 3+i, 2-5i) $\in$ **C**³. A: $\|(^{1}/_{2}, -^{1}/_{4}, ^{1}/_{3}, ^{1}/_{6})\| = \sqrt{}(<(^{1}/_{2}, -^{1}/_{4}, ^{1}/_{3}, ^{1}/_{6}),(^{1}/_{2}, -^{1}/_{4}, ^{1}/_{3}, ^{1}/_{6})>) = \sqrt{}(^{1}/_{4}+^{1}/_{16}+^{1}/_{9}+^{1}/_{36}) = {}^{\sqrt{(65)}}/_{12}$. (ii) $\|(1-2i, 3+i, 2-5i)\| = \sqrt{}((1-2i)(1+2i) + (3+i)(3-i) + (2-5i)(2+5i)) = \sqrt{}(5+10+29) = \sqrt{44} = 2\sqrt{11}$.

Q: Find the **angle** between (1,-2,3) and (2,1,5) in **R**³. Then find the angle between (1,i) and (1-i, 1) in **C**². The parallelogram law gives $\|u+v\|^2+\|u-v\|^2 = 2\|u\|^2+2\|v\|^2$. Draw a *picture* and use an example to explain it. A: $\cos\theta = {}^{<u,v>}/_{\|u\|\,\|v\|} = {}^{<(1,-3,2),\ (2,1,5)>}/_{\|(1,-3,2)\|\ \|(2,1,5)\|} = {}^{2-3+10}/_{\sqrt{(1+9+4)}\sqrt{(4+1+25)}} = {}^{9}/_{\sqrt{(14\times30)}} = {}^{9}/_{2\sqrt{105}}$. invcos$(^{3}/_{140}) \approx 64°$. (ii) ${}^{<(1,i),\ (1-i,1)>}/_{\|(1,i)\|\|(1-i,\ 1)\|} = {}^{1(1+i)+i(1)}/_{\sqrt{((1\times1+i\times-i)((1-i)(1+i)+1\times1))}} = {}^{1+2i}/_{\sqrt{2}\sqrt{3}}$.

The **parallelogram** rule is a generalisation of Pythagoras' rule on a rectangle. Example: let u = (10,1), and let v = (2,6), so that u+v = (12,7), and u-v = (8,-5). Remember that these are *vectors*, and do not have to start at the origin, but can be moved. The sum of the squares of the diagonals is equal to the sum of the squares of the edges (all four of them). So <u,u> = 100+1 = 101 = $\|u\|^2$. And <v,v> = 4+36 = 40 = $\|v\|^2$. Further, <u+v, u+v> = 144+39 = 193 = $\|u+v\|^2$. Finally, <u-v, u-v> = 64+25 = 89 = $\|u-v\|^2$. **So** $2\|u\|^2+2\|v\|^2$ = 202+80 = 282. **And** $\|u+v\|^2+\|u-v\|^2$ = 193+89 = 282. QED!



Q: Verify that the given set of vectors is orthogonal, and then convert it to an orthonormal set: $(^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5})$, (-2,2,0), $(^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})$. A: $<(^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5}),(-2,2,0)> = 0$. $<(^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5}),(^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})> = 0$. $<(-2,2,0),(^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})> = 0$. **Normalising**, $(^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5})/\|(^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5})\| = (^{1}/_{5}, ^{1}/_{5}, ^{1}/_{5})/(\sqrt{(3)}/5) = (^{1}/_{\sqrt{3}},^{1}/_{\sqrt{3}},^{1}/_{\sqrt{3}})$. And $(-2,2,0)/\|(-2,2,0)\| = (-2,2,0)/\sqrt{8} = (^{-1}/_{\sqrt{2}}, ^{1}/_{\sqrt{2}}, 0)$. **And** $(^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})/\|(^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})\| = (^{1}/_{9}, ^{1}/_{9}, -^{2}/_{9})/(\sqrt{(6)}/9) = (^{1}/_{\sqrt{6}}, ^{1}/_{\sqrt{6}}, ^{-2}/_{\sqrt{6}})$.

Q: Find the **projection** of v along w if (i) v = (1,-1,2) and w = (0,1,1) $\in$ **R**³; (ii) if v = (1-i, 2+3i) and w = (2-i, 3) $\in$ **C**². A: (i) $\text{proj}_W v = {}^{<(1,-1,2),(0,1,1)>}/_{<(0,1,1),(0,1,1)>}w = {}^{1}/_{2}(0,1,1) = (0,^{1}/_{2},^{1}/_{2})$. (ii) $\text{proj}_W v = {}^{<(1-i,\ 2+3i),\ (2-i),\ 3)>}/_{<(2-i,3),\ (2-i,\ 3)>}w = {}^{(1-i)(2+i)+(2+3i)(3)}/_{(2-i)(2+i)+(3)(3)}w = {}^{3+6+9i}/_{14}w = {}^{9}/_{14}+{}^{9i}/_{14}(2-i,\ 3) = (^{13}/_{7}+^{1}/_{2}i,\ ^{37}/_{14}+^{12}/_{7}i)$.

➢ 22nd November 1999

Tutorial

Q: Find the matrix P which *orthogonally diagonalises* A = $[{}^{4}2_{2}\ {}^{2}4_{2}\ {}^{2}2_{4}]$. A: Use λ-DRE. (λ) $[{}^{4-\lambda}2_{2}\ {}^{2}4\text{-}\lambda_{2}\ {}^{2}2_{4-\lambda}]$. (D) (4-λ)((4-λ)²-2×2)-2(2(4-λ)-2×2)+2(2×2-2(4-λ)). (R) 0 = (λ-2)²(λ-8). (E) λ = 2 and λ = 8 are the eigenvalues. For λ = 2, (4-2)x+2y+2z = 0; 2x+(4-2)y+2z = 0; and 2x+2y+(4-2)z = 0. So the basis for the *eigenspace is* $\{[{}^{-1}0_{1}], [{}^{-1}1_{0}]\}$. λ = 8: (4-8)x+2y+2z = 0; 2x+(4-8)y+2z = 0; 2x+2y+(4-8)z = 0, ..., -4x+2y+2z = 0; -3y+3z = 0; 3y-3z = 0. So the **basis** for the eigenspace is $\{[{}^{1}1_{1}]\}$. Now we have to find *orthonormal* bases for the eigenspaces.

Gram-Schmidt: $v_1 = (-1,0,1)$, $v_2 = (-1,1,0)$, $v_3 = (1,1,1)$. $u_1 = (-1,0,1)$. $(\bar{u}_1) = (-^1/_{\sqrt2}, 0, ^1/_{\sqrt2})$. $u_2 = (-1,1,0) - \langle(-1,1,0), (-^1/_{\sqrt2},0,^1/_{\sqrt2})\rangle(-^1/_{\sqrt2},0,^1/_{\sqrt2}) = (-1,1,0)-(^1/_{\sqrt2})(-^1/_{\sqrt2},0,^1/_{\sqrt2}) = (-1,1,0)-(-^1/_2,0,^1/_2) = (-^1/_2,1,-^1/_2)$. So $(\bar{u}_2)= (-^1/_{\sqrt6}, ^2/_{\sqrt2}, -^1/_{\sqrt6})$.

v_3 is *orthogonal* to $(\bar{u}_1)$ and $(\bar{u}_2)$ because it is in a **different** eigenspace. Normalising gives $(\bar{u}_3) = (^1/_{\sqrt3},^1/_{\sqrt3},^1/_{\sqrt3})$. So the *orthonormal* basis for the eigenspace with $\lambda = 2$ is $\{[^{-1/\sqrt2}0_{1/\sqrt2}], [^{-1/\sqrt6}2/\sqrt6_{-1/\sqrt6}]\}$. With $\lambda = 8$, it is $\{[^{1/\sqrt3}1/\sqrt3_{1/\sqrt3}]\}$. Therefore, $P = [^{-1/\sqrt2}0_{1/\sqrt2} \ {}^{-1/\sqrt6}2/\sqrt6_{-1/\sqrt6} \ {}^{1/\sqrt3}1/\sqrt3_{1/\sqrt3}]$ is an *orthogonal matrix* which diagonalises A, with $P^TAP = [^20_0 \ {}^02_0 \ {}^00_8]$.

Q: Find the matrix P which o*rthogonally diagonalises* $A = [^a{}_b \ {}^b{}_a]$ (where $b \neq 0$). A: $P = [^{1/\sqrt2}{}_{1/\sqrt2} \ {}^{-1/\sqrt2}{}_{1/\sqrt2}]$ is an orthogonal matrix. $P^TAP =[^{1/\sqrt2}{}_{1/\sqrt2} \ {}^{-1/\sqrt2}{}_{1/\sqrt2}][^a{}_b \ {}^b{}_a][^{1/\sqrt2}{}_{1/\sqrt2} \ {}^{-1/\sqrt2}{}_{1/\sqrt2}] = [^{a+b}{}_0 \ {}^0{}_{a-b}]$.

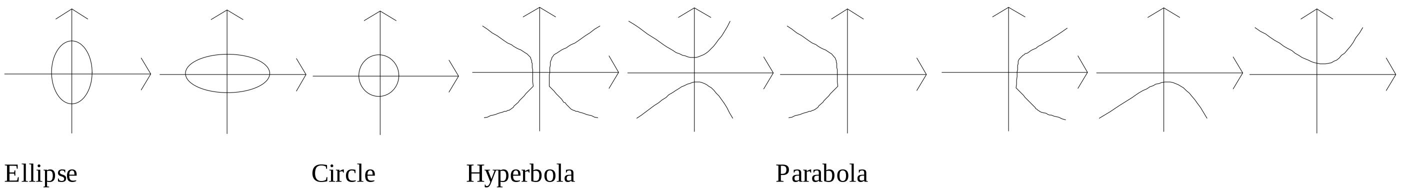
Q: Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(x,y) = (2x+3y, 3x+2y)$. Find an *orthonormal* basis for $\mathbf{R}^2$ so that the matrix representation of T w.r.t. this basis is <u>diagonal</u>. If the orthonormal basis is $\{u_1,u_2\}$, what is the matrix of the *transition* from the standard basis $\{e_1,e_2\}$ to $\{u_1,u_2\}$? What is $(4,^1/_6)$ as a co-ordinate vector in (i) the *standard* basis $\{e_1,e_2\}$, (ii) the *orthonormal* basis $\{u_1,u_2\}$?

A: Find a **matrix** representation of T. Use the *standard* basis and $T(x,y) = (2x+3y, 3x+2y)$ so that $T(e_1) = T(1,0) = (2,3)$ and $T(e_2) = T(0,1) = (3,2)$. Therefore, $[T]_E = [^2{}_3 \ {}^3{}_2]$ (the *transpose* of the coefficient matrix). From the **above** question, $[T]_E$ is orthogonally diagonalised by $P =[^{1/\sqrt2}{}_{1/\sqrt2} \ {}^{-1/\sqrt2}{}_{1/\sqrt2}]$ to give $[^{2+3}{}_0 \ {}^0{}_{2-3}] = [^5{}_0 \ {}^0{}_1]$ $(P^T[T]_EP = [T]_u)$, where $u_1 = (^1/_{\sqrt2},^1/_{\sqrt2})$ and $u_2 = (-^1/_{\sqrt2},^1/_{\sqrt2})$. An **orthonormal** basis of $\mathbf{R}^2$ which gives a *diagonal matrix* representation of T is $\{u_1, u_2\}$, where $[T]_u = [^5{}_0 \ {}^0{}_1]$. The matrix of *transition* is P, so that $P[v]_u = [v]_E$. Now let $v = (5,^1/_6)$, so that $[v]_E = [^5{}_{1/6}]$. Now $[v]_u = P^{-1}[v]_E = [^{1/\sqrt2}{}_{-1/\sqrt2} \ {}^{1/\sqrt2}{}_{1/\sqrt2}][^5{}_{1/6}]_E = [^{31/6\sqrt2}{}_{-29/6\sqrt2}]_u$ (using the *transition* matrix). Alternatively, $[v]_u = [^{\langle v,u_1\rangle}{}_{\langle v,u_2\rangle}]_u = [^{31/6\sqrt2}{}_{-29/6\sqrt2}]_u$ (by the *projection* formula).

➢ <u>25th November 1999</u>

<u>Conic Sections</u>

A graph of an **equation** of the form $ax^2+2bxy+cy^2+dy+ex+f = 0$ is called a CONIC SECTION or CONIC. There are *three* basic (non-degenerate) conic sections and by transforming a graph to its standard position, we can <u>classify</u> it. (1) **Ellipse** or **circle:** $^{x^2}/_{k^2}+^{y^2}/_{l^2} = 1$. (2) **Hyperbola:** $^{x^2}/_{k^2}-^{y^2}/_{l^2} = 1$ or $^{y^2}/_{l^2}-^{x^2}/_{k^2} = 1$. (3) **Parabola:** $y^2 = kx$ or $x^2 = ky$. To describe a given conic, you must change the *basis* (axes) and origin to put it in the <u>standard</u> position.



<u>Translation</u>. Example: $2x^2+y^2-12x-4y+18 = 0$. *Classify* the conic. In this case, there is no cross-product term (xy term) so we don't need to <u>rotate</u> the axes — only translate. So $(2x^2-12x)+(y^2-4y)+18 = 0$; $2(x^2-6x)+(y^2-4y)+18 = 0$; $2((x-3)^2-9)+(y-2)^2-4+18 = 0$; (completing the *square);* $2(x-3)^2+(y-2)^2 = 4$. Now change the axes by a <u>translation:</u> $x' = x-3$, $y' = y-2$. So $2x'^2+y'^2 = 4$, $^{x'^2}/_2+^{y'^2}/_4 = 1$, implying that we have an *ellipse*.

Example: $5x^2-4xy+8y^2-36 = 0$. (with an xy term). The matrix form of the equation is as *follows*: $[x\ y]A[{}^{x}{}_{y}]-36 = 0$, where $A = [{}^{5}{}_{-2}\ {}^{-2}{}_{8}]$ (in *general, we have* $[{}^{a}{}_{b}\ {}^{b}{}_{c}]$). We want to **change** the bases/axes used so that A is diagonal. So *orthogonally diagonalise* A. We have eigenvalues $\lambda = 4$ and $\lambda = 9$; *bases* for eigenspaces $\{[{}^{2}{}_{1}]\}$ and $\{[{}^{-1}{}_{2}]\}$; and orthonormal *bases* for eigenspaces $\{[{}^{2/\sqrt{5}}{}_{1/\sqrt{5}}]\}$ and $\{[{}^{-1/\sqrt{5}}{}_{2/\sqrt{5}}]\}$. There is no need to apply *Gran-Schmidt* because the bases are in different eigenspaces.

So the matrix which **orthogonally** diagonalises A is $P = [{}^{2/\sqrt{5}}{}_{1/\sqrt{5}}\ {}^{-1/\sqrt{5}}{}_{2/\sqrt{5}}]$. The determinant of P is 1, so that P is a rotation. The equation *becomes* (using the transformation matrix x = Px', where $x = [{}^{x}{}_{y}]$) $(P\underline{x}')^TA(P\underline{x}')-36 = 0$; $\underline{x}'^TP^TAP\underline{x}'-36 = 0$; $[x'\ y'][{}^{4}{}_{0}\ {}^{0}{}_{9}][{}^{x'}{}_{y'}]-36 = 0$; $4x'^2+9y'^2-36 = 0$; ${}^{x'^2}/_{9}+{}^{y'^2}/_{4} = 1$; implying that we have another *ellipse*.

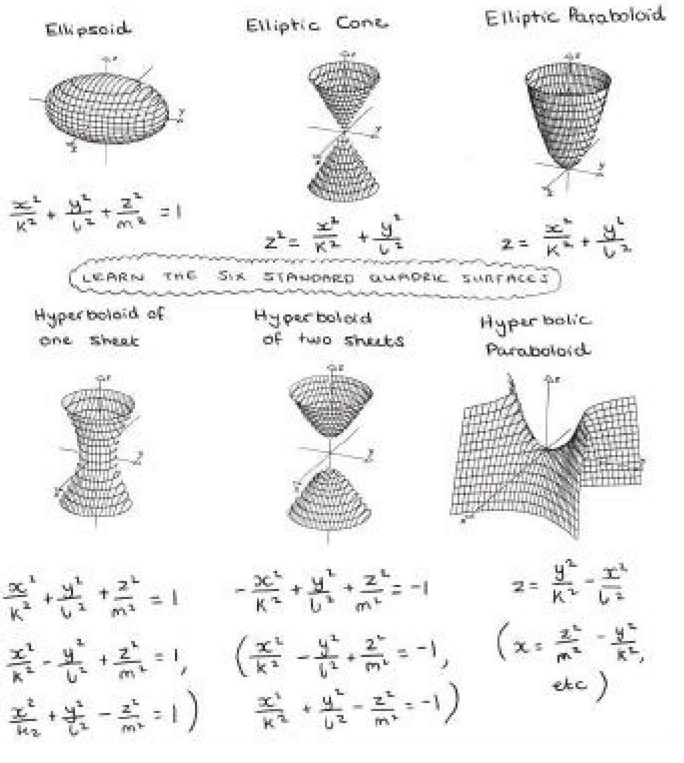
Example: Classify $5x^2-4xy+8y^2+{}^{20}/_{\sqrt{5}}x-{}^{80}/_{\sqrt{5}}y+4 = 0$. The **matrix** form is $\underline{x}^TA\underline{x}+kx+4 = 0$, where $A = [{}^{5}{}_{-2}\ {}^{-2}{}_{8}]$ and $k = [{}^{20}/_{\sqrt{5}}\ {}^{-80}/_{\sqrt{5}}]$. **Rotation**. *The eigenvalues* of A are $\lambda = 4$, $\lambda = 9$. The eigenvectors are $\{[{}^{2}{}_{1}]\}$ and $\{[{}^{-1}{}_{2}]\}$. The orthonormal *eigenvectors are* $\{[{}^{2/\sqrt{5}}{}_{1/\sqrt{5}}]\}$ and $\{[{}^{-1/\sqrt{5}}{}_{2/\sqrt{5}}]\}$. So $P = [{}^{2/\sqrt{5}}{}_{1/\sqrt{5}}\ {}^{-1/\sqrt{5}}{}_{2/\sqrt{5}}]$ orthogonally diagonalises A, so that $P^TAP = [{}^{4}{}_{0}\ {}^{0}{}_{9}]$ and $kP = [-8 -36]$. So we get $4x'^2+9y'^2-8x'-36y'+4 = 0$. **Translation**. We get $4(x'^2-2x')+9(y'^2-4y') = -4$. *Completing* the squares, $4(x'-1)^2+9(y'-2)^2 = 36$. So we *translate* the co-ordinate axes by $x'' = x'-1$, $y'' = x'-2$ to get $4x''^2+9y''^2 = 36$, so that ${}^{x''^2}/_{9}+{}^{y''^2}/_{4} = 1$ — yet another *ellipse*.

Principle Axis Theorem (in R²)

Given a **quadratic** in x & y, $ax^2+2bxy+cy^2+dx+ey+f = 0$, the co-ordinate axes can be rotated so that the *conic* has the equation $\lambda_1x'^2+\lambda_2y'^2+d'x'+e'y'+f = 0$, where λ_1 and λ_2 are the *eigenvalues* of $A = [{}^{a}{}_{b}\ {}^{b}{}_{c}]$. The rotation is *accomplished* by substituting $\underline{x} = P\underline{x}'$, where P orthogonally *diagonalises* A. (And det(P) = 1).

Classifying Quadrics (A quadric/quadric surface is the graph of a *quadratic* equation in x, y and z). This is the principal axes theorem in $\mathbf{R}^3$. A quadratic equation in x, y and z is of the form $ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz + j = 0$. So $\underline{x}^TA\underline{x} = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$ is the associated *quadratic* form, where the last three terms are cross products.

The *co-ordinate* axes can be rotated so that the equation of Q in the x', y' and z' has the form $\lambda_1x'^2+\lambda_2y'^2+\lambda_3z'^2+g'x'+h'y'+i'z'+j = 0$. The rotation can be *accomplished* by the substitution $\underline{x} = P\underline{x}'$, where P orthogonally diagonalises A. A further translation of the axes will put the conic in the **standard** position.



Example: Classify the *Quadric* $4x^2+4y^2+4z^2+4xy+4xz+2yz-3 = 0$. The matrix form of the equation $\underline{x}^T A\underline{x}-3 = 0$ is $[x\ y\ z][{}^{4}2_{2}\ {}^{2}4_{2}\ {}^{2}2_{4}][{}^{x}y_{z}]-3 = 0$. We eliminate *cross* products by a rotation i.e. we orthogonally diagonalise A. The **eigenvalues** of A are $\lambda = 2$ and $\lambda = 8$, while the orthonormal bases for the *eigenspaces* are $\{[{}^{-1/\sqrt{2}}1/\sqrt{2}_{0}], [{}^{-1/\sqrt{6}}-1/\sqrt{6}_{2/\sqrt{6}}]\}$, $\{[{}^{1/\sqrt{3}}1/\sqrt{3}_{1/\sqrt{3}}]\}$. So $P = [{}^{-1/\sqrt{2}}1/\sqrt{2}_{0}\ {}^{-1/\sqrt{6}}-1/\sqrt{6}_{2/\sqrt{6}}\ {}^{1/\sqrt{3}}1/\sqrt{3}_{1/\sqrt{3}}]$, and $P^TAP = [{}^{2}0_{0}\ {}^{0}2_{0}\ {}^{0}0_{8}]$. ($\det(P) = 1$). So the *transformation* $\underline{x} = P\underline{x}'$ is a rotation.

The **new** equation is $(\underline{x}')^T P^T AP\underline{x}'-3 = 0$; $[x'\ y'\ z'][{}^{2}0_{0}\ {}^{0}2_{0}\ {}^{0}0_{8}][{}^{x'}y'_{z'}]-3 = 0$, i.e. $2x'^2+2y'^2+8z'^2 = 3$, so that ${}^{x'^2}/_{3/2} + {}^{y'^2}/_{3/2} + {}^{z'^2}/_{3/8} = 1$. This implies that it is an *ellipsoid*. (If the rotation gives an equation with x^2 and x terms; y^2 and y terms, **or** z^2 and z terms, then the quadric needs to be *translated* to the standard position — this is the same procedure as for conics).

➢ 2nd December 1999

Maxima & Minima: Jacobean & Hessian Matrices

Given a **function** $f: \mathbf{R}^2 \to \mathbf{R}$, $f(x,y) = x \in R$, we have the *Jacobean* $({}^{\partial f}/_{\partial x}, {}^{\partial f}/_{\partial y}) = J(f)$; and the *Hessian* matrix $({}^{\partial^2 f/\partial x^2}{}_{\partial^2 f/\partial y\partial x}\ {}^{\partial^2 f/\partial x\partial y}{}_{\partial^2 f/\partial y^2})$, which is *symmetric* when f is sufficiently "nice". A quadratic form in n variables is an **expression** that can be written as $[x_1,\ldots,x_n]A[x_1,\ldots,x_n]^T = \underline{x}^T A\underline{x}$, where A is a *symmetric* $n\times n$ matrix.

Example: Consider the **quadratic** form $x^2+y^2+4xy = [x\ y][{}^{1}{}_{2}\ {}^{2}{}_{1}][{}^{x}{}_{y}]$. This is a **positive** definite quadratic form if $\underline{x}^T A\underline{x} > 0$ for all $\underline{x} \neq 0$. This is a **negative** definite quadratic form if $\underline{x}^T A\underline{x} < 0$. A positive definite **matrix** is a *symmetric* matrix A so that $\underline{x}^T A\underline{x}$ is a positive *definite* quadratic form. **Theorem**: A matrix is +ve definite if all its *eigenvalues* are > 0 (-ve definite if < 0). **Theorem**: $A = [{}^{a}{}_{b}\ {}^{b}{}_{c}]$, a real symmetric matrix, is positive *definite* $\Leftrightarrow \det(A) > 0$ & $\mathrm{tr}(A) > 0$; and is negative definite $\Leftrightarrow \det(A) > 0$ & $\mathrm{tr}(A) < 0$.

Theorem: Let $f: \mathbf{R}^2 \to \mathbf{R}$. If $J(f)(x_0,y_0) = (0,0)$, then (x_0,y_0) is a *stationary point of f*, and H_0 is the Hessian **matrix** evaluated at the point (x_0,y_0). (I) If $\det(H_0) > 0$ & $\mathrm{tr}(H_0) > 0$, *(H_0 is +ve definite)*, then f has a local **min** at (x_0,y_0). (II) If $\det(H_0) > 0$ & $\mathrm{tr}(H_0) < 0$, *(H_0 is -ve definite)*, then f has a local **max** at (x_0,y_0). (III) If $\det(H_0) < 0$, then f has a *saddle* point at (x_0,y_0). (IV) If $\det(H_0) = 0$, then the stationary point is *undetermined* by the Hessian.

Example: Let $f = x^3-3x^2+y^2$. Then $J(F) = [3x^2-6x\quad 2y^2]$. There are *2 solutions* to $J(f) = (0,0)$: $x = 0$, $y = 0$; $x = 2$, $y = 0$. The Hessian is $[{}^{6x-6}{}_{0}\ {}^{0}{}_{2}]$, with $\det(H) = (6x-6)2$. At $x = 0$, $y = 0$, $\det(H) = -12 < 0$, so we have a **saddle** point. At $x = 2$, $y = 0$, $\det(H) = 6 > 0$ and $\mathrm{tr}(H) = (12-6)+2 = 8 > 0$, so we have a local **minimum**.

Revision

Look in the notes for useful *note summaries*. Revision Questions: Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(x,y) = (2x+y, x+2y)$. *Matrix representation of linear transformations:* What is the matrix representation of T w.r.t. the standard basis E? Find a basis for the image and kernel of T. State the rank and the nullity of T.

A: **Because** $T(e_1) = 2e_1+e_2$ and $T(e_2) = e_1+2e^2$, taking the transpose of the *coefficient* matrix, we get $[T]_E = [{}^{2}{}_{1}\ {}^{1}{}_{2}]$. The **image** space is the space spanned by the columns. $[{}^{2}{}_{1}\ {}^{1}{}_{2}] \to [{}^{2}{}_{0}\ {}^{1}{}_{1}]$. So we have basis $\{[{}^{2}{}_{1}], [{}^{0}{}_{1}]\}$; rank 2. Kernel = solution space of $T\underline{x} = 0$. $[{}^{2}{}_{1}\ {}^{1}{}_{2}][{}^{x}{}_{y}] = [{}^{0}{}_{0}] \Rightarrow$ $2x+y = 0$ and $x+2y = 0$. So $x+y = 0$ and kernel = ϕ. Nullity = 0. Check: Rank + Nullity = dim $\mathbf{R}^2$ (Correct).

Diagonalisation: Calculate the eigenvalues and bases for the corresponding eigenspaces of $[T]_E$. Prove that the eigenvectors given above are a *basis* X for $\mathbf{R}^2$ (this implies that $[T]_E$ is diagonalisable). Find the matrix P which *diagonalises* $[T]_E$. Write down the matrix representation of T w.r.t. the **basis** X of eigenvectors.

A: The *characteristic equation* $(2-\lambda)(2-\lambda)-(1)(1) = 0$ implies that $\lambda = 1$ and $\lambda = 3$. We get bases for the eigenspaces $\{[{}^{-1}{}_{1}]\}$ and $\{[{}^{1}{}_{1}]\}$. Putting vectors as rows and doing row *reduction, we* obtain the matrix $[{}^{1}{}_{0}\ {}^{0}{}_{1}]$. The *eigenvectors* are a basis for $\mathbf{R}^2$. Therefore, $P = [{}^{-1}{}_{1}\ {}^{1}{}_{1}]$ diagonalises $[T]_E$. It follows that $[T]_x = P^{-1}[T]_E P = [{}^{1}{}_{0}\ {}^{0}{}_{3}]$.

Change of Basis: write down the transition matrix P which converts co-ordinate vectors w.r.t. E into co-ordinate vectors w.r.t. X. *Orthogonal Diagonalisation*: Find an orthonormal basis of eigenvectors Y and hence find a *matrix* Q which orthogonally diagonalises $[T]_E$.

A: P is the matrix which converts *co-ordinate* vectors w.r.t. E into co-ordinate vectors w.r.t. X, by $[v]_E = P[v]_x$ ($P' = P$). To find an *orthonormal* basis of eigenvectors for $\mathbf{R}^2$, and hence orthogonally *diagonalise* $[T]_E$, we must *normalise* the eigenvectors $X = ({}^{-1}/_{\sqrt{2}}, {}^{1}/_{\sqrt{2}})$; ${}^{(1,1)}/_{\|(1,1)\|} = ({}^{1}/_{\sqrt{2}}, {}^{1}/_{\sqrt{2}})$. So $Q = [{}^{-1/\sqrt{2}}{}_{1/\sqrt{2}}\ {}^{1/\sqrt{2}}{}_{1/\sqrt{2}}]$ (we know the vectors are *orthogonal* since they are in <u>different</u> eigenspaces).

$$\begin{tikzcd}
\mathbf{R}^2 \arrow[r, "P"] \arrow[rrr, bend left, "{[T]X}"] & \mathbf{R}^2 \arrow[r, "{[T]E}"] & \mathbf{R}^2 \arrow[r, "P-1"] & \mathbf{R}^2 \\
\text{basis X} & \text{basis E} & \text{basis E} & \text{basis X} \\
{[v]X} & {[v]E} & {[T(v)]E} & {[T(v)]X}
\end{tikzcd}$$

Exam Paper: January 2000

SECTION 1 (Compulsory)

(1) (a) Find the matrix of the linear transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by T(x, y) = (-2x+y, x+3y) with respect to:

(i) the standard basis of the domain and image;
(ii) the basis {(2, -1), (2, 2)} in the domain and the standard basis in the image;
(iii) the basis {(2, -1), (3, 2)} in both the domain and the image.

[10 marks]

(b) The linear transformation T: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}.$$

(i) Find the eigenvalues and eigenvectors of A.
(ii) Find a matrix P such that $P^{-1}AP$ is diagonal. **[10 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Let S denote the subset $\mathbb{R}^4$ of all vectors of the form (a, b, c, 2b-a-c).
(i) Show that S is a subspace of $\mathbb{R}^4$. **[4 marks]**
(ii) Show that {(1, 1, 0, 1), (1, -1, -3, 0), (0, 0, 1, -1)} forms a basis of S. **[5 marks]**
(iii) Apply the Gram Schmidt process to produce an orthonormal basis for S.
[6 marks]

(3) Let T: $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by T(x, y, z) = (x+y+z, 2x-y+z).
(i) Write down the matrix of T with respect to the standard bases in the domain, $\mathbb{R}^3$ and image, $\mathbb{R}^2$. **[2 marks]**
(ii) Prove that X = {(2, 3), (1, 2)} is a basis for $\mathbb{R}^2$. **[2 marks]**
(iii) Find the matrix T with respect to the standard basis of $\mathbb{R}^3$ and the basis X of $\mathbb{R}^2$.
[4 marks]
(iv) Show that Y = {(1, 2, 3), (1, 0, 1), (5, 1, 3)} is a basis of $\mathbb{R}^3$. **[4 marks]**
(v) Find the matrix of T with respect to the bases Y and X. **[3 marks]**

(4) The linear transformation T is represented by the matrix

$$A = \begin{pmatrix} 4 & 8 & 4 \\ -3 & 1 & 11 \\ 2 & 7 & 8 \end{pmatrix}$$

(i) Convert A to row echelon form. **[4 marks]**
(ii) Find the kernel of T. **[3 marks]**
(iii) Find the dimension of the image of T. **[2 marks]**
(iv) Find a basis for the image of T. Is (1, 1, 1) in the image? **[6 marks]**

(5) The symmetric matrix A has right eigenvectors v_1, v_2 where

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}, \; v_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Find a third eigenvector v_3 and eigenvalue λ_3 to provide an orthogonal basis of eigenvectors.

Write the quadric ϕ: $x^2+2y^2+3z^2-4xy-4yz+6y+6z-{}^{11}/_{2} = 0$ in matrix form. Describe the changes of basis and transformation needed to convert to standard form and hence classify the quadric. **[15 marks]**

(Questions done: 1, 2, 4)