G1M12 Analysis: Exam Paper, May 1999

SECTION 1 (Compulsory)

(1) (a) Express the parametric curve $x = 3 \cosh 3t$, $y = 2 \sinh 3t$ in cartesian coordinates and give its name. **[3 marks]**

Write down without simplification the equations of the tangent and of the normal at the point of the curve with parameter t. **[4 marks]**

(b) Let f be the real function given by $f(x) = x^2-4x+3$. Sketch the function, and determine the image by f of the sets $(-\infty, 2]$, $(1, 3]$, $[0, 3)$. **[6 marks]**

(c) For each of the following sets, state whether it is bounded above or bounded below; in each case determine the maximum, minimum, sup, inf of each set, when they exist.

(i) $(2, 8]$;
(ii) $(-\infty, -\sqrt{5})$;
(iii) $\{-1-1/n : n \in$ Œ and $n \neq 0\}$. **[7 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Find the arc length of the curve $x = 2t^2$, $y = 2t$ from $t = 0$ to $t = 1$. **[3 marks]**

(b) Let f be the real function given by $f(x) = x^4-2x^2+½$. Determine the intervals in which f is (i) increasing, (ii) decreasing, (iii) concave up and (iv) concave down. Locate the zeros of f at or between successive integers, and sketch the graph of f.
[7 marks]

(c) Find an n such that the error in using the Taylor series for sin x up to the term involving x^{2n+1} for $x = 1$ in less than 10^{-6}. **[5 marks]**

(3) Let $a > 0$. Give a name for the curve C(t) with parametric form $(at^2, 2at)$. Find the equation of the normal at the point with parameter t of C(t). Hence find the parametric form for the evolute E(t) of C(t), i.e. the envelope of the normals of C.
[5 marks]

Find also the radius of curvature of the curve C(t) at the point with parameter t and verify that this coincides with the distance from C(t) to E(t). [You may assume the formula for the radius of curvature $\rho = (\dot{x}^2+\dot{y}^2)^{(3/2)}/|\dot{y}\ddot{x} - \dot{x}\ddot{y}|$. Sketch this evolute. Your sketch should include the original curve C and some of its normals. **[4 marks]**

(4) (a) Prove that the function $f(x) = 2+(\cos x)^{1/2}$ is decreasing on the interval $[0, \pi/2]$. By dividing this interval into two equal parts show that $I = \int_0^{\pi/2} f(x)dx$ satisfies $3.8020 \leq I \leq 4.5874$. Does one obtain better upper and lower bounds by dividing this interval into unequal intervals $[0, 3\pi/8]$, $[3\pi/8, \pi/2]$?
[Use the rule $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$]. **[8 marks]**

(b) Let f be the real function given by $f(x) = 1+2x-x^2+x^3$. Prove that for $0 < \delta < 1$ we have $3-4\delta < f(1-\delta) < f(1) < f(1+\delta) < 3+5\delta$. Hence find a number $\delta > 0$ such that $|x-1| < \delta$ implies $|f(x)-3| < 10^{-6}$. **[7 marks]**

(5) (a) Find the general term of the series expansions, in terms of x, of $(1+2x)^{-1}$ and of $(3-x)^{-1}$. For which x are these series convergent? **[4 marks]**

By partial fractions, or otherwise, prove that the general term of the series expansion of $(1+2x)^{-1}(3-x)^{-1}$ is $\frac{1}{7}((-1)^n 2^{n+1}+3^{-n-1})x^n$. For which x is this series convergent? **[5 marks]**

(b) Let $f(x) = x(x-1)(x-2)$. Find a point c in [1, 3] which satisfies the Mean Value Theorem for f on this interval, and illustrate with a sketch graph. **[6 marks]**

<u>(Questions done: 1, 2, 4)</u>

G1M32 Algebra: Exam Paper, May 1999

SECTION 1 (Compulsory)

(1) (a) Find the greatest common divisor of a = 1357 and b = 2468 in the form ar + bs where r and s are integers. Find also the least common multiple of a and b. **[5 marks]**

(b) Prove by induction that $\sum_{k=1}^{n}(k-1)k = \frac{(n-1)n(n+1)}{3}$. **[4 marks]**

(c) Write down the numbers in $\mathbf{Z}_{18}$ that have an inverse under multiplication modulo 18. For each, give the corresponding inverse with reasons. **[4 marks]**

(d) Solve the equation $(3+i\sqrt{2})z = 1+3i$. **[4 marks]**

(e) Find the remainder when $x^5 + 2x^3 - 2x + 4$ is divided by $x^2 + 2$. **[3 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Solve the equation $z^6 = -1$ giving the solutions in the form a+ib, (do **not** leave them in polar form; you should evaluate the cos and sine terms). **[8 marks]**

(b) Prove by induction $\sum_{k=1}^{n}$ k.k! = (n+1)!-1. **[7 marks]**

(3) (a) Prove that there are infinitely many prime numbers. **[7 marks]**

(b) Find the image of the unit circle {z : |z| = 1} under the function f :Š → Š given by $f(z) = \frac{z-1}{z+2}$. **[8 marks]**

(4) Find all solutions in $\mathbf{Z}_{345}$ of the congruence $100x \equiv 65 \bmod 345$. **[15 marks]**

(5) Let p be a prime number. Prove the following:

(i) For r in the range 0 < r < p, the binomial coefficient $\binom{p}{r}$ is divisible by p; **[5 marks]**

(ii) For any integer n, $(n+1)^p \equiv n^p + 1$, by using the binomial theorem and (i); **[5 marks]**

(iii) For any integer $n \geq 0$, $n^p \equiv n \bmod p$, (by induction using (ii)). **[5 marks]**

(Questions done: 1, 3, 4)

G1M52 Sequences & Series: Exam Paper, May 1999

SECTION 1 (Compulsory)

(1) (a) Evaluate (i) $\lim_{x\to 0} \frac{(e^x-1)}{x}$, (ii) $\lim_{x\to 3} \frac{\sqrt{5x+1}-4}{\sqrt{7x+4}-5}$. **[6 marks]**

(b) Determine whether the following series are convergent or divergent, stating clearly the results used. (i) $\sum_{n=1}^{\infty} \frac{2n+3}{(n+1)^3}$, (ii) $\sum_{n=0}^{\infty} \frac{(3n+1)2^n}{(2n+1)3^n}$. **[8 marks]**

(c) Solve the differential equation $(x+y^2)^{dy}/_{dx} + 2(xy+y^3) = 0$ with boundary condition y = 1 when x = 1. **[6 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Solve the second-order, linear, recurrence relation
$u_0 = -1$, $u_1 = 0$, $u_n = 5u_{n-1} - 6u_{n-2}$ ($n \geq 2$). **[6 marks]**

(b) A sequence $S = \{u_n\}_{n\geq 1}$ is defined by
$u_1 = 3$, $u_n = \sqrt{2+u_{n-1}}$ ($n \geq 2$).
Prove that $1 \leq u_n \leq 2$ for all $n \geq 1$ and that S is increasing. Find $\lim_{n\to\infty} u_n$.
[9 marks]

(3) (a) Solve the following inequality, giving the solution set as a union of intervals:
$\frac{x^2-3}{x^2-4} \leq \frac{x^2-6}{x^2-9}$. **[8 marks]**

(b) Find the limit L as $n\to\infty$ of the sequence $\{u_n\}_{n\geq 1}$ when $u_n = \frac{n(3n+1)}{(n+1)(2n+1)}$.
Determine an integer N such that $|u_n - L| < 10^{-10}$ for all $n > N$. **[7 marks]**

(4) (a) Prove, from first principles, that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series. **[5 marks]**

(b) Using partial fractions, obtain expressions for the k-th partial sums s_k (the sum of terms from n = 2 to n = k) of the following series, and hence find the sum of each series: (i) $\sum_{n=2}^{\infty} \frac{1}{n^2+n}$, (ii) $\sum_{n=2}^{\infty} \frac{1}{n^2+n-2}$. **[8 marks]**

What can you say about the series $\sum_{n=2}^{\infty} \frac{1}{n^2+n-1}$? **[2 marks]**

(5) Find all the stationary points of the function $f(x, y) = 2x^3-3x^2y+2y^3-y^2-8y$. **[8 marks]**
Classify these points as maxima, minima or saddle points. **[7 marks]**

(Questions done: 1, 3, 5)

G1M70 Modelling: Exam Paper, May 1999

SECTION 1 (Compulsory)

(1) (a) The position vector **r** m of a particle P of mass 4 kg at time t s is given by $\mathbf{r} = (t^2+t)\mathbf{i} + (3t^2-2)\mathbf{j} + (2t^3-4t^2)\mathbf{k}$.

(i) Find the velocity of P at time t.
(ii) Find the speed of P at time t.
(iii) Find the acceleration of P at time t.
(iv) Find the kinetic energy of P at time t.
(v) Find the rate at which the force acting on P is working at time t.

[7 marks]

(b) The work done by the force $\mathbf{F} = 3\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ N in moving a particle from the point with position vector $2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ m to the point with position vector $a\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ m, where a is a constant, is 52 J. Find the value of a. **[5 marks]**

(c) Two roads intersect at 90° at a point P. A man A is cycling at $^{250}/_{9}$ ms^{-1} along one of the roads towards P and at a certain instant is 400 m from P. A then observes a second man B, 300 m from the junction, running towards it at $^{50}/_{3}$ ms^{-1}. Find the time when the men are nearest to each other and the distance between them.

[8 marks]

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Assuming Hooke's Law, show by integration that the work done in extending a spring of stiffness k a distance x beyond its natural length is $\frac{1}{2}kx^2$. **[2 marks]**

(b) A spring is held in a smooth horizontal tube with one end fixed and the other attached to a small bead which is held in equilibrium by a force of magnitude 60 N pressing it against the free end of the spring. The compression of the spring in this position is 0.03 m.

(i) Find the stiffness of the spring k. **[2 marks]**
(ii) The bead is released. Find, using the conservation of energy, the speed of the bead just as the spring attains its natural length. **[4 marks]**

(c) A uniform rod AB of weight 50 N and length 1.2 m is supported from above at the end A and from below at the point C where AC = 0.3 m. A particle of weight 15 N is placed at the point D where AD = 1.1 m. Find the forces exerted on the rod at A and C. **[7 marks]**

(3) The non-gravitational resistance to the motion of a car of mass 1000 kg moving with a speed of v ms^{-1} is known to be of the form ($kv + 0.05kv^2$) N, where k is a constant. When the car's engine is working at a rate of 11.25 kW the car can move at a steady speed of 25 ms^{-1} on a horizontal road.

(a) Find the value of k. **[2 marks]**

(b) Find the rate at which the car's engine works when the car is moving at a steady speed of 15 ms^{-1} up a hill inclined at an angle $\sin^{-1}(^1/_{49})$ to the horizontal.

[5 marks]

(c) When the car is moving with speed 25 ms^{-1} on a horizontal road the engine is switched off. Show that the speed v ms^{-1} of the car after travelling a distance of x m after the engine has been switched off satisfies the differential equation

$2500^{dv}/_{dx} = -20-v.$

Solve this differential equation to find the distance travelled before the car's speed falls to 5 ms^{-1}. **[8 marks]**

(4) A particle of mass 2 kg lies on a smooth inclined plane, whose angle with the horizontal is 30°. The mass is attached to the top of the plane by a light elastic string of stiffness 30 Nm and natural length 0.5 m.

(a) What is the length of the string in equilibrium? **[2 marks]**

(b) The particle is pulled 0.2 m from the equilibrium position and released. Find the period of the oscillation and calculate the maximum speed of the particle.

[5 marks]

(c) Describe the motion of the particle if the string becomes slack and calculate how far the particle has to be pulled down the plane for it just to reach the top.

[8 marks]

(5) (a) Three small spheres A, B and C lie in a straight line on a smooth table. Their masses are m, 2m and 4m respectively. Sphere A is projected towards sphere B with a speed of 8 ms^{-1}.

(i) If the coefficient of restitution is ¼, find the velocities of the three spheres after three collisions and show that there can be no more collisions.

[6 marks]

(ii) Calculate the total energy lost during the collisions. **[2 marks]**

(b) Consider the free motion under gravity of a particle projected with speed V at an angle α to the horizontal.

(i) Show that the maximum height attained by the particle is $h = \frac{V^2 \sin^2 \alpha}{2g}$.

[3 marks]

(ii) Show that the projectile's range is given by range = $\frac{V^2 \sin 2\alpha}{g}$. **[4 marks]**

(Questions done: 1, 3, 5)

G1M80 Probability: Exam Paper, June 1999

SECTION 1 (Compulsory)

(1) (a) Two different integers are chosen at random from the integers from 1 to 11 inclusive. If their sum is even, find the probability that both numbers are odd.

[5 marks]

(b) Let S be an equiprobable sample space with N elements, and let A and B be events on S with $P(A) = {}^1/_3$ and $P(B) = {}^1/_5$. What is $P(A \cup B)$ if

(i) A and B are independent events?
(ii) A and B are mutually exclusive events?
(iii) $(B \backslash \bar{A})$ contains exactly 4 points from S?
(iv) A and B are independent events and S is *not* equiprobable? **[5 marks]**

(c) Let S be a sample space and A and B be events on S. Prove that $\forall$ A, B, such that $A \subset B$, $P(B) \geq P(A)$. **[5 marks]**

(d) You play a chance game by taking one card from a deck of 52 cards without replacement and keep the drawn cards in a pile. When you draw a card so that in your pile you have either 2 Kings, a King and a Queen, or 2 Queens, you win and the game is over.

(i) What is the probability that you win by drawing exactly 10 cards?
(ii) What is the probability that you need to draw at most 3 cards to win?

[5 marks]

SECTION 2 (Answer 2 out of 4 questions)

(2) A biased coin with P(B) = 0.3 is tossed 3 times. Let (X, Y) be a two-dimensional random variable with X = the number of heads thrown and Y = the number of tails in the first throw.

(a) Give the probability mass function (p.m.f.) of (X, Y).

(b) Find the marginal p.m.f.'s.

(c) Find the marginal distribution functions.

(d) Are X and Y dependent?

(e) Find the expected value of (X, Y).

[15 marks]

(3) Six married couples are standing in a room. If two people are chosen at random, find the probability that

(a) They are married to each other.

(b) One is male and the other is female.

(c) If a committee of 5 is chosen at random, what is the probability that the committee does not include two people who are married to each other?

[15 marks]

(4) Let X be a continuous-valued random variable with a probability density function f(x) > 0 for $x \in [a, b]$ and 0, elsewhere.

(a) Prove that the cumulative distribution function F(x) is strictly increasing in (a, b).

(b) Find the value of the constant k for

$$f(x) = \begin{cases} kxe^{-x^2/4}, & x \in [a,b] \\ 0, & elsewhere. \end{cases}$$

(c) For a = 0, b = 1, find x such that $P(X \geq x) = 0.7$.

[15 marks]

(5) (a) Of 100 boxes of fuses, 5 fuses per box, 20 boxes contain fuses from factory A, 30 boxes from factory B, and 50 boxes from factory C. Fuses from factory A are, on average, 5% defective; from factory B, 4%, and from factory C, 2%. The fuses and boxes all look alike and are piled without regard of place of manufacture. A box is selected at random and a fuse in it is tested and found to be defective. What is the probability that it was produced at factory B?

(b) Five red books and 3 green books are placed at random on a shelf. Find the probability that the green books will all be together.

[15 marks]

<u>(Questions done: 1, 2, 5)</u>