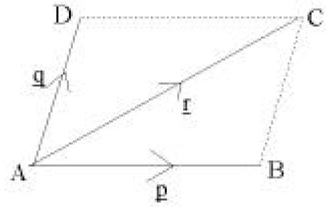
Vector Algebra

Scalars. Many **physical** quantities are unrelated to any direction in space and are defined entirely by numerical magnitude (in appropriate units) e.g. *temperature, mass, energy*. **Vectors**. However, other physical quantities have both a *direction* and a *magnitude* e.g. velocity, acceleration, force.

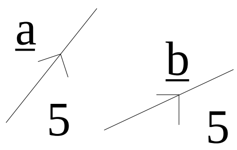
A vector obeys the same **laws** as displacements in Euclidean space and so may be represented geometrically as directed line segments, where the length of the line represents the magnitude and the line direction represents the **direction**. Parallelogram law of addition: AB+AD = AC; $\underline{p}+\underline{q} = \underline{r}$.



Notation

Vectors are normally **written** in text books in bold type, but can be denoted by **a**, $\vec{a}$, or $\underline{a}$ (used here). The magnitude of a vector is denoted by $|\underline{a}|$, or just a. **Unit** vectors. Vectors with unit magnitude have $|\underline{a}| = 1$. The unit vector in the same direction as $\underline{a}$ is denoted by $\underline{\hat{a}}$. Usually $\underline{a} = |\underline{a}|\underline{\hat{a}}$; $\underline{a} = a\underline{\hat{a}}$. Here are some **properties** of vectors:

(1) **Equality**. Two vectors are said to be *equal* if and only if their magnitude and direction are identical. For example, $a \neq b = 5$ in the diagram (because they have different directions).



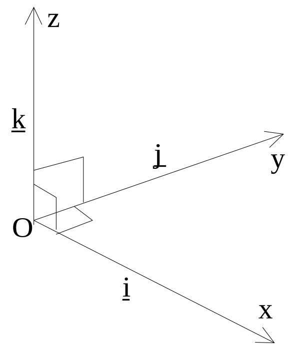
(2) **Addition**. The parallelogram law of addition has the following properties. (a) $\underline{a}+\underline{b} = \underline{b}+\underline{a}$ (Commutative). (b) $(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$ (Associative).

(3) **Subtraction**. $\underline{AB} + \underline{BA} = \underline{AA} = 0$. The 'null' or 'zero' vector has zero magnitude and no direction. Let us define $-\underline{a} = \underline{BA} = -\underline{AB}$ as the vector of **equal** magnitude but **opposite** direction to $\underline{a}$, so that $\underline{a}-\underline{a} = 0$.

(4) **Multiplication**. $S\underline{a}$ is a vector in the **same** direction as $\underline{a}$, but with magnitude **multiplied** by S. This operation obeys the following *distributive* law: $S(\underline{a}+\underline{b}) = S\underline{a}+S\underline{b}$.

Components of a Vector

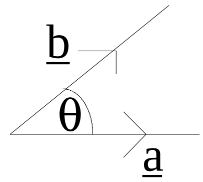
An **important** reference system is the right handed rectangular Cartesian **co-ordinate** frame Oxyz. It is usual to denote the unit vectors along Ox, Oy and Oz as $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively. **Consequently** the position vector $\underline{r}$ of a point P is given by $OP = \underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$.



Similarly, if $\underline{a}$ is any vector $\underline{a}$, then $\underline{a} = a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$, where a_1, a_2 and a_3 are the **components** of $\underline{a}$ in the $\underline{i}$, $\underline{j}$ and $\underline{k}$ directions. By **Pythagoras'** theorem, $a = |\underline{a}| = \sqrt{(a_1^2+a_2^2+a_3^2)}$.

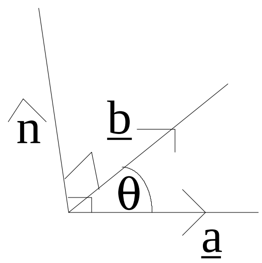
Products of Vectors

There are **2** "product" operations on vectors...



(1) **<u>Scalar Product</u>** (<u>inner</u> product or <u>dot</u> product). If $\underline{a}$ and $\underline{b}$ are two vectors inclined to each other at an angle θ, their *scalar* product is defined as $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta = ab\cos\theta$. **Note** that $a\cos\theta$ is the component of $\underline{a}$ in the direction of $\underline{b}$. Similarly, $b\cos\theta$ is the component of $\underline{b}$ in the direction of $\underline{a}$.

<u>Rules & Notes.</u> (i) $\underline{a}.\underline{b} = \underline{b}.\underline{a}$ (*Commutative*). (ii) $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b}+\underline{a}.\underline{c}$. (*Distributive*) (iii) $(S\underline{a}).(t\underline{b}) = St(\underline{a}.\underline{b})$. (iv) If $\theta = \pi/2$, $\cos(\theta) = 0$, so in **this** case $\underline{a}.\underline{b} = 0$. (v) If $\theta = 0$, $\cos(\theta) = 1$, so $\underline{a}.\underline{b} = ab$. In this **case**, if $\underline{b} = \underline{a}$ then $\underline{a}.\underline{a} = aa = a^2$. (vi) $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$. $\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$. (vii) In terms of *Cartesian* components, if $\underline{a} = a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$; $\underline{b} = b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$, then $\underline{a}.\underline{b} = a_1b_1+a_2b_2+a_3b_3$. <u>Proof</u>: in $(a_1\underline{i}+a_2\underline{j}+a_3\underline{k}).(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$, because of (vi), most **disappear**. (viii) The angle between **two** vectors $\underline{a}$ and $\underline{b}$ is given by $\theta = \cos^{-1}(\underline{a}.\underline{b}/ab)$. (ix) If $\underline{a}.\underline{b} = 0$, then $\underline{a} = 0$ and/or $\underline{b} = 0$ and/or $\theta = \pi/2$.



(2) **<u>Vector Product</u>** (Cross Product). If $\underline{a}$ and $\underline{b}$ are two vectors and θ is the angle between them measured from the **direction** of $\underline{a}$, then the vector product is defined as $\underline{a}\times\underline{b} = |\underline{a}||\underline{b}|\sin\theta\hat{n} = ab\sin\theta\hat{n}$, where $\hat{n}$ is the **unit** vector normal to the plane *containing* $\underline{a}$ and $\underline{b}$ such that $\underline{a}$, $\underline{b}$ and $\hat{n}$ form a right handed **triad** as shown on the right.

<u>Rules & Notes.</u> (i) $\underline{b}\times\underline{a} = -\underline{a}\times\underline{b}$ (<u>NOT</u> Commutative). (ii) $\underline{a}\times(\underline{b}+\underline{c}) = \underline{a}\times\underline{b}+\underline{a}\times\underline{c}$ (**Distributive**). (iii) $(S\underline{a})\times(t\underline{b}) = St(\underline{a}\times\underline{b})$. (iv) $\theta = 0$ implies $\sin\theta = 0$, then $\underline{a}\times\underline{b} = \underline{0}$. (v) $\underline{a}\times\underline{a} = \underline{0}$. (vi) $\underline{i}\times\underline{i} = \underline{j}\times\underline{j} = \underline{k}\times\underline{k} = \underline{0}$. (vii) $\underline{i}\times\underline{j} = |\underline{i}||\underline{j}|(\sin\pi/2)\hat{n} = 1\times1\times1\times\underline{k} = \underline{k}$. Further, $\underline{j}\times\underline{k} = \underline{i}$ and $\underline{k}\times\underline{i} = \underline{j}$. (viii) In **Cartesian** component form, $\underline{a}\times\underline{b} = \underline{i}(a_2b_3-b_2a_3) - \underline{j}(a_1b_3-b_1a_3) + \underline{k}(a_1b_2-b_1a_2)$, **or** the determinant shown in <u>yellow</u> on the right. (ix) If $\underline{a}\times\underline{c} = \underline{b}\times\underline{c}$ this *does not imply that* $\underline{a} = \underline{b}$.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vector Functions of One Real Variable

If for **every** value of a scalar variable t in a given interval *there exists a vector* $\underline{a}(t)$, then $\underline{a}$ is said to be a vector **function** of t. Then $\underline{a}$ is the dependent variable and t is the independent variable. In **rectangular** Cartesian components, we have $\underline{a}(t) = a_1(t)\underline{i} + a_2(t)\underline{j} + a_3(t)\underline{k}$, where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the **components** of $\underline{a}(t)$ in the *x, y and z* directions respectively.

Derivative of a Vector

The **derivative** of $\underline{a}(t)$ with respect to t is **defined** as $\frac{d}{dt}(\underline{a}(t)) = \lim_{\delta t\to0}\{\frac{a(t+\delta t)-a(t)}{\delta t}\} = \lim_{\delta t\to0}\{\frac{\delta y}{\delta x}\}$. In terms of **Cartesian** components, $\frac{d}{dt}(\underline{a}(t)) = (d\underline{a_1}/dt)\underline{i} + (d\underline{a_2}/dt)\underline{j} + (d\underline{a_3}/dt)\underline{k}$. <u>Notes</u>. (i) If c is a **constant** scalar, $\frac{d}{dt}(c\underline{a}(t)) = c(d\underline{a}/dt)(t)$. (ii) $\frac{d}{dt}(\underline{a}(t)+\underline{b}(t)) = (d\underline{a}/dt) + (d\underline{b}/dt)$. (iii) $\frac{d}{dt}(\underline{a}(t).\underline{b}(t)) = \underline{a}.(d\underline{b}/dt)+(d\underline{a}/dt).\underline{b}$. (iv) $\frac{d}{dt}(\underline{a}(t)\times\underline{b}(t)) = \underline{a}\times(d\underline{b}/dt)+(d\underline{a}/dt)\times\underline{b}$. (v) $\underline{a}.(d\underline{a}/dt) = a(da/dt)$.

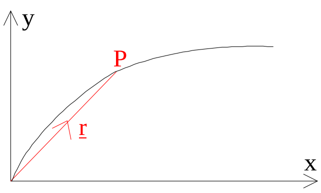
Integral of a Vector

If $(d\underline{a}/dt)(t) = \underline{b}(t)$, then $\int(d\underline{a}/dt)dt = \int\underline{b}(t)dt + \underline{c}$ ($\underline{c}$ = **constant** of integration, a vector). Therefore, $\underline{a}(t) = \int\underline{b}(t)dt+\underline{c}$. Using **definite** integrals, $\int_{t_0}^{t} (d\underline{a}/dt)(t)dt = \int_{t_0}^{t} \underline{b}(t)dt$. **So** $[\underline{a}(t)]_{t_0}^{t} = \int_{t_0}^{t} \underline{b}(t)dt$. And $\underline{a}(t)-\underline{a}(t_0) = \int_{t_0}^{t} \underline{b}(t)dt$.

Kinematics (Geometry of Motion)

Consider a **particle** (a theoretical point mass) moving in space. **Position**. The position may be represented by the vector $\underline{OP}$, where O is a fixed origin. The vector $\underline{OP}$ is usually denoted by $\underline{r}$ or $\underline{x}$. In *Cartesian components*, $\underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$. If the particle's position **changes** with respect to time, then we write $\underline{r} = \underline{r}(t)$. In **Cartesian** components, $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$. This is the **parametric** form of the curve traced out by the particle's path *with co-ordinates at time t* given by (x(t), y(t), z(t)). Eliminating t **from** the expression for x, y and z gives the equation of the curve as a relation *between* x, y and z.

Example. Suppose that the equation of a *curve* is given by $\underline{r} = at^2\underline{i} + 2at\underline{j}$. So $x(t) = at^2$ and $y(t) = 2at$. (a is a scalar constant). We **eliminate** t by rearranging y(t) as $t = {}^{y}/_{2a}$ and **substituting** into x(t) to give $x = a({}^{y}/_{2a})^2 = {}^{ay^2}/_{4a^2} = {}^{y^2}/_{4a}$. Therefore, $y^2 = 4ax$, which is the shape of a **parabola** (see the *diagram*).

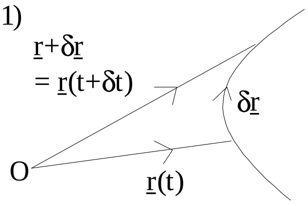


8th February 1999

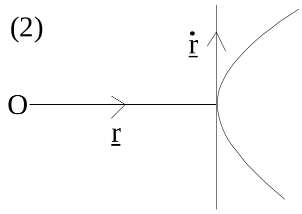
Kinematics

Displacement. Consider two fixed points P & Q. The *displacement* from P to Q is represented by the vector $\underline{s} = \underline{PQ}$. The displacement from Q to P is $\underline{QP} = -\underline{s}$. **Velocity**. The velocity is defined as the *rate of change* of position with respect to time, i.e. $d\underline{r}/dt$, denoted by $\underline{v}$, and also by $\dot{\underline{r}}$ (or 'r dot underlined'). **Hence** $d\underline{r}/dt = \lim_{\delta t\to 0}\{\frac{\delta r}{\delta t}\}$. **Looking** at diagram (2), we see that it is *tangential* to the curve. In Cartesian components, $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$. So $\dot{\underline{r}} = \dot{x}\underline{i}+\dot{y}\underline{j}+\dot{z}\underline{k}$, or $\underline{v} = {}^{dx}/_{dt}\underline{i}+{}^{dy}/_{dt}\underline{j}+{}^{dz}/_{dt}\underline{k}$.

(1)

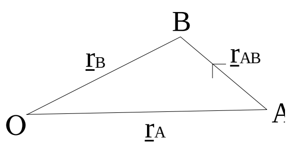


(2)



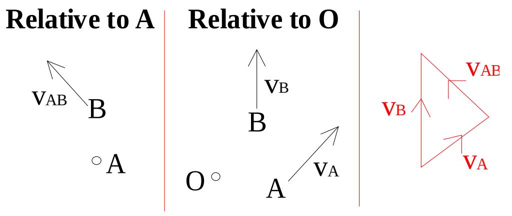
Speed. It is the magnitude of velocity (a scalar): $v = |\underline{v}|$, but $r \neq |\dot{\underline{r}}|$. Acceleration. The acceleration is the **rate** of change of velocity with respect to time i.e. $d\underline{v}/dt$ or $d^2\underline{r}/dt^2$, denoted by $\underline{a}$, $\dot{\underline{v}}$ (or 'v dot underlined') or $\ddot{\underline{r}}$ (or 'r double dot underlined').

Relative Position. If A and B have position vectors $\underline{r}_A$ and $\underline{r}_B$ respectively, (*relative* to an origin O), then B has **position** vector $\underline{AB}$ relative to A given by $\underline{r}_{AB} = \underline{r}_B-\underline{r}_A$. (Think of it as **starting** at A, **going** to O $(-\underline{r}_A)$, then to $B(\underline{r}_B)$, so that $\underline{r}_{AB} = -\underline{r}_A+\underline{r}_B$). Therefore, A *observes* B to have **position** vector $\underline{r}_{AB}$ relative to an origin positioned at A. **Conversely**, for an observer at B, A has *position vector* $\underline{BA}$ relative to B, so that $\underline{r}_{BA} = \underline{r}_A-\underline{r}_B$.

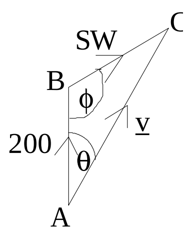


Resultant position. If A has *position vector* $\underline{r}_A$ relative to an origin O, and B has position vector $\underline{r}_{AB}$ *relative* to A, then B has position vector $\underline{r}_B$ relative to O given by $\underline{r}_B = \underline{r}_A+\underline{r}_{AB}$. *Similarly*, $\underline{r}_A = \underline{r}_B+\underline{r}_{BA}$. The above statements may be **repeated** for velocity vectors. **Relative velocity**. If A has velocity $\underline{v}_A$, and B has velocity $\underline{v}_B$, we say that $\underline{v}_{AB} = \underline{v}_B-\underline{v}_A$. Further, $\underline{v}_{BA} = \underline{v}_A-\underline{v}_B$ is A's velocity **relative** to B.

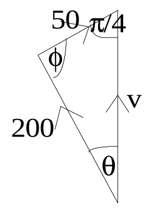
Resultant velocity. If B has velocity $\underline{v}_{AB}$ relative to A, which itself has a *velocity* $\underline{v}_A$, then B's **velocity** is given by $\underline{v}_B = \underline{v}_A+\underline{v}_{AB}$. *Similarly*, $\underline{v}_A = \underline{v}_B+\underline{v}_{BA}$. **Relative** and **Resultant** problems occur often — for example, the velocity of a *boat* / *swimmer* / *aeroplane* / *athlete* (relative) is affected by the **water** / **air** in which it is travelling. This accounts for differences in *journey times* because of e.g. prevailing winds and tides.



Example. An aeroplane wishes to fly **north** in a wind whose speed is 50kmh^{-1} from the SW. If the plane's engines create a *forward velocity of* 200kmh^{-1} in still air, (a) what happens to the plane if it steers a course **due north**?; (b) what direction should the pilot steer to **travel** north? A: Angle ABC = ϕ = 135° = $^{\pi}/_{2}+^{\pi}/_{4}$ = $^{3\pi}/_{4}$. $v^2 = 200^2+50^2-(2\times200\times50\times\cos(^{3\pi}/_{4}))$. (Using the **Cosine** rule $c^2 = a^2+b^2-2bc\times\cos(c)$). So v = 238kmh^{-1}. Now using the **Sine rule,** $^{50}/_{\sin\theta} = {}^{238}/_{\sin(3\pi/4)}$; …., $\sin\theta$ = 8.5°.



(b) Again using the **Sine** rule for the **new** diagram, $^{200}/_{\sin(\pi/4)} = {}^{50}/_{\sin\theta}$; so θ = 10.2°. Now ϕ = 180°-10.2°-45° = 124.8°. So using the Sine rule **again**, $^{v}/_{\sin(\phi)} = {}^{50}/_{\sin(\theta)}$; v = 232.3kmh^{-1}.



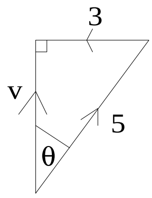
Exercises 1

Q: The position vector $\mathbf{r}(t)$ of a particle at time t is given by $\mathbf{r}(t) = (t^2+t)\mathbf{i} + (3t-2)\mathbf{j} + (2t^3-4t^2)\mathbf{k}$. At the instant t = 2, find the (a) velocity; (b) speed; (c) acceleration; (d) magnitude of acceleration. (a) Velocity = $(2t+1)\mathbf{i}+(3)\mathbf{j}+(6t^2-8t)\mathbf{k} = 5\mathbf{i}+3\mathbf{j}+8\mathbf{k}$. (b) Speed = $|\mathbf{v}| = \sqrt{(5^2+3^2+8^2)} = \sqrt{98}$. (c) Acceleration = $2\mathbf{i}+0\mathbf{j}+(12t-8)\mathbf{k} = 2\mathbf{i}+16\mathbf{k}$. (d) Magnitude of acceleration = $\sqrt{(2^2+16^2)} = \sqrt{260} = 2\times\sqrt{65}$.

Q: If $\mathbf{r}(t) = \mathbf{a}\cos(\omega t)+\mathbf{b}\sin(\omega t)$, where **a** and **b** are any constant non-collinear vectors, and ω is a **constant** scalar, prove that $\mathbf{r}\times{}^{d\mathbf{r}}/_{dt} = \omega(a\times b)$ and that $^{d^2\mathbf{r}}/_{dt^2} = -\omega^2\mathbf{r}$. A: $^{d\mathbf{r}}/_{dt} = -\mathbf{a}\omega\sin(\omega t)+\mathbf{b}\omega\cos(\omega t)$. So $\mathbf{r}\times{}^{d\mathbf{r}}/_{dt} = [\mathbf{a}\cos(\omega t)+\mathbf{b}\sin(\omega t)]\times[-\mathbf{a}\omega\sin(\omega t)+\mathbf{b}\omega\cos(\omega t)]$ = (*using* $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$) $= \mathbf{a}\cos(\omega t)\times-\mathbf{a}\omega\sin(\omega t) + \mathbf{b}\sin(\omega t)\times-\mathbf{a}\omega\sin(\omega t) + \mathbf{a}\cos(\omega t)\times\mathbf{b}\omega\cos(\omega t) + \mathbf{b}\sin(\omega t)\times\mathbf{b}\omega\cos(\omega t) = 0 + \mathbf{b}\sin(\omega t)\times-\mathbf{a}\omega\sin(\omega t) + \mathbf{a}\cos(\omega t)\times\mathbf{b}\omega\cos(\omega t) + 0 = \mathbf{b}\sin(\omega t)\times-\mathbf{a}\omega\sin(\omega t) + \mathbf{a}\cos(\omega t)\times\mathbf{b}\omega\cos(\omega t) = \mathbf{a}\omega\sin(\omega t)\times\mathbf{b}\sin(\omega t) + \mathbf{a}\cos(\omega t)\times\mathbf{b}\omega\cos(\omega t) = \sin(\omega t)\sin(\omega t)(\mathbf{a}\omega\times\mathbf{b}) + \cos(\omega t)\cos(\omega t)(\mathbf{a}\times\omega\mathbf{b}) = \sin^2(\omega t)(\mathbf{a}\omega\times\mathbf{b}) + \cos^2(\omega t)(\mathbf{a}\times\omega\mathbf{b}) = \sin^2(\omega t)(\mathbf{a}\times\mathbf{b})\omega + \cos^2(\omega t)(\mathbf{a}\times\mathbf{b})\omega = \omega(\mathbf{a}\times\mathbf{b})(\sin^2(\omega t)+\cos^2(\omega t)) = \omega(\mathbf{a}\times\mathbf{b})(1) = \omega(\mathbf{a}\times\mathbf{b})$. (ii) $^{d^2\mathbf{r}}/_{dt^2} = -\mathbf{a}\omega^2\cos(\omega t)-\mathbf{b}\omega^2\sin(\omega t) = -\omega^2(\mathbf{a}\cos(\omega t)+\mathbf{b}\sin(\omega t)) = -\omega^2 r$.

Q: A **swimmer** swims $5ms^{-1}$ vertically in *still* water, and needs to cross a 20m wide river which flows horizontally at $3ms^{-1}$. Find the time to **cross**; the distance the swimmer drifts down the **river**; the direction needed to set off at in order to swim across *directly*; and why is it not possible to swim directly if the speed of the river is > $5ms^{-1}$?

A: The river flow doesn't affect the **vertical** velocity (it is perpendicular to it) so we can say that it takes 4 seconds to cross ($v = {}^{S}/_{t}$, $t = {}^{S}/_{v} = {}^{20}/_{5} = 4s$). *In these 4 seconds*, the swimmer drifts 12m down the river ($S = tv = 4\times3 = 12m$). To cross **directly**, look at the picture. Applying the **Sine** rule, ${}^{Sin90°}/_{5} = {}^{sin\theta}/_{3}$; $\sin\theta = {}^{sin90°\times3}/_{5} = {}^{3}/_{5}$ so $\theta = 36.9°$. The swimmer cannot **cross** directly when the flow is > $5ms^{-1}$ because in our *expression*, ${}^{sin90°}/_{5} = {}^{sin\theta}/_{v}$, ($v > 5ms^{-1}$), we **have** ${}^{1}/_{5}v = \sin\theta$. If $v > 5ms^{-1}$, we **have** $\sin\theta > 1$, for which we *cannot* get a value for θ.



12th February 1999

One-Dimensional Rectilinear Motion

One dimensional motion is a simplification — all variables *are scalars*. All quantities may be expressed as negative or positive values along the direction of motion. Take the line of motion to be the **x-axis**, represented by the unit vector **i**. So $\mathbf{r}(t) = x(t)\mathbf{i}$, $\mathbf{v}(t) = {}^{d\mathbf{r}}/_{dt} = {}^{dx}/_{dt}\mathbf{i}$ or $\dot{x}\mathbf{i}$; and $\mathbf{a}(t) = {}^{d^2\mathbf{r}}/_{dt^2} = {}^{dv}/_{dt} = \ddot{x}\mathbf{i}$ or $\dot{v}\mathbf{i}$ or $a\mathbf{i}$.

It is usual for **i** to be omitted and for *direction to be represented by* a +ve or -ve sign. Displacement: x or s. Velocity: $v = {}^{dx}/_{dt} = \dot{x}$ or $\dot{s}$. Acceleration: $a = {}^{dv}/_{dt} = {}^{d^2x}/_{dt^2} = \ddot{x}$. If the velocity is *known as a function* of t, v(t), it may be integrated to give the **displacement** x. Since ${}^{dx}/_{dt} = v$, then $x = \int v.dt$ + constant. The **constant** of integration for the above indefinite integral is determined by knowing the values of x at some *time* t.

If we require the **distance** travelled in some time *interval*, say $t_1 \le t \le t_2$, we can use the **definite** integral $\int_{t_1}^{t_2} v.dt$. Similarly, if the *acceleration is known* as a function of t, the velocity may be obtained from the following: since ${}^{dv}/_{dt} = a(t)$, then $v = \int a.dt$ + constant.

Quite often, the *acceleration is known* as a function of x. In this case, we use the **identity** ${}^{dv}/_{dt} = v.{}^{dv}/_{dx}$ **(*)** to obtain the *velocity* from $v.{}^{dv}/_{dx} = a(x)$. **Integrating** with respect to x, we get $\int v.dv = \int a.dx$ + constant, with ${}^{1}/_{2}v^2 = \int a.dx$ + constant. Proof of **(*):** use the *chain* rule of differentiation: ${}^{dv}/_{dt} = {}^{dv}/_{dx}\times{}^{dx}/_{dt} = {}^{dv}/_{dx}.v$ (as required).

Example. If $v(t) = t^3-3t+1$, find an **expression** for x(t) if x = 2 when t = 1. A: $x(t) = \int t^3-3t+1 = (t^4/4) - (3t^2/2) + t$ + constant. **Substitute** in x = 2 when t = 1 to find out that the *constant* is ${}^{9}/_{4}$. So $x(t) = \frac{1}{4}t^4 - {}^{3}/_{2}t^2 + t + {}^{9}/_{4}$.

Graphical Representation

Time Displacement Graph. The *gradient* of the graph is the velocity, $v = {}^{dx}/_{dt}$, which is +ve or -ve. **Time Velocity Graph.** The *gradient* of the graph is the acceleration, $a = {}^{dv}/_{dt}$. The area **under** the graph is the displacement, $x = \int v.dt$.

15th February 1999

Constant / Uniform Acceleration

Constant acceleration almost never occurs in real life, but it gives easy *formulae* for v and x, usually denoted by s. Let $a = a_0$ be the *constant* acceleration. Then the definition $a = {}^{dv}/_{dt}$ gives $v = \int a_0 dt = a_0 t$ + constant. If the **initial** velocity is u, i.e. v = u at t = 0, then $v = a_0 t + u$. (Equation (i)). Therefore, in a **velocity-time** graph, the y-intercept is u and the *slope* is a_0.

Integrating again gives $s = \int v.dt$ + constant = $\int (a_0 t + u)dt$ + constant. So $s = \frac{1}{2}a_0 t^2 + ut$ + constant. If the *initial* displacement is zero, i.e. s = 0 at t = 0, then $s = \frac{1}{2}a_0 t^2 + ut$. (Equation (ii)). A 3rd equation may be obtained by either (a) *eliminating t from (i) & (ii)* or (b) *integrating* the acceleration with respect to x, using $v\,{}^{dv}/_{ds} = a_0$.

Using (b), $\int v dv = \int a_0 ds$ + constant; $\frac{1}{2}v^2 = a_0 s$ + constant. If the initial velocity is u when the *displacement* is zero, i.e. v = u when s = 0, then $\frac{1}{2}v^2 = a_0 s + \frac{1}{2}u^2$. So $v^2 = u^2 + 2a_0 s$. (Equation (iii)). The equations (i), (ii) and (iii) are known as the *constant acceleration formulae*. Often, **f** is used to denote acceleration.

Mechanics

Mechanics is the study of the **interaction** between *forces & matter*. There are 2 main sub areas: (i) Statics — no motion; forces in *equilibrium*. (ii) Dynamics — with *motion*; is equivalent to Mechanics + Kinematics. Here, we are concerned with **particle** dynamics.

Newton's Laws of Motion

Newton **Principia**, 1687: *empirical observational laws*. These are the axioms for (classical Newtonian) mechanics. (1) Every particle will remain in a **state** of *rest* or of *uniform/constant velocity* (a vector) unless acted upon by a **force**. (2) The *rate of change of momentum* (momentum = mass × velocity) is proportional to the **force** acting on the particle (vectorially). (3) To every **action,** there is an *equal and opposite* reaction.

Notes. From Newton's **2nd** law, ${}^{d(m\mathbf{v})}/_{dt} = \mathbf{F}$. If the mass m is constant, (as it *often* is), then $\mathbf{F} = m\,{}^{d\mathbf{v}}/_{dt} = m\,{}^{d^2\mathbf{r}}/_{dt^2} = m\mathbf{a}$. $\mathbf{F} = m\mathbf{a}$ is the equation most associated with the 2nd law. An *illustration* of Newton's 3rd law is the reaction of the floor against a person's object **weight**.

16th February 1999

Tutorial: Exercises 1

Q: Describe the **motion** of $\mathbf{r}(t) = a\sin(\omega t)\mathbf{i} + a\cos(\omega t)\mathbf{j}$. Find the *velocity* and *acceleration* of the particle, and show that their magnitudes are *constant*. A: The parametric form of the particle's path is $x(t) = a\sin\omega t$; $y(t) = a\cos\omega t$. This is a **circle** centred at (0,0) with radius a. It *starts* at (0,a) and moves in a **clockwise** direction. To eliminate t from *these 2 equations*, square both equations and add them: $x^2+y^2 = a^2\sin^2\omega t + a^2\cos^2\omega t = a^2(\sin^2\omega t+\cos^2\omega t)$; $x^2+y^2 = a^2$.

$\mathbf{v} = \dot{\mathbf{r}}$ = aωcos(ωt)**i** - aωsin(ωt)**j**. And $\mathbf{a} = \dot{\mathbf{v}}$ = $-\omega^2$(asin(ωt)**i**+acos(ωt)**j**). *Therefore,* **a** = $-\omega^2$**r**. (Note: a is not equal to |**a**|, where a is the *constant* in this question). v = |**v**| = $\sqrt{((a\omega\cos\omega t)^2+(a\omega\sin\omega t)^2)}$ = $a\omega\sqrt{(\cos^2\omega t+\sin^2\omega t)}$. **Therefore,** v = aω ('a' is a constant). |**a**| = |$-\omega^2$**r**| = ω^2|-**r**| = ω^2a (another constant).

Q: **Describe** the motion of **r**(t) = acos(ωt)**i**+asin(ωt)**j**+bt**k**. *Show* that $^{d^2\mathbf{r}}/_{dt^2}$ = -**ω**×$^{d\mathbf{r}}/_{dt}$, where **ω** = ωk is the *angular velocity* of the particle **about** the Oz axis. A: If the **k** component was zero, the motion *would be a circle of radius a; centred at the origin,* in the x-y plane. The particle would start on the x-axis and move in an anticlockwise direction. However, the **k** component increases *linearly* with time, so the path is a helix with axis along the z-axis (like a screw). If you were to look at it from above, you would just see a **circle**.

v(t) = -ωasin(ωt)**i**+ωacos(ωt)**j**+b**k**; **a**(t) = $-\omega^2$acos(ωt)**i**-ω^2asin(ωt)**j**. Taking **ω** = ωk, then -**ω**×$^{d\mathbf{r}}/_{dt}$ is worked out using the *determinant* method (in terms of components) to give the **desired** result.

19th February 1999

Analysis of Exercises

In the "*river*" exercise, in general, if the river's speed is U, then sinθ = $^u/_s$. Now |sinθ| ≤ 1 for all **real** θ, so |$^u/_s$| ≤ 1; |u| ≤ s. Exercises 2, Q1: Show this **Gallileo** assertion is correct: "...The distance *traversed* during equal intervals of time, by a constantly accelerating body started from rest, stand to one another in the **same** ratio as the odd numbers *beginning* with unity".

A: Let Δt be the *constant* interval of time. Let S_n be the distance **travelled** after n intervals, and $\Delta S_n = S_n - S_{n-1}$ be the distance **travelled** in the n^{th} interval. Let the constant *acceleration* be a_0, so the **formula** S = $\frac{1}{2}a_0t^2$+ut holds. The body starts from **rest**, so u = 0, and S = $\frac{1}{2}a_0t^2$. So $S_1 = \frac{1}{2}a_0(\Delta t)^2$; $S_2 = \frac{1}{2}a_0(2\Delta t)^2$, $S_3 = \frac{1}{2}a_0(3\Delta t)^2$, and so on until $S_n = \frac{1}{2}a_0(n\Delta t)^2$.

So $\Delta S_1 = S_1 - 0 = \frac{1}{2}a_0\Delta t^2$. $\Delta S_2 = S_2 - S_1 = \frac{1}{2}a_0(2\Delta t)^2 - \frac{1}{2}a_0\Delta t^2 = \frac{1}{2}a_0 3\Delta t^2$. $\Delta S_3 = S_3 - S_2 = \frac{1}{2}a_0(3\Delta t)^2 - \frac{1}{2}a_0(2\Delta t)^2 = \frac{1}{2}a_0 5\Delta t^2$, and so **on** until $\Delta S_n = S_n - S_{n-1} = \frac{1}{2}a_0(n\Delta t)^2 - \frac{1}{2}a_0((n-1)\Delta t)^2 = \frac{1}{2}a_0(n^2-(n^2-2n+1))\Delta t^2 = \frac{1}{2}a_0(2n-1)\Delta t^2$. **Now** $\Delta S_2/\Delta S_1 = {}^3/_1$; $\Delta S_3/\Delta S_2 = {}^5/_3$, ..., $\Delta S_n/\Delta S_{n-1} = {}^{2n-1}/_{2n-3}$. So **Gallileo's** assertion is correct. Remember in questions to consider the *general* and *particular* situations.

Q: A **point** moving with *constant acceleration* covers distances S_1 and S_2 in successive intervals T_1 and T_2. Prove that the *acceleration* is $2(S_2T_1-S_1T_2)/(T_1+T_2)T_1T_2$. A: Let the **initial** speed be u_1 and the constant *acceleration* be a_0. Then S = $\frac{1}{2}a_0t^2$+ut **holds**. So $S_1 = \frac{1}{2}a_0T_1^2+uT_1$ (1); $(S_1+S_2) = \frac{1}{2}a_0(T_1+T_2)^2+u(T_1+T_2)$ (2).

Eliminate u from these two equations by doing T_1(2)-(T_1+T_2)(1) which *gives* $T_1(S_1+S_2) - (T_1+T_2)S_1 = \frac{1}{2}a_0T_1(T_1+T_2)^2 - \frac{1}{2}a_0T_1^2(T_1+T_2)$. So $S_1T_1+S_2T_1-S_1T_1-S_1T_2 = \frac{1}{2}a_0(T_1+T_2)(T_1+T_2-T_1)T_1$; $S_2T_1-S_1T_2 = \frac{1}{2}a_0(T_1+T_2)(T_1T_2)$. Therefore, $a_0 = 2(S_2T_1-S_1T_2)/(T_1+T_2)(T_1T_2)$. **QED**.

Mechanics

Newton's 2nd law is $^d/_{dt}$(m**v**) = **F**, so $^d/_{dt}(m\dot{x}) = F_x$, $^d/_{dt}(m\dot{y}) = F_y$, and $^d/_{dt}(m\dot{z}) = F_z$, where **F** = F_x**i**+F_y**j**+F_z**k**. If m is a *constant, then* **F** = m$^{d\mathbf{v}}/_{dt}$ = m$^{d^2\mathbf{r}}/_{dt^2}$ = m**a**. The **basic** problem in all particle dynamics is: given the *total* force **F** = **F**(**r**, $\dot{\mathbf{r}}$, t), find the *path* of the particle **r** = **r**(t). In general, this is *impossible* analytically, but we shall study certain tractable problems.

23rd February 1999

Newton's **laws** of motion hold in an *inertial* frame of reference. For calculations or measurements on an everyday scale, the Earth's surface may be taken to be an **inertial** frame. Any frame which is moving with a constant velocity (vectorially) relative to an inertial frame is inertial. However, if it's moving with *relative* acceleration, a naive application of Newton's laws will give **incorrect** results, unless the accelerating frame is taken into account. For example, in a *lift or fairground ride, centripetal and centrifugal forces may apply.*

Example: How much does an object **weigh** in an accelerating lift? If it is stationary, its weight is *W* = *mg*, where m = object's mass; g = acceleration due to gravity = 9.81ms^{-2}. By Newton's **3rd** law, the reaction of the floor is *equal* & *opposite* to the weight: R = mg. The lift is accelerating with acceleration **a**. All calculation must be done relative to an *inertial* frame of reference outside the lift, at an origin O, say. Using Newton's **2nd** law of motion, (F = ma), we get mg-R = ma (We take the +ve direction to be downwards; mg-R is the total force, and ma is the *rate of change of momentum*). So m(g-a) = R. Note that the **occupants** of the lift are unaware of the acceleration a, and would calculate mg = R, which is *incorrect*. If a = 0, then mg = R. If a > 0, then the lift accelerates downwards, and R < mg — the object feels **lighter**. If a = g, then R = 0, and we have **no** reaction (or weightlessness). If a > 0, then we are accelerating upwards, and R > mg — the object feels **heavier**.

Impulse

The **impulse** of a force F is *defined* by **F** = $\int_{t_1}^{t_2}$ **F**dt. (The force can, in general, *vary* with time). By Newton's 2nd law, **F** = $^{d(m\mathbf{v})}/_{dt}$, so the above *integral* becomes I = $\int_{t_1}^{t_2} {}^{d(m\mathbf{v})}/_{dt}\, dt = [m\mathbf{v}]_{t_1}^{t_2}$, which is the *change of momentum* during the time interval from t_1 to t_2.

Units: Using SI units, *variables* have units as follows:

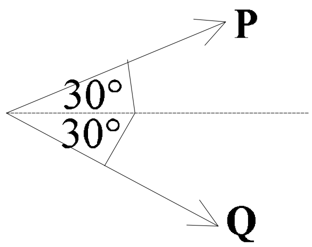
Notes	*Variable*	*Dimension*	*Unit*	*Symbol*
The	mass	M	kilograms	kg
Fundamental	length	L	metres	m
Units	time	T	seconds	s
	velocity / speed	L/T	metres per second	ms^{-1}
	acceleration	L/T^2	metres per second per second	ms^{-2}
N = $kgms^{-2}$	force	ML/T^2	Newtons	N
$kgms^{-1}$	momentum	ML/T	Newton second	Ns
	impulse	"	"	"
Nm	work		Joule	J
	energy		"	"
(Js)	power		Watts	W

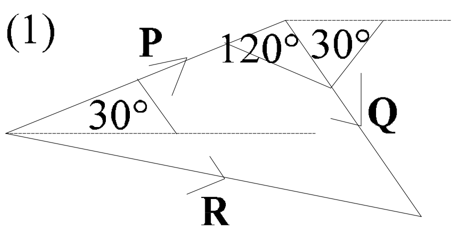
26th February 1999

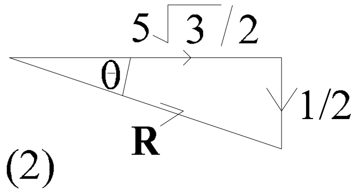
Force as a Vector

When specifying a **force** in 2 or 3 dimensions, we must specify its *magnitude*, *direction* and *point of application* on the body. (It is a **localised** or non-free vector). While we model particles only, as opposed to general *bodies* of finite extend, the point of application will always be the particle's position — a point.

Forces may be **resolved** into two or more components in convenient directions (usually along the i, j and k directions). Forces may be *combined* vectorially (by vector triangles or by summing components) to produce a **single** resultant force. Q: Find the Resultant force **R** for **P** & **Q**, where $|\mathbf{P}| = 2$N and $|\mathbf{Q}| = 3$N.







A: First by a **vector** triangle (Picture (1)): by the *cosine* rule, $\mathbf{R}^2 = \mathbf{P}^2+\mathbf{Q}^2-2\mathbf{PQ}\times\text{Cos}120°$; ..., $\mathbf{R}^2 = \sqrt{19}$. Then by the *sine* rule, ${}^{\mathbf{P}}/_{\sin\theta} = {}^{\mathbf{R}}/_{\sin 2\pi/3}$; ..., $\theta = 23.4°$. So the angle to the horizontal is $-(30°-23.4°) = -6.6°$. By using *components* in the **i** and **j** directions, $\mathbf{P} = 2\cos{}^{\pi}/_{6}\mathbf{i}+2\sin{}^{\pi}/_{6}\mathbf{j}$; $\mathbf{Q} = 3\cos{}^{\pi}/_{6}\mathbf{i}-3\sin{}^{\pi}/_{6}\mathbf{j}$. So $\mathbf{P} = \sqrt{(3)}\mathbf{i} + \mathbf{j}$ and $\mathbf{Q} = {}^{3\sqrt{3}}/_{2}\mathbf{i} - {}^{3}/_{2}\mathbf{j}$. Therefore, $\mathbf{R} = \mathbf{P}+\mathbf{Q} = (\sqrt{3}+\sqrt{3}\times{}^{3}/_{2})\mathbf{i} + (1-{}^{3}/_{2})\mathbf{j}$; $\mathbf{R} = {}^{5}/_{2}\sqrt{3}\mathbf{i} - ½\mathbf{j}$. So **R** = (Picture (2)) = $\sqrt{[({}^{5}/_{2}\sqrt{3})^2+(-{}^{1}/_{2})^2]}$ = ... = $\sqrt{19}$. And $\theta = \tan^{-1}({}^{y}/_{x}) = \tan^{-1}({}^{1/2}/_{5\sqrt{3}}) = 6.6°$. Note: **resolving** forces into perpendicular components has many *advantages* if more than two forces are present.

Assignment 1

Q: A **particle** has an acceleration *proportional* to the square of its speed. Assuming rectilinear motion, find expressions for the **velocity** and the **distance** moved in a time t. A: $a \propto v^2$, so $a = kv^2$ with $k > 0$. Using ${}^{dv}/_{dt} = kv^2$, then $\int {}^{dv}/_{v^2} = \int k.dt$; $-{}^{1}/_{v} = kt+c$; $v = -{}^{1}/_{kt+c}$. If $v = v_0$ at $t = 0$, then $v_0 = -{}^{1}/_{c_1}$; therefore $c_1 = -{}^{1}/_{v_0}$. **Now** $v = -{}^{1}/_{kt-(1/v_0)} = -{}^{v_0}/_{v_0kt-1} = {}^{v_0}/_{1-v_0kt}$. (**Note**: If $v_0 = 0$, then $v = 0$ for all t — hence *no* motion).

Integrating again from ${}^{dx}/_{dt} = v$ gives $\int dx = \int {}^{v_0}/_{1-v_0kt}.dt$; $x = -{}^{1}/_{v_0k}.v_0.\ln|1-v_0kt| + c_2$. If $x = x_0$ at $t = 0$, then $x_0 = -{}^{1}/_{k}\ln|1| + c_2$; *therefore* $c_2 = x_0$. Now $x = x_0-{}^{1}/_{k}\ln|1-v_0kt|$. **Notes**: If $v_0 = 0$, then $x = x_0$ and $v = 0$ for all t. If $v_0 > 0$, then x and v *tend* to ∞ as v_0kt tends to 1, i.e. as t **approaches** ${}^{1}/_{v_0k}$. And if $v_0 < 0$, then $v < 0$ for all t, and v *approaches* 0 as t approaches ∞ **while** x approaches -∞ as t approaches ∞.

In a **bearing** type question, (like the ship one), split things into *components*; establish whether you're dealing with constant velocities or not (which equations to use) and consider things like **pythagoras' theorem** and **alternative** methods.

1st March 1999

Equilibrium

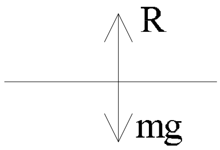
If the **resultant** of a system of forces is zero, then the forces are said to be in *equilibrium*. For example, if **P** = 4**i**, **Q** = 2**j**, and **S** = -(4**i**+2**j**), then **P**+**Q**+**S** = **0**. Note that the *resultant* of **P** and **Q**, **R** = **P**+**Q**, satisfies **R** = -**S**. This allows the *solution* of any problem in statics where the system of forces acts on a single point.

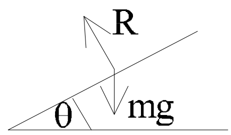
Type of Forces

Forces may be caused by **different** physical phenomena. A few *common* types are:

(1) **Weight**. The acceleration due to gravity at the Earth's surface is g, where g is roughly $9.81ms^{-2}$. This causes a force on *each* body with a magnitude mg, acting downwards, where m is the **mass** of the body.

(2) **Normal Reaction** of a *surface* on a body. The normal reaction essentially *prevents* the body from accelerating through the plane. If no other forces act in the normal direction, R = mgcosθ, which follows by **resolving** the forces normal to the plane.

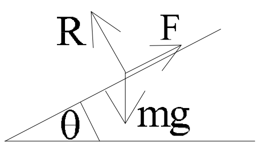




(3) **Surface Friction**. This force arises from the *tangential* interaction between two rough (not smooth) surfaces. If the force P is **introduced** and is acting horizontally, it is found that an opposing resistance force F arises. This is called friction. *Unless* P is large, it is usually found that the body does **not** move initially. The frictional force is such so as to cancel P, thus maintaining equilibrium. This is *static friction.*

If P is **increased** gradually, its value just before the body begins to slide is called the '**limiting** friction'. Experimentally, this is shown to be *related* to the normal reaction, $\mu_S R$, where μ_S is the **constant** for the two surfaces called the *coefficient* of static friction. So the frictional force will be created by the two surfaces, and will increase up to a **limiting** value, i.e. $F \leq \mu R$.

If the force P is **increased** further, the body will move horizontally. There is still an *opposing* frictional force, now called dynamic friction, given by $F = \mu_D R$, where μ_D is the coefficient of *dynamic* friction. Usually, $\mu_D < \mu_S$ by a *small* amount. Consider a rough inclined plane, where friction gets to oppose the body's motion **down** the plane. If the body is stationary under the acting forces, then resolving *perpendicularly* and *along* the plane gives: Normal direction: R = mgcosθ (i). *Tangential* direction: F = mgsinθ (ii) But the **frictional** force obeys $F \leq \mu_S R$, which by (i) gives $F \leq \mu_S mg\cos\theta$; and using (ii) **gives** $mg\sin\theta \leq \mu_S mg\cos\theta$; $\tan\theta \leq \mu_S$; $\theta \leq \tan^{-1}(\mu_S)$.



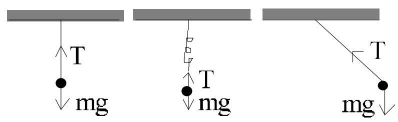
The block will **begin** to slide down the slope if θ is increased, so that it is bigger than $\tan^{-1}(\mu_S)$. For *smaller* angles, friction will increase **accordingly** to cancel out the component of the body's weight down the slope.

2nd March 1999

(4) **Drag**. This is friction due to air/water/fluid resistance. This arises from complicated interactions between the motion of a solid body and the surrounding fluid. For particle dynamics, it is usually **modelled** by a frictional force that opposes the motion, and has magnitude that is either proportional to the square of the speed, or proportional to the speed itself.

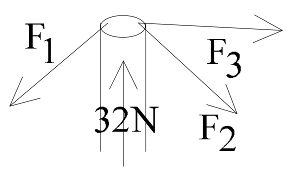
So $F = kv^2$, which **really** means $F = -kv^2$ for $v > 0$ and $F = kv^2$ for $v < 0$, (we need force in the *opposite* direction to velocity; it could be written as $F = -k|v|v$), or $F = -kv$, where k is a **constant**.

(5) **Tension in a string; Tension/Thrust in a rod or spring** When a body is suspended by a *string, rod, or spring*, its weight is supported by an upward force called the tension. A spring or rod can **exert** a thrust.

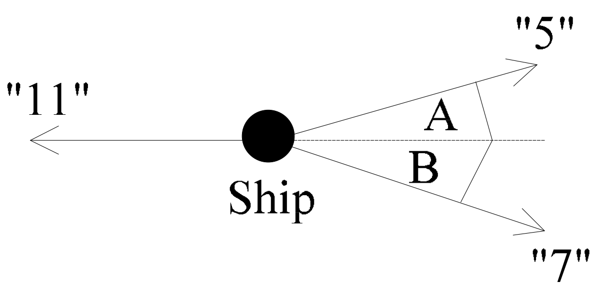


Exercises 3

Q: Three **forces** F_1, F_2 and F_3 act at the top of a radio mast as *illustrated* in the diagram. The mast is in **equilibrium**. If $F_1 = 5\mathbf{i}-2\mathbf{j}-12\mathbf{k}$, and $F_2 = -5\mathbf{i}-2\mathbf{j}-12\mathbf{k}$, and the magnitude of the *thrust* in the mast is 32N, find F_3 in terms of the *component vectors*, where **k** is the vertically upwards unit vector. A: *Resolving* in the "**k**" direction, $32\mathbf{k}-12\mathbf{k}-12\mathbf{k}+F_{3k} = 0$; $F_{3k} = -8\mathbf{k}$. *Resolving* in the "**i**" direction, $5\mathbf{i}-5\mathbf{i}+0+F_{3i} = 0$; $F_{3i} = 0$. *And* in the "**j**" direction, $-2\mathbf{j}-2\mathbf{j}+0+F_{3j} = 0$; $F_{3j} = 4\mathbf{j}$. So $F_3 = 0\mathbf{i}+4\mathbf{j}-8\mathbf{k} = 4\mathbf{j}-8\mathbf{k}$.



Q: A **ship** is being pulled by *two other ships* with tensions 5×10^6N and 7×10^6N in the ropes. If the sea has *resistance* 11×10^6N, what are the angles A and B if the ship travels at constant speed? A: Because it is travelling at a **constant** speed, relative force in a direction is 0. Note on *calculation*: The "10^6's" cancel through so I have *omitted* them.

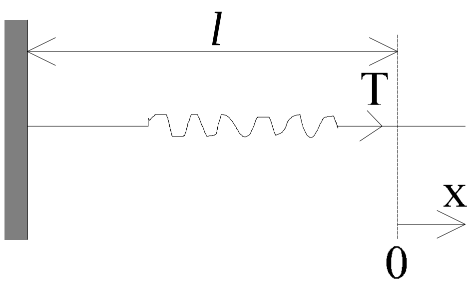


Resolving **vertically**, $5\sin A = 7\sin B$ (---(1)) and **horizontally**, $11 = 5\cos A+7\cos B$ (---(2)). *Squaring* (1), $25\sin^2 A = 49\sin^2 B$ (---(3)). *Squaring* (2) in the form $11-5\cos A = 7\cos B$, $(2)^2$ is $121-110\cos A+25\cos^2 A = 49\cos^2 B$ (---(4)). *Adding* (3) and (4), $25\sin^2 A+121-110\cos A+25\cos^2 A = 49\cos^2 B+49\sin^2 B$; $25+121-110\cos A = 49$; $110\cos A = 97$; $A = \cos^{-1}(^{97}/_{110}) = 28.1°$. Squaring (2) *differently* gives $121-154\cos B+49\cos^2 B = 25\cos^2 A$ (---(5)), and **adding** this onto (3) gives $121-153\cos B+49 = 25$; $154\cos B = 145$; $B = \cos^{-1}(^{145}/_{154}) = 19.7°$.

8th March 1999

For **inextensible** strings and springs, (the length *cannot* be changed), the tension or thrust is set up within the material so as to prevent stretching/extension or compression/contraction. For extensible springs or elastic strings however, the tension or thrust set up within the material **depends** on...

(1) The **stiffness** of the material; (2) The **magnitude** of the extension/compression. An empirical (experimental) rule that describes the above observations is *Hooke's law* (1667?) which states that the <u>tension</u> in a spring is proportional to its **extension** or **contraction**, and acts so as to *oppose* the change in the spring's length.



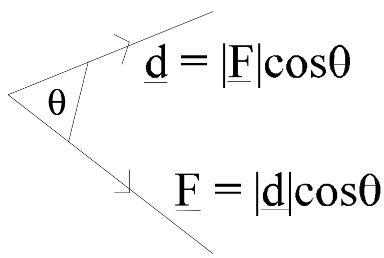
The <u>equilibrium</u> length (natural length) of the spring is l. The displacement from equilibrium is x. **Hooke's** law: T = -kx. Here, k is the *constant of proportionality*, called the stiffness**.** Alternatively, $T = -{}^{\lambda}/_{l}x$, where λ is a constant called the *modulus of elasticity*, related to the **stiffness** by $k = {}^{\lambda}/_{l}$. Note that x > 0 **implies** that T < 0, and x < 0 **implies** that T > 0.

<u>Work, Power and Energy</u>

Energy is the *capability of a body to do work*. Energy can take <u>many</u> forms, e.g. heat, light, sound, electrical, magnetic, nuclear, etc. In this course, we are concerned with **mechanical** energy. This is of two types: (i) <u>Kinetic</u> Energy — the energy of a body by *virtue of its motion*. (ii) <u>Potential</u> energy — the energy of a body by *virtue of its position.*

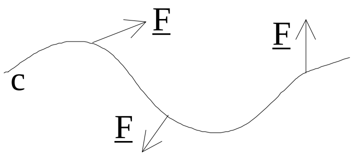
<u>Work</u>

This is essentially **force** multiplied by the distance moved, with the unit of joules. (1J = 1Nm). If a force **F** acts on a particle, then the <u>work</u> done by the force on the particle is given by (i) the **component** of forces in the direction of motion multiplied by the **distance** moved, or alternatively (ii) the force multiplied by the distance **moved** in the direction of the force.

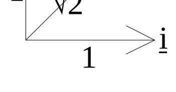


If the force **F** is constant (i.e. the magnitude and the direction is constant) throughout the motion, and the particle *moves in a straight line* represented by the vector **d** which is at an angle θ to **F**, then **W** = |**F**|cosθ|**d**|, which is the *scalar* product **W** = **F**.**d**.

If the force does not remain constant *and/or* the path of the motion is curved, then <u>integration</u> is used to sum the total work done. If δW is the work done by the **force F** during a small *displacement* δ**r**, then δW = **F**.δ**r**, and W, the work done, is *approximately* Σ**F**.δ**r** or Σ**F**(**r**).δ**r**. Taking the *limit* as δ**r** tends to 0 gives $\int_c \boldsymbol{F}.\boldsymbol{dr}$ which is a *line* integral, giving the work done by F moving <u>along</u> the curve c.



<u>Example</u>: A force is given by **F** = 10**i**. What is the work done in *moving a particle 2m* in the <u>following</u> directions: (i) **i**, (ii) **j**, (iii) **k**, (iv) **i**+**j**, (v) **i**+**j**+**k** (vi) -**i**. We know that W = **F**.**d**. Now (i) W = 10**i**.2**i** = 20(**i**.**i**) = 20J. (ii) W = 10**i**.2**j** = 10(**i**.**j**) = 0J. *Similarly* (iii) = 0J. (iv) <u>Here,</u> $d = {}^{2(\mathbf{i}+\mathbf{j})}/_{\sqrt{2}}$; d = √(2)(**i**+**j**). So W = 10**i**.√(2)(**i**.**j**) = 10√(2)(**i**.(**i**+**j**)) = 10√(2)(1+0) = 10√(2)J. (v) $d = {}^{2(\mathbf{i}+\mathbf{j}+\mathbf{k})}/_{\sqrt{2}}$; so $W = {}^{20}/_{\sqrt{2}}$(**i**.**i**+**i**.**j**+**i**.**k**) = ${}^{20}{}_{\sqrt{2}}$J. (vi) W = 10**i**.-2**i** = -20J.



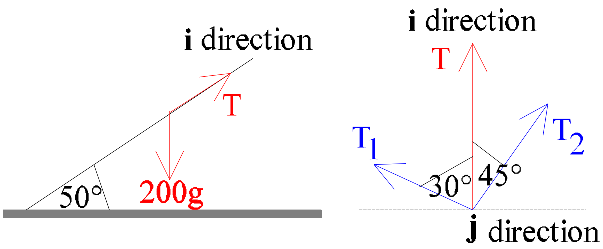
Q: A **force** of 5N acts in the direction 0.6**i**+0.8**j**. What is the *work done in moving* a particle a distance of 5m in (i) the **i** direction, (ii) the **j** direction, (iii) the **-i** direction. A: The *unit* vector is $\sqrt{[(^3/_5)^2+(^4/_5)^2]}$ = 1. So the *force* vector is 5(0.6**i**+0.8**j**) = 3**i**+4**j**. Now (i) W = (3**i**+4**j**).5**i** = 15**i**.**i** + 20**j**.**i** = 15J. (ii) W = (3**i**+4**j**).5**j** = 15**i**.**j** + 20**j**.**j** = 20J. (iii) W = (3**i**+4**j**).5-**i** = -15J. Taking the motion in (i) & (iii) *sequentially*, this results in **no** work done.

Power

Power is *defined* as the rate at which work is being done. In **general**, P = $^{dW}/_{dt}$. The SI unit for Power is the **Watt**, given the symbol W. (1W = $1Js^{-1}$),. A *light bulb* is 60W, 100W, etc. An electric fire or kettle is 3kW. A **power** station's output in measured in MW or GW.

Assignment 1

Q: A climber is *lowered* down a 50° slope. At the instant the climber is at **rest**, two ropes which hold him make angles of 30° & 45° to the line of greatest slope. The climber and his stretcher has a mass of 200kg. Find the **tension** in each rope if no resistances act. A: Resolving in the direction of the greatest slope, T = 200gsin50°. In the **2nd** figure, T = T_1cos30°+T_2cos45°. So 200gsin50° = T_1cos30°+T_2cos45°. Resolving **perpendicular** to the line of greatest slope, T_1sin30° = T_2sin45°. Now because sin45° = cos45°, we can write T_1sin30° = T_2cos45°. **Substituting** for T_2cos45° from the above, 200gsin50° = T_1cos30°+T_1sin30°; ..., T_1 = 1099N. Substituting *back*, we get T_2 = 777N.



12th March 1999

W = $\int_c$**F**.d**r**. (W = **F**.**d**). Power P = $^{dW}/_{dt}$. The power may be expressed as the *scalar* product of the force and the velocity. To prove this, if δW is the work done by the force F during a small *displacement* δ**r** which occurs during a small *time* interval of δt, then δW = **F**.δ**r**, and *dividing* by δt, $^{\delta W}/_{\delta t}$ = **F**.$^{\delta \mathbf{r}}/_{\delta t}$. Taking the limit as δt tends to 0, $^{dW}/_{dt}$ = **F**.$^{d\mathbf{r}}/_{dt}$. So P = **F**.**v**.

Work in 1-D Motion

In general, W = $\int_c$F.dr. The **simplest** case is if the force is *constant*, and the path is a straight **line**, so then P = **F**.**d**, where **d** is the displacement. A slightly more *complicated* case is if the force varies with position, but acts in the same line as the displacement which is constrained to be in a **straight** line, i.e. 1 dimensional motion. If c is the *direction of motion*, the line integral becomes an ordinary integral, W = $\int$F dx, where in **general**, F = F(x, $\dot{x}$, t). If F depends just on position i.e. F = F(x), then the *work integral* may be integrated to give the work function W(x) = $\int$f(x)dx.

Examples: (i) Calculate the *work function* for the force required to raise a particle of mass m against gravity. (ii) The force required to **extend** a spring is F = kx (k = spring stiffness). Calculate the work *function* for this force. A: (i) W(x) = $\int$mg.dx = mg$\int$dx = mgx + constant (which is **zero** because at zero displacement, *no* work is done. (ii) W(x) = $\int$kx.dx = $^{kx^2}/_2$ + constant.

15th March 1999

Energy

There are **two** forms of mechanical energy: *Kinetic* Energy & *Potential* Energy.

(1) KINETIC ENERGY. The kinetic energy (K.E. or T) of a particle is the energy *derived* from its motion. Energy is the capacity to do work. In order to relate the work done to the motion of the particle, we use **Newton's** 2nd law of motion. Firstly, in 1-D motion, $W = \int_{x_0}^{x_1} F\,dx$ is the work done in *moving* from $x = x_0$ to $x = x_1$.

From Newton's **second** law of motion, $F = m.\frac{d^2x}{dt^2}$, where *m* is the mass of the particle which we will use in the form $F = mv\frac{dv}{dx}$, and substitute into the work integral to give $[W]_{x_0}^{x_1} = \int_{x_0}^{x_1} mv\frac{dv}{dx}dx = \int_{x_0}^{x_1} v.dv = [\frac{1}{2}mv^2]_{x_0}^{x_1}$. The expression $\frac{1}{2}mv^2$ is called *the Kinetic Energy* of the particle. The above expression says that the **change** in the K.E. of the particle between $x = x_0$ and $x = x_1$ is equal to the *work done by F* in this displacement.

More **generally**, for 2 and 3 dimensions with *general particle path C*, we use vector calculus: $[W]_c = \int_c \mathbf{F}.d\mathbf{r}$. Now $\mathbf{F} = m\ddot{\mathbf{r}}$ gives $[W]_c = \int_c m\ddot{\mathbf{r}}.d\mathbf{r} = \int_c m\ddot{\mathbf{r}}.\frac{d\mathbf{r}}{dt}dt = \int_c m\frac{1}{2}\frac{d}{dt}(\frac{d\mathbf{r}}{dt}.\frac{d\mathbf{r}}{dt})dt = \int_c m\frac{1}{2}\frac{d}{dx}(\mathbf{v}.\mathbf{v})dt = [\frac{1}{2}m|\mathbf{v}|^2]_c = [W]_c$. So the *work done* by **F** is equal to the change in KE ($= \frac{1}{2}m|\mathbf{v}|^2$) as C is *traversed*. [Note: some write $|\mathbf{v}|^2 = \mathbf{v}.\mathbf{v}$ as just $\mathbf{v}^2$.] *Aside*: $\frac{d}{dt}(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \dot{\mathbf{r}}.\frac{d\dot{\mathbf{r}}}{dt} + \frac{d\dot{\mathbf{r}}}{dt}.\dot{\mathbf{r}} = 2\dot{\mathbf{r}}.\frac{d\dot{\mathbf{r}}}{dt} = 2\ddot{\mathbf{r}}.\dot{\mathbf{r}}$.

(2) POTENTIAL ENERGY. Denoted by P.E. or V. If the force F acts on a *particle* and the work function $W(r) = \int \mathbf{F}.d\mathbf{r}$ exists and is a *single-valued* function of position, the force is termed "conservative". For such a force, the P.E. is defined as the **negative** of the work function, $P.E. = V = -W = -\int \mathbf{F}.d\mathbf{r}$ For *one-dimensional* forces, all forces which depend *only* on position are conservative, i.e. $F = F(x)$. This, of course, includes **constant** forces such as gravity.

Examples. (a) P.E. due to gravity, $\mathbf{F} = -mg\mathbf{k}$. In *one-dimension*, $F(z) = -mg$. So $V(z) = -\int_{z_0}^{z} -mg.dz$. **Therefore**, $V(z) = mg(z-z_0)$. If $z_0 = 0$, then $V(z) = mgz$. The P.E. of the *particle* is the work done **against** gravity required to position it at some specified height. (b) P.E. due to a *spring* of stiffness k, $\mathbf{F} = -kx\mathbf{i}$. In *one-dimension*, $F(x) = -kx$. $V(x) = -\int_{x_0}^{x} F\,dx = -\int_{x_0}^{x} -kx\,dx = [\frac{1}{2}kx^2]_{x_0}^{x}$. **Taking** $x_0 = 0$ gives $V(x) = \frac{1}{2}kx^2$.

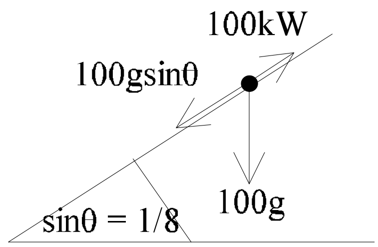
Conservation of Energy. Putting the *two sections* on K.E. and P.E. together gives the principle of the conservation of energy. In one-dimension, Newton's second law is $m\ddot{x} = F(x)$, or $mv\frac{dv}{dx} - F(x) = 0$. Now *integrate* w.r.t. x giving $\int mv\frac{dv}{dx}dx - \int F(x)\,dx = E$. This *gives* $\frac{1}{2}mv^2 + V(x) = E$, or K.E.+P.E. = E, where E is a *constant of integration*, which we shall call the total energy.

16th March 1999

Tutorial

Q: A car of **mass** 100kg climbs a hill of gradient θ, where $\sin\theta = \frac{1}{8}$. The maximum speed is $40ms^{-1}$. If the power of the car is 100kW, what is the **frictional** force on the car? If the frictional force is then halved, what's the new *top* speed? A: (see over)...

A: Resolving **parallel** to the slope, 100gsinθ + Friction = Force of Car. So 100gsinθ + Friction = $^{P}/_{V}$; Friction = $^{100000}/_{40}$ - 100×9.8×$^{1}/_{8}$ = 2377.5N. If the frictional force is *halved* to 1188.75N, then 1188.75 = $^{100000}/_{v}$-100×9.8×$^{1}/_{8}$; ..., v = 76.3ms^{-1}.



Q: A **lorry** of weight W N can generate a power P and has a max. speed of u ms^{-1} on *level* ground, but v ms^{-1} on an *upslope* α. If the power and *resistance* remain unchanged, prove that uvWsinα = P(u-v). A: On **level** ground, F = R, so $^{P}/_{u}$ = R. On the *upslope*, Wsinα + R = F. But we *know that* R = $^{P}/_{u}$, so Wsinα + $^{P}/_{u}$ = F. Here, F = $^{P}/_{v}$, so Wsinα + $^{P}/_{u}$ = $^{P}/_{v}$. Multiplying *through* by uv gives Wuvsinα + Pv = Pu; Wuvsinα = P(u-v). QED.

Q: A car of mass M kg works at a **constant** rate of Mk J. If there is constant resistance, and the maximum speed attainable is u ms^{-1}, show that the speed v of the car at time t satisfies the *equation* $^{1}/_{v}$-$^{1}/_{u}$ = $^{1}/_{k}$$^{dv}/_{dt}$. A: P = Mk, which is *constant.* And P = F_{car}v. (Force imparted by car's engine).

Now after reaching a *steady* speed, v = u, F_{car} = Friction; and P = F_{car}u. So P = Friction, Friction = $^{P}/_{u}$. Now we know that the *acceleration* is given at a time t by F_{car} - Friction. So F_{car} - Friction = M$^{dv}/_{dt}$. But Friction = $^{P}/_{u}$, and F_{car} at a time t is $^{P}/_{v}$. So $^{P}/_{v}$-$^{P}/_{u}$ = M$^{dv}/_{dt}$. Finally, we know that P = Mk, so it follows that $^{Mk}/_{v}$ - $^{Mk}/_{u}$ = M$^{dv}/_{dt}$; $^{k}/_{v}$-$^{k}/_{u}$ = $^{dv}/_{dt}$, $^{1}/_{v}$-$^{1}/_{u}$ = $^{1}/_{k}$$^{dv}/_{dt}$. QED.

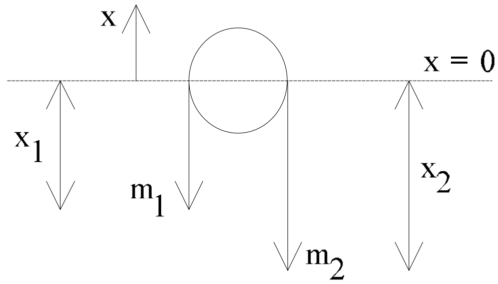
19th March 1999

Conservation of Energy

K.E. + P.E. = E. For a **conservative** force, the total amount of mechanical energy is conserved. During the *particle's motion*, K.E. may be converted into P.E. and vice-versa, but the sum remains constant. The conservation of energy is obtained as a first integration of Newton's 2nd law of motion. Using the **conservation** of energy equation enables many problems to be solved far *quicker* than when starting with the equations of motion.

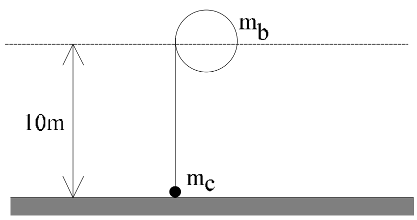
Examples. (1) A boy throws a ball vertically upwards with a speed of 15ms^{-1}. How high does it go? Let the P.E. *Datum* be at ground level. The conservation of energy gives ½mv^2+mgx = E. To obtain E, substitute from the initial conditions — v = 15, x = 0, to give ½m$(15)^2$+0 = E; E = $^{225}/_{2}$m. So the conservation of energy for the **particle** is ½mv^2+mgx = $^{225}/_{2}$m; v^2+2gx = 225. At maximum height, v = 0, so 0+2gx_{max} = 225, x_{max} = $^{225}/_{2g}$ = approximately 11.47m.

Example with **connected** particles. (Particles connected by a light *inextensible* string, e.g. a pulley). NOTE: The speed of the particles are the same. The conservation of energy for the system is ½m_1v^2 + ½m_2v^2 + mgx_1+mgx_2 = E. x is measured up from the *pulley* height (so x_1 & x_2 are always -ve).



Q: A bucket is connected by a *light rope over* a pulley to a counterweight that is half the weight of the bucket. Initially, the bucket is held at the pulley, with the counterweight 10m below, just touching the ground. What is the **speed** of the bucket when (a) the bucket passes the counterweight; (b) the bucket **hits** the floor. A: (see over)....

We know that $m_b = 2m_c$. The speed of the **counterweight** and the pulley is v. The conservation of energy gives $\frac{1}{2}m_bv^2 + \frac{1}{2}m_cv^2 + m_bgx_b + m_cgx_c = E$. Put $m_c = m$, so that $m_b = 2m$. Now $\frac{1}{2}(2m)v^2 + \frac{1}{2}mv^2 + 2mgx_b+mgx_c = E$; ${}^3/_2mv^2 + mg(2x_b+x_c) = E$; ${}^3/_2v^2+g(2x_b+x_c) = {}^E/_m$. Initially, v = 0, $x_b = 0$, and $x_c = -10$, so $g(-10) = {}^E/_m$; ${}^E/_m = -10g$. So the **conservation** of energy equation becomes ${}^3/_2v^2+g(2x_b+x_c) = -10g$. The bucket is *heavier* than the counterweight, so the bucket goes <u>down</u>.

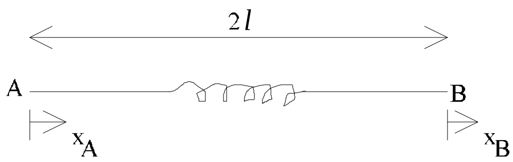


(a) They pass when they are at the **same** height: $x_b = -5$, $x_c = -5$. So ${}^3/_2v^2+g(2(-5)-5) = 10g$; ${}^3/_2v^2-15g = -10g$; ${}^3/_2v^2 = 5g$. $v^2 = {}^{10}/_3g$; $v = \sqrt{({}^{10g}/_3)}$ = *approximately* $5.71ms^{-1}$. (b) Here, $x_b = -10$, and $x_c = 0$. So ${}^3/_2v^2+g(2(-10)+0) = -10g$; ${}^3/_2v^2-20g = -10g$. ${}^3/_2v^2-10g = -10g$; ${}^3/_2v^2 = 10g$; $v = \sqrt{({}^{20g}/_3)}$ = *approximately* $8.09ms^{-1}$.

22nd March 1999

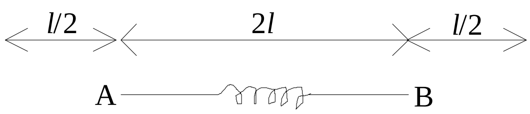
<u>Particles Connected by Light Springs</u>

$V(x) = \frac{1}{2}kx^2$. The P.E. in a **spring** is given by $\frac{1}{2}k(\text{extension})^2$. Question: 2 identical masses A & B (of mass m) lie on a *horizontal table*, and are connected by a light spring of natural length *l*; <u>stiffness</u> $k = {}^{mg}/_l$. The masses are held a **distance** of 2*l* apart. (a) The mass A is released. What is its *speed* when it is at a distance *l* from B? (b) If both masses are <u>released,</u> what would be the speed of A when it is at a **distance** *l* from B? We use the conservation of **energy** equation K.E.+P.E. = E, and assume distances of x_A and x_B from the **initial** position.



Extension of spring = $2l-l+x_B-x_A = l+x_B-x_A$. So the **P.E.** in the spring is $\frac{1}{2}k(\text{extension})^2 = \frac{1}{2}{}^{mg}/_l(l+x_B-x_A)^2$. Also, the **total** K.E. of the system is $\frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2$. So the *conservation* of energy equation is $\frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}{}^{mg}/_l(l+x_B-x_A)^2 = E$. **Initially,** $v_A = v_B = 0$ and $x_A = x_B = 0$, so $0+0+\frac{1}{2}{}^{mg}/_l(l+0-0)^2 = E$; $E = {}^{mgl}/_2$. So the *conservation of energy equation* is $\frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}{}^{mg}/_l(l+x_A-x_B)^2 = \frac{1}{2}mgl$.

(a) A is **released**, B is kept **stationary**. So $x_B = 0$, $v_B = 0$, $x_A = l$, v_A = ? Now $\frac{1}{2}mv_A^2 + 0 + \frac{1}{2}{}^{mg}/_l(l+0-l) = \frac{1}{2}mgl$; $\frac{1}{2}mv_A^2 = \frac{1}{2}mgl$; $v_A^2 = gl$, $v_A = \sqrt{(gl)}$. (b) Both A and B are **released** and we consider the situation shown. By *symmetry*, $v_B = -v_A$ and $x_B = -x_A$. When the particles are at a <u>distance</u> *l* apart, $x_A = {}^l/_2$ and $x_B = -{}^l/_2$. The conservation of energy gives $\frac{1}{2}mv_A^2 + \frac{1}{2}mv_A^2 + \frac{1}{2}{}^{mg}/_l(l-{}^l/_2-{}^l/_2) = \frac{1}{2}mgl$. (Note: $\frac{1}{2}mv_A^2$ is non negative because *squaring* makes it +ve). So $mv_A^2 = \frac{1}{2}mgl$; $v_A = \sqrt{({}^{gl}/_2)}$.

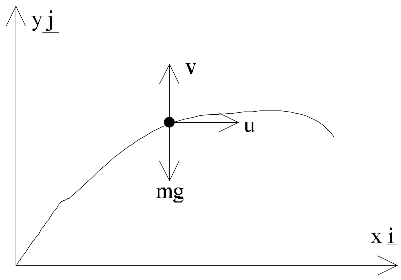


<u>Motion of Projectiles</u>

Here, we **study** the free motion under gravity of a particle projected in some direction which is *not* vertical. <u>Assumptions</u> for modelling projectile motion: (1) The projectile body will be modelled as a **particle;** (i.e. a point mass); *rigid* body rotation is neglected. (2) There is no <u>drag</u> — we neglect air resistances, etc.; no hydrodynamic effects such as **lift**, **swerve** etc., and no *wind* and/or *current* effects (We essentially consider motion in a <u>vacuum</u>). (3) **Acceleration** due to gravity is a constant, g. Note that (2) implies *2 dimensional* motion.

Independence of Horizontal & Vertical Motion

We set up *x and y axes* in the plane of motion as follows: **Newton's** 2nd law of motion is $m\ddot{\mathbf{r}} = \mathbf{F}$. For *2-dimensional* projectile motion, $m\ddot{x}\mathbf{i} + m\ddot{y}\mathbf{j} = \mathbf{0i} - mg\mathbf{j}$. **Taking** x and y components *separately* gives $\ddot{x} = 0$ and $\ddot{y} = -g$. These are two **constant** acceleration equations. (See the previous *section* on constant (uniform) acceleration — the formulae obtained there could be used **directly** in the x and y directions, but we will integrate from *scratch*).



$u = \dot{x} = c_1$; $v = \dot{y} = -gt+c_2$, where u and v **denote** the x and y components of the *velocity*. If the particle is projected at time t = 0, with an initial **speed** V at an angle α to the horizontal, the x and y *components* are x = Vcosα and y = Vsinα. These are used to determine the constants of integration c_1 and c_2, to **give** u = Vcosα and v = -gt+Vsinα.

26th March 1999

Projectiles

Throughout the motion, the x component of the velocity, u, is *constant*, and the y component of velocity, v, decreases linearly with time. During the particle's motion, its speed is $\sqrt{(u^2+v^2)}$ and its *direction* of motion is at an angle of $\tan^{-1}(v/u)$ to the horizontal. This angle may be **written** as $\tan^{-1}\left(\frac{dy/dt}{dx/dt}\right)$ or $\tan^{-1}\left(\frac{dy}{dx}\right)$.

The expressions for the *velocity* components may be integrated again w.r.t. time. (to give $x = (V\cos\alpha)t+c_3$; $y = -\frac{1}{2}gt^2+(V\sin\alpha)t+c_4$). The *constants* of integration c_3 & c_4 are determined by the particle's **initial** position. This is usually the *origin* at time zero i.e. x = 0 and y = 0 at t = 0. So $c_3 = 0$ and $c_4 = 0$. This gives x = (Vcosα)t and $y = -\frac{1}{2}gt^2+(V\sin\alpha)t$.

This is just the **constant** acceleration formula $\frac{1}{2}a_0t^2+ut$. In the y-direction, the *following* equation is often useful: $v^2 = (V\sin\alpha)^2-2gy$. This **comes** from $v^2 = u^2+2a_0S$. It may also be seen to be *equivalent* to the conservation of energy equation $\frac{1}{2}mv^2 + mgy = \frac{1}{2}m(V\sin\alpha)^2$. (Note: The $\frac{1}{2}m(V\cos\alpha)^2$ terms **cancel** on each side of the equation).

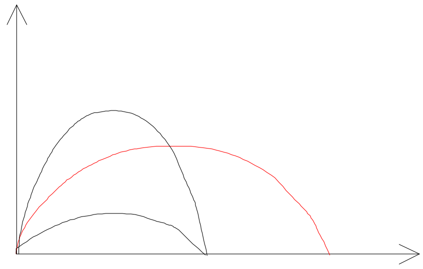
Maximum Height and Horizontal Range

Greatest Height. Consider *vertical* motion. We know that $v^2 = (V\sin\alpha)^2-2gy$. At the maximum height, the y component of *velocity* is instananeously zero, so $0 = (V\sin\alpha)^2-2gh$; $h = \frac{(V\sin\alpha)^2}{2g}$; $h = \frac{V^2\sin^2\alpha}{2g}$. The **largest** possible h is achieved with $\alpha = \pi/2$ (projecting *vertically*).

Horizontal Range. Consider vertical motion again. We **have** $y = -\frac{1}{2}gt+(V\sin\alpha)t$. When the particle *starts* or *returns* to the horizontal, y = 0. So $0 = -\frac{1}{2}gt^2+(V\sin\alpha)t$; $0 = t(-\frac{1}{2}gt+V\sin\alpha)$. So either t = 0 or $t = \frac{2V\sin\alpha}{g}$. This is the **time** taken by the particle to travel from y = 0 up to y = h and back *down* to y = 0 again.

The **length** of the range may be obtained from the *equation* for horizontal position: $x = (V\cos\alpha)t$; $x = (V\cos\alpha)^{2V\sin\alpha}/_g = {}^{2V^2\cos\alpha\sin\alpha}/_g = {}^{V^2\sin2\alpha}/_g$. So the **range** is ${}^{V^2\sin2\alpha}/_g$. This reaches a *maximum* when $\sin2\alpha = 1$, i.e. $2\alpha = {}^{\pi}/_2$, $\alpha = {}^{\pi}/_4$, which gives the **maximum** possible range of ${}^{V^2}/_g$.

To obtain a **particular** range (*less* than the maximum range) there is a choice of 2 angles of projection, symmetric **either** side of $\alpha = {}^{\pi}/_4$.



Easter Holiday

19th April 1999

Projectiles

Example: (a) A fielder is 80m away from the catcher. Can he *throw the ball* to the catcher in one go, if he *throws* at $25ms^{-1}$? (b) Throwing the ball at $25ms^{-1}$, what are the possible **angles** of projection if he wants to throw the ball a distance of 40m? (c) In order to return the ball tossed 40m as quickly as possible, what angle of projection should be used?

A: (a) **Maximum** range = $^{V^2}/_g = ^{25^2}/_{9.81}$ = 63.78m. So he **cannot** throw the ball a distance of 80m. (a) Applying the formula for range, we want $40 = ^{25^2\sin2\alpha}/_{9.81}$; $2\alpha = \sin^{-1}(^{40g}/_{625})$. So 2α = 38.89° or 141.11° (For $0\leq2\alpha\leq\pi$). So α = 19.45° or α = 70.55° (for $0\leq\alpha\leq^{\pi}/_2$). (c) The **horizontal** velocity for an angle of projection of 19.45° is 25cos19.45° = $23.57ms^{-1}$. For an angle of projection of 70.55° , the horizontal velocity is 25cos70.55° = $8.32ms^{-1}$.

This gives the travel times of the ball to be (using $t = ^{range}/_{V\cos\alpha}$) $^{40}/_{23.56}$ = 1.7s and $^{40}/_{8.32}$ = 4.8s. Clearly, the **lower** trajectory takes much less time. Alternatively, (c) could be solved using the formula for the **time** in flight, $t = ^{2V\sin a}/_g$, given *previously*.

The Equation for the Path of a Projectile / Trajectory

The equation of a *trajectory* is a relation of the form y = y(x). We have previously derived the x and y positions of a *projectile* in terms of t, i.e. $x = (V\cos\alpha)t$ and $y = -^1/_2gt^2+(V\sin\alpha)t$. To obtain y in terms of x, **eliminate** t from these two equations. The first gives $t = ^x/_{V\cos\alpha}$. Substituting in the 2nd for t gives $y = -^1/_2g(^x/_{V\cos\alpha})^2+V\sin\alpha(^x/_{V\cos\alpha})$. Therefore we have the following: $\mathbf{y = -^{g\sec^2\alpha}/_{2V^2}\, x^2 + \tan\alpha\, x}$.

Thus y is a *quadratic function of x*, i.e. it has the shape of a parabola. Its **exact** shape depends on V and α. Notes: Think of the *shape* of $y = x^2$, then $y = -x^2$, and then $y = -x^2+x$. This gives the kind of shape we want. Because the path is a **parabola**, the following points are true: The *maximum* height is attained at $x = ^{range}/_2$. The **height** is symmetric about $x = ^{range}/_2$. The *direction* of the particle's **motion** is anti-symmetric about $x = ^{range}/_2$.

20th April 1999

Exercises 5

Q: When projected at $\tan^{-1}(^3/_4)$, a projectile falls 40m *short* of a target. When projected at 45°, it overshoots the target by 50m. Show that the target is 2200m away and evaluate the **correct** angle of projection. A: Using Range = $^{V^2\sin2\alpha}/_g$, for an angle $\tan^{-1}(^3/_4)$, $R-40 = ^{V^2\times2\sin\alpha\cos\alpha}/_g = ^{V^2\times2\times(3/5)\times(4/5)}/_{9.81}$. For 45°, we have $R+50 = ^{V^2\sin90°}/_g$. These give 2 simultaneous equations which we *solve* to get R = 2200m and V = $148.568ms^{-1}$.

Q: If a particle is projected in a **room** of height 5m, what is the greatest possible range if the *speed of projection* is $20ms^{-1}$? A: Greatest Height = $^{V^2\sin^2\alpha}/_{2g}$. Here it is 5m. So $5 = ^{20^2\sin^2\alpha}/_{2\times9.81}$; $\sin^2\alpha = 0.24525$; $\sin\alpha = 0.49522$, so α = 29.7° (or 150.3°). So the range is $^{V^2\sin2\alpha}/_g = ^{20^2\sin59.4°}/_{9.81}$ = 35.1m.

<u>*24th April 1999*</u>

<u>Impulses & Collisions</u>

This section is concerned with problems where there's a **sudden** (instantaneous) change in the motion of particles as a result, say, of the *collision* between 2 particles, or the impact of a particle and a <u>fixed</u> barrier. Collisions are everywhere in the real world — *snooker, vehicle accidents, Brownian motion of molecules.*

We shall assume that the bodies are **particles**, and that the surface of the bodies are **smooth,** so that no frictional forces are present during the impact. (This precludes analysis of, say, **spin** shots in snooker). The collision may be *direct* (head on) or *oblique*. In many cases, the <u>time</u> of contact during the collision is very small and the forces involved are **large**. We use Newton's 2nd law of motion to investigate *impulse,* which measures the product of <u>force</u> and <u>time</u>.

<u>Impulse</u>. The impulse of a force (which can in general *vary* with time) is defined as $\underline{I} = \int_{t_1}^{t_2} \underline{F}dt$. Using Newton's *2nd* law, $\underline{F} = d(m\underline{v})/dt$. For a particle, the above expression may be **integrated** w.r.t. time to give $\underline{I} = \int_{t_1}^{t_2} [d(m\underline{v})/dt]dt$. **Therefore**, $\underline{I} = [m\underline{v}]_{t_1}^{t_2}$, which is the change of *momentum* that has occurred during the time <u>interval</u> t_1 to t_2.

If the mass is **constant**, and we denote $\underline{v} = \underline{v}(t_2)$ and $\underline{u} = \underline{v}(t_1)$, the above may be written as $\underline{I} = m\underline{v}-m\underline{u}$. If F is a **constant** force during the impact, then $\underline{I} = \underline{F}(t_2-t_1)$. For 1-D motion, i.e. **head-on** collision, the above vector quantities $\underline{I}$, $\underline{F}$ and $\underline{v}$ as scalars. However, **care** is required with the +ve & -ve *signs*.

<u>Example</u>: A force of 25N acts for 2s in the direction of *motion*, on a body of mass 10kg travelling with **velocity** $5ms^{-1}$. Calculate (a) the **impulse**, and (b) the **final** velocity of the body. Solution: (a) $I = F\times t = 25\times 2 = 50Ns$. (b) $I = mv-mu$; $50 = 10v-10\times 5$; $100 = 10v$; $v = 10ms^{-1}$.

Q: A **particle** of mass 2kg is moving in a *flat horizontal plane* so that it is travelling at $3ms^{-1}$ in a direction with bearing 120º. If an **impulse** of magnitude 2Ns is applied to the particle towards the North, find the new *direction* of motion and its speed. A: Think of the diagram, and then by resolving we get the **particle's** initial velocity as $u = {}^{3\sqrt{3}}/_2\underline{i} - 3.{}^1/_2\underline{j}$. The **impulse** is $I = 2\underline{j}$. Now, $\underline{I} = m\underline{v}-m\underline{u}$; so $2\underline{j} = 2\underline{v}-2({}^{3\sqrt{3}}/_2\underline{i}-{}^3/_2\underline{j})$; $\underline{v} = {}^{3\sqrt{3}}/_2\underline{i}-({}^3/_2-1)$; $\underline{v} = {}^{3\sqrt{3}}/_2\underline{i}-{}^1/_2\underline{j}$, from which we can get the **magnitude** & the **direction** of the motion.

<u>*26th April 1999*</u>

<u>Conservation of Momentum for Colliding Particles</u>

Consider 2 particles moving at *differing* velocities. The <u>equations</u> of motion for these two particles are $m_A\ddot{x}_A = \mathbf{F}_A$ and $m_B\ddot{x}_B = \mathbf{F}_B$, where $\mathbf{F}_A$ and $\mathbf{F}_B$ are the forces acting on the two particles *separately*. These forces are zero except for a <u>short</u> time interval $[t_1, t_2]$ during which time the particles are in <u>contact</u>.

During this time, by Newton's 3rd law, "**action** and **reaction** are equal and opposite". So $\mathbf{F}_B = -\mathbf{F}_A = \mathbf{F}$. Therefore, $m_A\ddot{x}_A = -\mathbf{F}$ and $m_B\ddot{x}_B = \mathbf{F}$.

Integrating w.r.t. time gives **over** the time interval $[t_1, t_2]$ the following: $[m_A\dot{\mathbf{x}}_A]^{t_2}_{t_1} = -\int^{t_2}_{t_1} \mathbf{F}dt = -\mathbf{I}$, and $[m_B\dot{\mathbf{x}}_B]^{t_2}_{t_1} = +\int^{t_2}_{t_1} \mathbf{F}dt = \mathbf{I}$. *Adding* these together gives $[m_A\dot{\mathbf{x}}_A]^{t_2}_{t_1} + [m_B\dot{\mathbf{x}}_B]^{t_2}_{t_1} = \mathbf{0}$. *Taking* m_A and m_B as constants, and **denoting** m_A = M and m_B = m, then $\dot{\mathbf{x}}_A(t_1) = \mathbf{U}$; $\dot{\mathbf{x}}_B(t_1) = \mathbf{u}$; (velocities *before* collision); $\dot{\mathbf{x}}_A(t_2) = \mathbf{V}$, $\dot{\mathbf{x}}_B(t_2) = \mathbf{v}$. (velocities *after* collision). The above becomes $M\mathbf{U}+m\mathbf{u} = M\mathbf{V}+m\mathbf{v}$.

This states that the *total linear momentum* after the collision is equal to the total linear momentum **before** the collision. This important equation has been obtained without knowing the time of contact or the <u>force</u> of impact. The impulse experienced by the particle M is $-\mathbf{I} = M\mathbf{V}-M\mathbf{U}$. Similarly, for the particle m, the impulse is $\mathbf{I} = m\mathbf{v}-m\mathbf{u}$ (*equal* and *opposite*).

For one-dimensional problems, the vectors <u>reduce</u> to just scalars: $M\mathbf{U}+m\mathbf{u} = M\mathbf{V}+m\mathbf{v}$. Note that the *conservation* of momentum equation is in itself not enough to solve for general collisions. For example, in 1-D, we may <u>know</u> U and u, and want to calculate V and v. But <u>one</u> equation is not enough. However, for certain cases, *one* equation suffices, e.g. when we also happen to know **V** or **v**, or if the two particles *coalesce* (when **V** = **v**), or if one particle *breaks apart*.

<u>Example</u>: A bullet of mass 10g is fired *horizontally* into a block of wood of mass 1kg which rests on a <u>smooth</u> horizontal plane. If the bullet's velocity is $1000ms^{-1}$, find the velocity of the block if the bullet becomes *embedded* in the block. A: By the conservation of momentum, $(0.01\times1000)+(1\times0) = 1.01\times v$. Therefore, $v = {}^{10}/_{1.01} = 9.90099ms^{-1}$.

<u>Newton's Empirical Law</u>

We shall introduce the law is **one dimension** initially. Consider velocities as before. Newton's *Empirical* law is stated as: *"If two particles collide, their relative velocities after impact is equal to -e times their relative velocity before impact"* Symbolically, $\mathbf{V}-\mathbf{v} = -e(\mathbf{U}-\mathbf{u})$. The constant e is called the <u>coefficient of restitution</u> for the two bodies concerned. It is a positive constant, $0\le e\le1$. This empirical law takes into account **energy** losses at collision. The two extremes are: e = 0: *inelastic* collision, particles *coalesce*; e = 1: *perfectly* elastic, no *energy* loss (<u>unattainable</u> in practice). To solve problems, use the *conservation of momentum* equation together with the <u>restitution</u> law to give 2 equations for V and v. (Which you can solve **simultaneously**).

<u>27th April 1999</u>

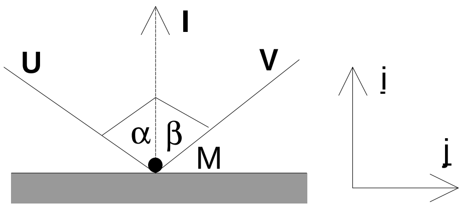
<u>Direct Collision (Between a Particle and a Barrier)</u>

Consider a particle of **mass** M in collision with a fixed barrier. Let U be the particle's velocity *before* the collision, and let V be the particle's velocity *after* the collision. Let I denote the impulse <u>experienced</u> by the particle, and the impulse relationship gives I = MV-MU. The impulse **experienced** by the barrier is equal and opposite, but no information is gained since the barrier is **fixed** (u = v = 0) and the momentum is lost.

The restitution law *still holds,* and models the loss in energy, giving V-0 = -e(U-0); V = -eU. This is a <u>reduced</u> restitution law, and is sufficient to obtain the **rebounded** velocity of the particle.

Oblique Collisions (Between a Particle and a Fixed Barrier)

Before collision: velocity U at an angle α to the normal. **After** collision: velocity V at an angle β to the normal. The particle and the *barrier* are assumed to be smooth, so (i) the impulse is in the normal direction, and (ii) the **restitution** law models energy losses which occur only in the normal direction — and must therefore be applied only in this direction.



Let the impulse vector be **I**. The impulse *relation* for the particle gives **I** = M**V**-M**U**. Taking *components* in the **i** and **j** directions gives **I** = M(**V**sinβ**i**+**V**cosβ**j**) - M(**U**sinα**i**-**U**cosα**j**). The **j** component gives **I** = M**V**cosβ + M**U**cosα, and in the **i** direction, M**v**sinβ = M**U**sinα (i). Now apply the *restitution* law in the normal direction: **V**cosβ = e.**U**cosα (ii).

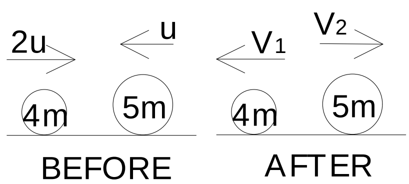
Given the **direction** α and the **magnitude** of the incoming velocity U, the outgoing direction β is given by $^{(i)}/_{(ii)}$ to give $\tan\beta = {}^{\tan\alpha}/_{e}$. And $V = U\sqrt{(e^2\cos^2\alpha+\sin^2\alpha)}$ is the **magnitude** of the *outgoing* velocity.

30th April 1999

Tutorial

Q: Two masses of mass m and 2m are *travelling* in opposite directions with speed u. They collide and unite: find the magnitude and direction of the velocity of the combined particle, and determine the **loss** in energy. A: Using the conservation of momentum, mu-2mu = 3mv; -u = 3v; so $v = -^{u}/_{3}$. The loss in energy is given by KE Before - KE After = $(\frac{1}{2}mu^2 + \frac{1}{2}.2m.u^2) - (^1/_2.3m.(^u/_3)^2) = {}^3/_2mu^2 - {}^{mu^2}/_6 = {}^4/_3mu^2$ J.

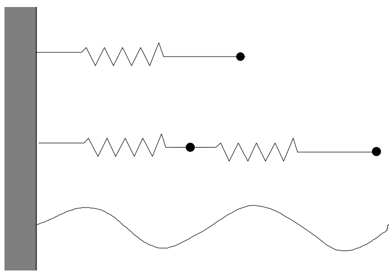
Q: A 4m mass is **travelling** at 2u, and *collides* with a 5m mass ball moving with speed u in the opposite direction. If the coefficient of restitution is ½, find the resulting velocities of the balls after impact, and the loss of energy *due* to the collision. A: Using the conservation of momentum, we have 4m.2u - 5mu = $-4mV_1+5mV_2$; 3mu = $-4mV_1+5mV_2$, 3u = $-4V_1+5V_2$. Using the restitution law, V-v = -e(U-u); $-V_1-V_2 = -^1/_2(2u+u)$; $2V_1+2V_2$ = 3u. Equating for V_2, $3u-2V_1/2 = 3u+4V_1/5$; $15u-10V_1 = 6u+8V_1$; $9u = 18V_1$; $V_1 = {}^u/_2$. So $V_2 = 3u-2V_1/2 = u$. The loss of Energy as *before* is given by KE Before - KE After = $7.5mu^2$.



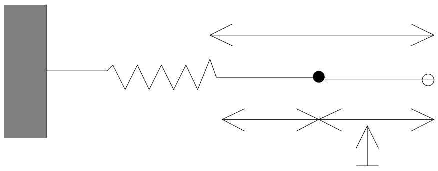
4th May 1999

Vibrations/Oscillations

Introduction: Any matter will vibrate in a 'to and fro' motion The *mechanism* is the restoring force set up which opposes deformation. This has its origins in forces between molecules or atoms. We shall restrict ourselves to 1-dimensional vibrations of a single particle. The prototype model is a *mass* attached to a spring on a smooth **horizontal** table.

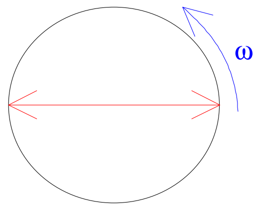


If the spring is at its *natural length*, (neither stretched or compressed), the particle will not be subject to any force, and the system will therefore be at its <u>equilibrium</u> position. If the particle is displaced from its equilibrium position and released, it will vibrate back and forth. The spring will try to **restore** the particle to its equilibrium position, but it will overshoot. In the absence of damping and friction, it will *vibrate* forever.



Amplitde, the distance between the equilibrium position & the maximum displacement

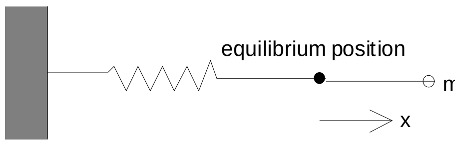
The **time** taken to perform one cycle (one *complete* oscillation) is called the period, usually denoted by T. The number of <u>cycles</u> per second is called the frequency, f, where $f = {}^{1}/_{T}$. If the cycle (one) is associated with 2π radians, the angular *frequency* ω is the number of radians per second, where $\omega = 2\pi f$; or $\omega = {}^{2\pi}/_{T}$. To see the association with 2π, consider motion in a *circle*.



<u>Hooke's Law</u> gives $F = -kx$. (The force in the string is *proportional* to the spring's extension). The constant k is the spring's <u>stiffness</u>.

<u>Simple Harmonic Motion</u>

Measure the displacement x from the *equilibrium* position, so that the spring's extension is x. By Newton's 2nd law, $m\ddot{x} = F$. Using Hooke's law, (for a **linear** spring), $m\ddot{x} = -kx$. It is convenient to rewrite this as ${}^{d^2x}/_{dt^2} = -{}^{k}/_{m}x$, and write ${}^{d^2x}/_{dt^2} = -\omega^2 x$. This is the SHM or *oscillator* equation, where $\omega = \sqrt{({}^{k}/_{m})}$ is the natural **angular** frequency.



This is a **linear** 2nd order D.E., $\ddot{x} + \omega^2 x = 0$. The *general* solution may be obtained by assuming a solution of the form $x = e^{\lambda t}$ and *substituting* into the o.d.e.: $\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0$; $\lambda^2 + \omega^2 = 0$; $\lambda = \pm i\omega$. So the *general* solution is $x = ae^{i\omega t} + be^{-i\omega t}$, where a & b are constants of integration, i.e. $x = i(a-b)\sin\omega t + (a+b)\cos\omega t$. [a+b real; i(a-b) real]. For **real** x, a & b must be complex *conjugates*, which gives as general solution $x = C\sin\omega t + D\cos\omega t$, where C & D are **constants** of integration.

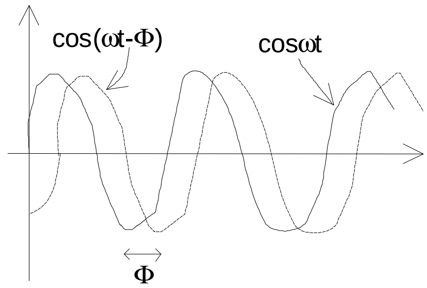
<u>7th May 1999</u>

The constants C and D are **obtained** by using the conditions on x and ${}^{dx}/_{dt}$ at some particular time, usually t = 0. *Example*: Calculate the displacement x in terms of ω and t, if at time t = 0, x = 3 and the *velocity* is zero. Solution: The general solution is $x = C\sin\omega t + D\cos\omega t$. Differentiating gives $v = \dot{x} = \omega C\cos\omega t - \omega D\sin\omega t$. At t = 0, $\dot{x} = 0$. So $0 = \omega C$, therefore C = 0. Hence the *general* solution is $x = D\cos\omega t$. At t = 0, x = 3, so 3 = D. Therefore, $x = 3\cos\omega t$.

<u>Phase Angle and Amplitude</u>

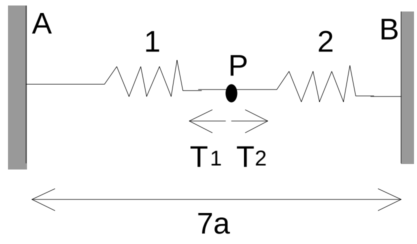
The **general** solution for a linear oscillation is $x = C\sin\omega t + D\cos\omega t$. This may be rewritten as $x = \sqrt{(C^2+D^2)}[{}^{C}/_{\sqrt{(C^2+D^2)}\sin\omega t} + {}^{D}/_{\sqrt{(C^2+D^2)}\cos\omega t}]$. Here, $\sin\Phi = {}^{C}/_{\sqrt{C^2+D^2}}$ and $\cos\Phi = {}^{D}/_{\sqrt{C^2+D^2}}$. This may be written as $x = A\cos(\omega t - \Phi)$, where $A = \sqrt{(C^2+D^2)}$. So $A\cos(\omega t - \Phi) = A[\cos\omega t\cos\Phi + \sin\omega t\sin\Phi]$, and Φ is the *phase angle* given by $\Phi = \sin^{-1}({}^{C}/_{\sqrt{(C^2+D^2)}}) = \cos^{-1}({}^{D}/_{\sqrt{(C^2+D^2)}})$, or $\Phi = \tan^{-1}({}^{C}/_{D})$.

In this *form*, $v = \dot{x} = \omega A\sin(\omega t-\Phi)$. *Alternatively*, $x = A\sin(\omega t-\Theta)$, where $A = \sqrt{(C^2+D^2)}$ and $\Theta = \sin^{-1}(D/\sqrt{(C^2+D^2)}) = \cos^{-1}(C/\sqrt{(C^2+D^2)}) = \tan^{-1}(D/C)$.



Energy: the equation of *motion* is $m\frac{d^2x}{dt^2} = -kx$, or $mv\frac{dv}{dx} = -kx$; $\int mv\,dv = -\int kx\,dx$. Therefore, $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$ (K.E. + P.E. in spring = total energy), i.e. *conservation* of energy. If the initial conditions are so that the <u>displacement</u> is x_0 when $v = 0$, then $0 + \frac{1}{2}kx_0^2 = E$. Therefore, $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}k_0^2$. It follows that $v^2 = \frac{k}{m}(x_0^2-x^2)$; $v^2 = \omega^2(x_0^2-x^2)$. Now $v^2 \geq 0$ as $x^2 \leq x_0^2$, so $-x_0 \leq x \leq x_0$.

<u>Example</u>: A particle P of mass m lies on a *smooth* horizontal table, and is attached to 2 points A & B (where AB = 7a) by two springs of **natural** lengths 3a and 2a; stiffness k and 3k. Show that the *particle* is in equilibrium when AP = $\frac{9a}{2}$. The particle is held at rest with AP = 5a and then released. Find the period of the **motion** and the particle's **maximum** speed.



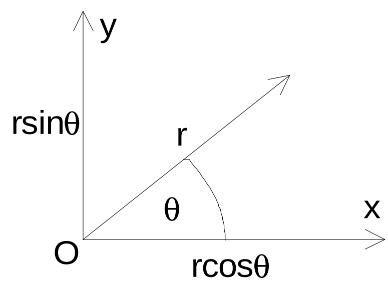
A: Spring 1: *natural* length 3a, stiffness k. Spring 2: *natural* length 2a; stiffness 3k. Measure the particle's <u>displacement</u> x from A. **Hooke's** law for the springs gives T = stiffness × extension. So $T_1 = k(x-3a)$ and $T_2 = 3k(5a-x)$. <u>ANALYSE THIS!</u> The equilibrium position is where the net force is zero, i.e. $T_2 - T_1 = 0$. Therefore, $3k(5a-x)-k(x-3a) = 0$; $15ka-3kx-kx+3ka = 0$; ..., $x = \frac{9a}{2}$, the *equilibrium* position.

<u>10th May 1999</u>

<u>Circular Motion</u>

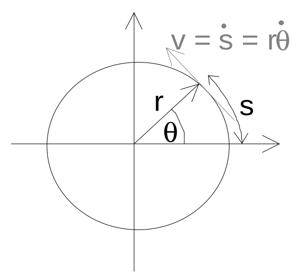
Real life **examples**: washing machines, spin dryers, cornering in a car, CD ROM, etc. Circular motion is a *particular* case of 2-D motion. In, say, <u>projectile</u> motion, Cartesian co-ordinates (x & y) were ideal. In circular motion however, (plane) polar co-ordinates are far more convenient.

PLANE POLAR COORDINATES. In Cartesian co-ordinates, a *position* in space is given by the (x, y) co-ordinate pair. In <u>plane</u> polar co-ordinates. a position in space is given by the (r, θ) pair, where r is the distance **from** the origin O, and θ is the angle measured in radians from a *fixed* line in space.



If θ is measured from the x-axis, the **formulae** for converting from (x,y) co-ordinates to (r,θ) *co-ordinates* are $x = r\cos\theta$, $y = r\sin\theta$. And those from (r,θ) to (x,y) are $r = \sqrt{(x^2+y^2)}$; $\theta = \tan^{-1}(y/x)$.

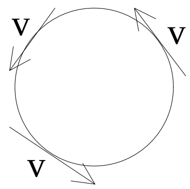
<u>Kinematics of Circular Motion</u>



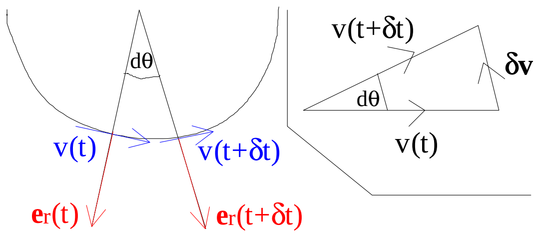
For **motion** in a circle, i.e. r = constant, the distance *travelled* by a particle during an angular <u>displacement</u> θ rad is given by $s = r\theta$. The *tangential* linear velocity of a particle moving on a <u>circle</u> is $v = \dot{s} = \frac{ds}{dt} = r\frac{d\theta}{dt}$.

At each instant, it is **directed** along the tangent to the circle. $\dot{\theta}$ is the angular velocity and is the *rate* of change of angular displacement, measured in rad s^{-1}. It is often given the **symbol** ω (especially if $\dot{\theta}$ is a constant) — then we *write* v = rω. If $\dot{\theta}$ does **not** change with time, then $^{dv}/_{dt} = r^{d^2\theta}/_{dt^2} = r\ddot{\theta}$. Here, $^{dv}/_{dt}$ is (in ms^{-2}) the *linear* acceleration in rad s^{-2}.

Acceleration Towards the Centre



If a particle is *moving* along a circle, the direction of the velocity (a vector) is changing all the time. Thus, it must be **accelerating**. This is true even if its speed remains *constant*, i.e. v = rω = constant. Now consider motion with *constant* angular velocity ω. (Hence v = rω is constant). The vector acceleration is **defined** as $\mathbf{a} = {}^{d\mathbf{v}}/_{dt} = \lim_{\delta t\to 0} {}^{d\mathbf{v}}/_{dt} = \lim_{\delta t\to 0} {}^{\mathbf{v}(t+dt)-\mathbf{v}(t)}/_{dt}$.



Consider the change in the *vector* **v** as the particle goes *around* the circle. Denote the radial unit vector as e_r. Consider the **triangle** of vectors. This is an *isosceles* triangle since v = |**v**| = *constant*. From the triangle, δ**v** = $\mathbf{v}(t+\delta t)-\mathbf{v}(t) \approx -v\sin\delta\theta[\mathbf{e}_r(t+{}^{\delta t}/_2)]$. So $\mathbf{a} = \lim_{\delta t\to 0} -{}^{v\sin\delta\theta}/_{dt}(\mathbf{e}_r) = \lim_{\delta t\to 0} -v{}^{d\theta}/_{dt}\,\mathbf{e}_r$. *Therefore,* $\mathbf{a} = -v\omega\mathbf{e}_r$.

Therefore, the *acceleration* is directed towards the centre, and has magnitude a = vω. Since v = rω, this may be *written* as $a = r\omega^2$ or $a = {}^{v^2}/_r$. Thus a force must be *present* in order to maintain circular motion. If a particle has **mass** m, the force has magnitude ma, directed towards the centre of the circle.

Horizontal Circular Motion: Example

A particle of mass 2kg is attached to a *horizontal* string of length 0.5m. Calculate the tension in the string required to maintain circular motion with an angular velocity of 60 revolutions per minute (60 rpm). **Solution**: m = 2, r = ½, ω = 60rpm = 1rps = 2π $rads^{-1}$. So tension = $mr\omega^2 = 2\times{}^1/_2\times(2\pi)^2 = 4\pi^2$ N.

Assignment 4

Q: A zoologist has an *arrow* with a maximum range of 100m. If a rhinoceros charges at him at $30kmh^{-1}$, and he aims his bow 20° *above* the horizontal, how far away should the rhinoceros be when he releases the arrow? A: When $a = {}^{\pi}/_4$, range = ${}^{v^2}/_g$. So $100 = {}^{v^2}/_g$, $v = 10\sqrt{(g)}$. When a = 20°, R = ${}^{v^2}/_g\sin 40 = 100\sin 40°$.

We now want the **time** of flight of the projectile. Time of flight = ${}^{\text{Horizontal Range}}/_{\text{Horizontal Velocity}} = {}^{100\sin 40°}/_{10\sqrt{(g)}\cos 20°}$ = 2.18397s. *When* t = 2.18397s, we want the Rhinoceros at a distance of 100sin40° m away. In this time, the rhinoceros will have *travelled* a distance of 18.19975m at a speed of $30kmh^{-1} = 8.3333ms^{-1}$. Conclusion: the rhinoceros should be 100sin40° = 64.2788 + 18.19975 = 82.48m away when he **fires**.

Q: A projectile is launched at a speed V in a **direction** perpendicular to a surface having slope α to the horizontal. *Determine* the range R. A: Consider the projection point at the origin of the graph. The equation for the **projectile** path is $y = -\frac{g\sec^2\alpha}{2V^2}x^2 + (\tan\alpha)x$. We equate this with the equation of the *slope*, $y = -\tan\alpha x$, to get an **expression** for the range.

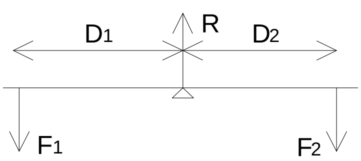
In collision questions, apply the conservation of *momentum* and then the *restitution* equation to get 2 simultaneous equations which you can solve to get any unknowns. Watch out for numerical slips! When you have more than **one** collision, there are no further collisions if all the particles are still; or if the particle that is moving away is travelling faster than the particle behind it.

11th May 1999

Statics

We now consider the effect of **forces** on rigid bodies, in cases where they cannot be treated as particles. So instead of *point* masses, we have bodies with a finite extent. If a force acts on the body, we must be careful to specify not only the magnitude and direction of the force, but also where it acts. It is a **localised** vector, not a free vector.

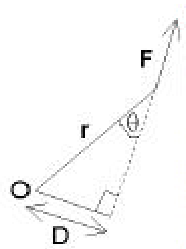
We are then led to *consider* the moment of the force, which is a measure of its tendency to cause rotation of the body. We shall consider rigid bodies in equilibrium under the action of forces — no motion — **statics**. Consider the see-saw, or balance, which *consists* of a rigid horizontal bar with a pivot at its mid-point.



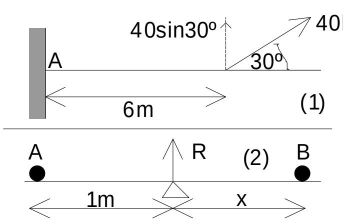
The **magnitude** of the moment (or the torque) of the force F_1 about O is $M_{O_1} = F_1D_1$, where D_1 is the perpendicular distance from O to the line of action of F_1. Its sense is *anticlockwise* about O, and by convention, this is +ve. Moment has units Nm. For force 2, the *moment* is $M_{O_2} = -F_2D_2$ in a **clockwise** sense about O, which is -ve by convention. The total moment in the example is $M_O = M_{O_1}+M_{O_2} = F_1D_1-F_2D_2$.

Equilibrium Conditions

For a rigid body to be in **equilibrium,** (i) the sum of the *forces* must be zero; (ii) the sum of the *moments* must be zero. In the above example, if R is the reaction of the pivot, then (i) implies that $R = F_1+F_2$, and (ii) implies that $M_O = F_1D_1-F_2D_2 = 0$. **Calculation of moments**. Let D be the *perpendicular* distance. The dotted line is the line of *action* of the force. In vector notation, $\mathbf{M}_O = \mathbf{r}\times\mathbf{F}$; $|\mathbf{M}_O| = |\mathbf{r}||\mathbf{F}|\sin\theta$; $|\mathbf{M}_O| = D|\mathbf{F}|$.



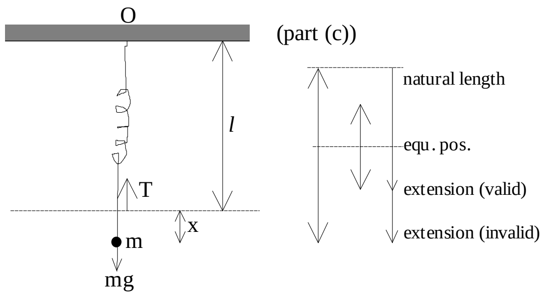
Example (1): What is the *moment* of the 40kN force about the point A as shown? A: $M_A = 40000\times\sin30^o\times6$ (resolve the force into 2 perpendicular directions) = 12000Nm = 120kNm. *Example* (2): Find the distance x so that the **system** is in equilibrium A: In *equilibrium*, $M_O = 0$, i.e. $70g = 50g.x$; $x = \frac{70g}{50g} = \frac{7}{5}$m.



14th May 1999

Tutorial: Oscillations, Circular Motion & Statics

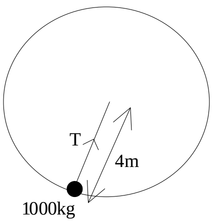
Q: A particle of mass m is hanging **freely** via a light piece of elastic from a fixed *point* O. If the piece of elastic is considered to be a spring of stiffness k and natural length *l*, find (a) the length of the elastic when the mass hangs in **equilibrium**; (b) the **period** of oscillation about this equilibrium position if the particle is disturbed; (c) the *maximum* distance the mass may be pulled down and released if the motion is to be simple harmonic motion.



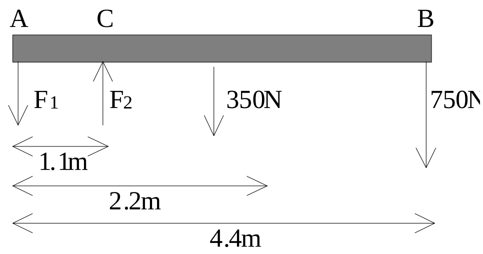
A: **Hooke's** law states that T = kx. However, we also have T = mg. In *equilibrium*, the resultant force is zero, so that 0 = mg-kx; kx = mg; $x = {}^{mg}/_{k}$. This is the **extension** at equilibrium. (b) The forces acting on the particle (the sum of) are mg-T = F; mg-kx = F. Newton's 2nd law *states* that F = ma, hence $m\ddot{x} = mg - kx$; $\ddot{x} = g - {}^{kx}/_{m}$; $\ddot{x} + \omega^2 x = g$. (Where $\omega = \sqrt{({}^{k}/_{m})}$). From the **notes**, $T = {}^{2\pi}/_{\omega} = 2\pi\sqrt{({}^{m}/_{k})}$ = *period* of oscillation.

(**Aside**: For $\ddot{x}+\omega^2 x = 0$, the *solution* is $x = A\cos(\omega t+\phi)$. For $\ddot{x}+\omega^2 x = g$, we **also** need a *particular* solution. The complementary function is $A\cos(\omega t+\phi)$. The particular integral is ${}^{g}/_{\omega^2}$ by ***inspection***. So the *general* solution is $A\cos(\omega t+\phi)+{}^{g}/_{\omega^2} = {}^{gm}/_{k}+A\cos(\omega t+\phi) = x$).

(c) The **maximum** distance is ${}^{2mg}/_{k}$ because if the *oscillation* goes beyond the natural length, it will not compress (it's a string), and therefore we do not have SHM. Therefore, keep the oscillation between the **natural** length position and the **extension** position.



Q: A car on a **fairground** roundabout has a mass of 1000kg and is connected to the central spindle at a length of 4m. If the car goes *around* the circle once every 2s, find the tension in the arm. A: We need the **centripetal** force, $F = mr\omega^2 = mv\omega = {}^{mv^2}/_{r}$. Use $mr\omega^2$. Here, $\omega = \pi$, because we *have* 2π rad in 1 revolution; it goes around **twice**, (2 revolutions), so π rad per second. So $F = mr\omega^2 = 1000\times4\times(\pi)^2 = 4000\pi^2$N.



Q: A diver of weight 750N is standing at the **end** B of a diving board AB of length 4.4m and weight 350N. The board is supported from above at A and from *below* at C, where AC = 1.1m. The force exerted at A is down, and at C, it is up. Modelling the diving board as an uniform rod, and the diver as a **particle**, find the forces exerted on the diving board at A & C.

A: For the forces to be in **equilibrium**, we have $F_2 = F_1+350+750$; $F_2 = F_1+1100$. For the **moments** to be in equilibrium, taking moments about C, we have $1.1\times F_1 = 350\times1.1 + 750\times3.3$; $F_1 = 350+(750\times3)$; $F_1 = 2600$N. So $F_2 = 2600+1100 = 3700$N by *back substitution*.

REMEMBER TO STATE & USE THE CORRECT **UNITS.**

Exam Paper: May 1999

SECTION 1 (Compulsory)

(1) (a) The position vector **r** m of a particle P of mass 4 kg at time t s is given by
$\mathbf{r} = (t^2+t)\mathbf{i} + (3t^2-2)\mathbf{j} + (2t^3-4t^2)\mathbf{k}$.

(i) Find the velocity of P at time t.
(ii) Find the speed of P at time t.
(iii) Find the acceleration of P at time t.
(iv) Find the kinetic energy of P at time t.
(v) Find the rate at which the force acting on P is working at time t.

[7 marks]

(b) The work done by the force $\mathbf{F} = 3\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ N in moving a particle from the point with position vector $2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ m to the point with position vector $a\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ m, where a is a constant, is 52 J. Find the value of a. **[5 marks]**

(c) Two roads intersect at 90° at a point P. A man A is cycling at $^{250}/_{9}$ ms^{-1} along one of the roads towards P and at a certain instant is 400 m from P. A then observes a second man B, 300 m from the junction, running towards it at $^{50}/_{3}$ ms^{-1}. Find the time when the men are nearest to each other and the distance between them.

[8 marks]

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Assuming Hooke's Law, show by integration that the work done in extending a spring of stiffness k a distance x beyond its natural length is $\frac{1}{2}kx^2$. **[2 marks]**

(b) A spring is held in a smooth horizontal tube with one end fixed and the other attached to a small bead which is held in equilibrium by a force of magnitude 60 N pressing it against the free end of the spring. The compression of the spring in this position is 0.03 m.

(i) Find the stiffness of the spring k. **[2 marks]**
(ii) The bead is released. Find, using the conservation of energy, the speed of the bead just as the spring attains its natural length. **[4 marks]**

(c) A uniform rod AB of weight 50 N and length 1.2 m is supported from above at the end A and from below at the point C where AC = 0.3 m. A particle of weight 15 N is placed at the point D where AD = 1.1 m. Find the forces exerted on the rod at A and C. **[7 marks]**

(3) The non-gravitational resistance to the motion of a car of mass 1000 kg moving with a speed of v ms^{-1} is known to be of the form ($kv + 0.05kv^2$) N, where k is a constant. When the car's engine is working at a rate of 11.25 kW the car can move at a steady speed of 25 ms^{-1} on a horizontal road.

(a) Find the value of k. **[2 marks]**

(b) Find the rate at which the car's engine works when the car is moving at a steady speed of 15 ms^{-1} up a hill inclined at an angle $\sin^{-1}(^1/_{49})$ to the horizontal.

[5 marks]

(c) When the car is moving with speed 25 ms^{-1} on a horizontal road the engine is switched off. Show that the speed v ms^{-1} of the car after travelling a distance of x m after the engine has been switched off satisfies the differential equation

$2500^{dv}/_{dx} = -20-v.$

Solve this differential equation to find the distance travelled before the car's speed falls to 5 ms^{-1}. **[8 marks]**

(4) A particle of mass 2 kg lies on a smooth inclined plane, whose angle with the horizontal is 30°. The mass is attached to the top of the plane by a light elastic string of stiffness 30 Nm and natural length 0.5 m.

(a) What is the length of the string in equilibrium? **[2 marks]**

(b) The particle is pulled 0.2 m from the equilibrium position and released. Find the period of the oscillation and calculate the maximum speed of the particle.

[5 marks]

(c) Describe the motion of the particle if the string becomes slack and calculate how far the particle has to be pulled down the plane for it just to reach the top.

[8 marks]

(5) (a) Three small spheres A, B and C lie in a straight line on a smooth table. Their masses are m, 2m and 4m respectively. Sphere A is projected towards sphere B with a speed of 8 ms^{-1}.

(i) If the coefficient of restitution is ¼, find the velocities of the three spheres after three collisions and show that there can be no more collisions.

[6 marks]

(ii) Calculate the total energy lost during the collisions. **[2 marks]**

(b) Consider the free motion under gravity of a particle projected with speed V at an angle α to the horizontal.

(i) Show that the maximum height attained by the particle is h = $\frac{V^2 \sin^2\alpha}{2g}$.

[3 marks]

(ii) Show that the projectile's range is given by range = $\frac{V^2 \sin 2\alpha}{g}$. **[4 marks]**

<u>(Questions done: 1, 3, 5)</u>