

G1M31 Matrix Algebra: Exam Paper, January 1999

SECTION 1 (Compulsory)

(1) Given square matrices A and B where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 & 2 \\ 4 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \text{ calculate}$$

- (i) $A + B$
- (ii) $A^t + B^t$
- (iii) AB
- (iv) $\det(A)$
- (v) $\text{adj}(A)$
- (vi) A^{-1}
- (vii) AA^{-1}

Use Gaussian elimination to

(viii) solve $Ax = d$ where $d = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$. Explain why the solution is unique.

[20 marks]

SECTION 2 (Answer 2 out of 4 questions)

(2) Find the general solution of the following equations for all real values of λ and μ , indicating those values for which the equations have (i) a unique solution, (ii) more than one solution, (iii) no solution.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= \mu \\ \lambda x_1 + x_2 + x_3 &= 0 \\ (2\lambda - 1)x_1 + (3 - \lambda)x_2 + 3x_3 &= \mu \end{aligned}$$

[15 marks]

(3) (a) If $\mathbf{a} = (-4, 1, -3)$ and $\mathbf{b} = (2, 5, -1)$ find a third vector \mathbf{c} perpendicular to both \mathbf{a} and \mathbf{b} , so that $\{\mathbf{abc}\}$ form a basis set of right handed axes. Express the vector \mathbf{d} given by $\mathbf{d} = (4, -7, -12)$ in terms of this basis. Show that \mathbf{d} , \mathbf{c} , and $2\mathbf{a} - 2\mathbf{b}$ are coplanar.

[8 marks]

(b) Let the points A, B, C, D have position vectors

$$(1, -1, 0) \quad (-1, 2, 1) \quad (5, -1, 2) \quad (11, -7, 1) \text{ respectively.}$$

Find the equations of the line joining A to C and of the line joining B to D. Do these lines meet?

[7 marks]

- (4) (a) State the conditions under which a non empty subset of a vector space is a subspace. **[2 marks]**
- (b) Prove that the subspace U of \mathbb{R}^4 of vectors (a, b, c, d) satisfying $a = b = c+d$ is a subspace. State the dimension of U and find a basis. **[6 marks]**
- (c) The linear transformation $f: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is defined by $f(x) = Ax$ where A is the matrix
- $$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$
- Find the dimension of the kernel and image of f. Find bases for these subspaces. **[7 marks]**
- (5) Explain the meaning of the terms, linearly dependent, spanning set and basis. Show that the vectors $\mathbf{x}_1 = (1, -1, 3, 2)$; $\mathbf{x}_2 = (-1, 1, 1, -2)$ and $\mathbf{x}_3 = (2, -1, 1, 3)$ are linearly independent. Show that one of the vectors $\mathbf{x}_4 = (1, 2, 0, 2)$ and $\mathbf{x}_5 = (-1, 0, 6, -1)$ belongs to the subspace spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and one does not. Express the one which does as a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. **[15 marks]**

(Questions done: 1, 3, 5)

G1M41 Differential Calculus: Exam Paper, January 1999

SECTION 1 (Compulsory)

- (1) (a) If $y = \tan^{-1}(x)$ show that $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$. **[6 marks]**
- (b) If $x = \tan(t)$, $y = \tan(pt)$, where p is a constant, show that $(1 + x^2)\frac{d^2y}{dx^2} = 2(py - x)\frac{dy}{dx}$. **[7 marks]**
- (c) Find the particular solution of the differential equation $\frac{dy}{dx} + \frac{2}{x}y = 1$, $x > 0$ which satisfies $y = 0$ when $x = 1$. **[7 marks]**

SECTION 2 (Answer 2 out of 4 questions)

- (2) Consider the function $f(x) = x^4e^{-x}$, $-\infty < x < \infty$.
- (a) Determine the stationary points and stationary values of f . **[3 marks]**
- (b) Find which stationary values are turning values and determine their nature. **[4 marks]**
- (c) Examine f for inflexion points and determine their position. **[4 marks]**
- (d) Sketch f showing the positions of the stationary points, turning points and inflexion points. **[4 marks]**
- (3) By writing $y = \operatorname{cosech}^{-1}(x)$ in the form $\operatorname{cosech}(y) = x$ and using the definition $\sinh(y) = (e^y - e^{-y})/2$, show that $\operatorname{cosech}^{-1}(x) = \log_e\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$, $x \neq 0$. **[8 marks]**
- Hence, or otherwise, show that $\frac{d}{dx}(\operatorname{cosech}^{-1}(x)) = \frac{-1}{\sqrt{x^4 + x^2}}$, $x \neq 0$. **[7 marks]**
- (4) Show that if $I_n = \int_0^{\pi/2} \cos^n(x)dx$ then $I_n = \left(\frac{n-1}{n}\right)I_{n-2} - \frac{1}{n^2}$, $n \geq 2$. **[10 marks]**
- Deduce that $I_4 = \frac{1}{64}(3\pi^2 - 16)$.
(Hint: write $\cos^n(x) = \cos(x)\cos^{n-1}(x)$ and use integration by parts.) **[5 marks]**
- (5) (a) Find the solution to the differential equation $\frac{dy}{dx} + y \tan(x) = \sin(2x)$ given that at $x=0$, $y=0$. **[8 marks]**
- (b) Find the general solution to the differential equation $2x^3\frac{dy}{dx} = y^2 + 3xy^2$, $x > 0$. Hence show that if $y = 4/3$ when $x=1$, then $y \rightarrow 1$ as $x \rightarrow \infty$. **[7 marks]**

(Questions done: 1, 2, 4)

G1M90 Ideas in Mathematics: Exam Paper, January 1999

SECTION 1 (Compulsory)

- (1) (a) Give diagrams illustrating two cases of the theorem “The angle at the centre is twice the angle at the circumference”, and prove one of these cases. **[4 marks]**
- (b) State and prove a converse to your theorem in (a). **[6 marks]**
You are given two facts about subsets of the set X . Firstly for any subset A of X , $(A^c)^c = A$. Secondly for any two subsets B, C of X , $(B \cap C)^c = B^c \cup C^c$. Using these facts and nothing more prove that for any two subsets, D, E of X , $(D \cup E)^c = D^c \cap E^c$. **[5 marks]**
- (c) Using a truth table, or otherwise, prove that the following is a tautology.
 $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$. **[5 marks]**

SECTION 2 (Answer 2 out of 4 questions)

- (2) Let O, O' be the centres of circles meeting in points D, E and let P be the mid of DE . Prove that O, P, O' are colinear. **[6 marks]**

Let ABC be a triangle and let L, M, N be points on BC, CA, AB respectively. Prove that the circles AMN, BNL, CLM have a common point, and that the centres of these circles form a triangle similar to ΔABC . **[9 marks]**

[Any facts on isosceles triangles and cyclic quadrilaterals used should be stated clearly.]

- (3) A hexagon H^2 lying in a plane is moved at right angles to the plane through one unit of length, forming a solid figure H^3 . Determine the number of points, edges and faces of H^3 . Now H^3 is moved in a fourth dimension through one unit of length, forming a figure H^4 . Determine the number of points, edges and 2-dimensional faces of H^4 . Explain your working clearly. **[7 marks]**

Let H^n be obtained similarly from H^{n-1} for any $n \geq 3$. Let $H(n, r)$ be the number of r -dimensional faces of H^n . Let $h_n(x) = \sum_{r=-1}^{\infty} H(n, r)x^r$. Explain why $h_{n+1}(x) = (x+2)h_n(x)$. Hence find a formula for $H(n, r)$. **[8 marks]**

- (4) (a) Suppose $f: A \rightarrow B$ is a function and T and U are subsets of B . Define $f^{-1}(T) = \{a \in A \mid f(a) \in T\}$, similarly for $f^{-1}(U)$.
- (i) Prove that if $T \subseteq U$ then $f^{-1}(T) \subseteq f^{-1}(U)$. **[5 marks]**
- (ii) Give an example of a function $f: A \rightarrow B$ and subsets T, U of B such that $f^{-1}(T) \subseteq f^{-1}(U)$ but T is not a subset of U . (Hint: Use your favourite function that is not one-to-one.) **[5 marks]**
- (b) Find the number of onto functions from $\{1,2,3,4\}$ to $\{a,b\}$. You should briefly justify your method of finding them and how you know you have the exact number. **[5 marks]**
- (5) (a) You are given n random lines in the plane ' 2 ' and these divide the plane into r_n regions. Prove that $r_{n+1} = r_n + (n+1)$. **[7 marks]**
 [Lines are 'random' if no two are parallel and no three meet in a point.]
- (b) You are given n random planes in 3 dimensional space ' 3 '. These divide ' 3 ' into ρ_n regions. Prove that $\rho_{n+1} = \rho_n + r_n$. **[8 marks]**
 [Planes are 'randomly placed' if no two are parallel, no three meet in a line and no four meet in a point.]

(Questions done: 1, 2, 3)

G2S02 CAL Statistics: Exam Paper, January 1999

SECTION 1 (Compulsory: Answer ALL Questions)

- (1) The following data are the wool weights (ounces) of lambs born and bred at the UWB farm in 1980: 32 37 40 41 44 45 46 49 50 52 53 63 79
- (a) Test whether the sample comes from a population with a mean of 45 ounces at the 5% significance.
- (b) What is the 99% confidence interval for the population mean?
- (b) Is there any evidence that one or more of the observations are outliers?
- (c) Explain the terms significance level and confidence interval in your own words.
- (2) You are about to undertake the following experiment. You have a large number of plant pots available. You are waiting to compare the height to which grass will grow over a certain time period when planted in one of two fertilisers. One fertiliser is the standard fertiliser currently used which has given a mean and standard deviation of 35mm and 10mm respectively. It is hoped the new fertiliser will give an increase of 5mm in height over the standard treatment.
- (a) State the name of the statistical test you intend to use to analyse the data you will collect.
- (b) State the assumptions that the test chosen in question (a) above makes and state how you intend to check the assumptions.
- (c) How many plant pots do you intend to use in your experiment. Justify your choice.

- (3) The file `o:\statdata\data\capefear.mtw` contains data on the experiment described below. The data summarises a piece of research to identify soil characteristics and relationships in the Cape Fear estuary in North Carolina.

There are 3 factors, Type and Location.

Type is a label for 3 areas, 1 - revegetated areas, 2 - short grass area, 3 - tall grass area.

Location is a label for 3 geographic locations, 1 - Oak Island, 2 - Smith Island, 3 - Snows Marsh.

There are 5 chemical measurements:

soil salinity - measured in part per thousand.

pH - acidity as measured in water

Kk - potassium in part per million

Na - sodium in part per million

Zn - zinc in part per million

There is one biological measurement, Biomass, which is the aerial biomass of the new grass in gm^{-2} .

Answer **ONE** of the following questions.

- (a) Is the salinity concentration found in the plots the same or different when compared at the three locations?
- (b) Is it possible to predict the concentration of zinc from that of the pH?

In your answer, include summary statistics from any output you use, give clear statements about the statistical tests you have used. I only expect one analysis to be performed answering the central question, together with any checks of the assumption. If the assumptions do not hold, make sensible recommendations about what you would do. Do not reanalyse the data.

Credit will be given for a clearly written and reasoned conclusion.