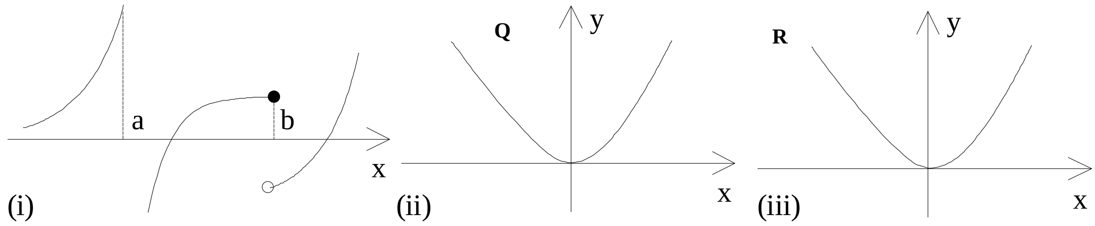
1st February 1999

Limits & Continuity

A **graph** with no holes/jumps/breaks/gaps is continuous.



(i) **Not** continuous at x = a, x = b. (ii) f: **Q**→**Q**, x→x². **Q** = {$^n/_d$ | n∈ $\mathbb{Z}$, $d \in \mathbb{Z}^+$ }. This is *nowhere* continuous. (iii) f: $\mathbb{R} \to \mathbb{R}$, $x \to x^2$. Continuous *everywhere*.

Reals & Rationals

Real numbers are represented by all *points* on a line. Fixing 0 and 1 on the line determine a value for every point. A real number is represented by an infinite decimal, a = $a_5a_4a_3a_2a_1.d_1d_2d_3d_4$... Snag: a real number may have **2** such representations e.g. 1.00000..... and 0.99999..... Any *infinite* string of 9's can be replaced by an infinite string of 0's provided we add 1 to the digit on the left.

A **rational** number is a real number where the *decimal expansion repeats*. We write e.g. $^1/_3$ = 0..33333.... = $0.\dot{3}$; $^1/_7$ = $0.\dot{1}4285\dot{7}$. **Algorithm** for converting a repeating decimal to a fraction. Example: Let x = $0.\dot{9}$. *Multiply* by $10^{(\text{repeat length})}$ (in this example 10) so $10(0.\dot{9}) = 9.\dot{9}$. **Subtract**, 10x-x = $9.\dot{9}$ - $0.\dot{9}$; 9x = $9.\dot{0}$; x = $1.\dot{0}$; x = 1. **Another example**: x = $3.4\dot{5}\dot{2}\dot{1}$. **Multiply** by 10² and *subtract* giving 100x-x = $345.\dot{2}\dot{1}$ - $3.4\dot{5}\dot{2}\dot{1}$. So 99x = $341.76\dot{0}$; 9900x = $34176\dot{0}$; x = $^{34176}/_{9900}$ = $^{2848}/_{825}$.

Evaluating Limits of Functions

The **expression** $\lim_{x\to 2}\left[\frac{\sqrt{x+7}-3}{\sqrt{11-x}-3}\right]$ is like $^0/_0$ at x = 2 and so is *not* defined. Note: To sketch a graph of a function like **this**, first sketch the numerator, then the denominator, then form the quotient. You'll also need **information** such as: domain = [-7,11] = {x∈ $\mathbb{R}$ | -7≤x≤11}; at x = 11, value is 1-√2; at x = -7, value is -1-√2.

$\lim_{x\to 3}\left(\frac{\sqrt{x+6}-3}{\sqrt{x+1}-2}\right)$ (at x = 3, it is *undefined*).

$= \lim_{x\to 3}\left(\frac{\sqrt{x+6}-3}{\sqrt{x+1}-2}\right)\left(\frac{\sqrt{x+6}+3}{\sqrt{x+6}+3}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)$

$= \lim_{x\to 3}\left(\frac{(x+6)-3^2}{(x+1)-2^2}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+6}+3}\right) = \lim_{x\to 3}\left(\frac{x-3}{x-3}\right)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+6}+3}\right)$

When x ≠ 3, ($^{x-3}/_{x-3}$) = 1, so we **have**

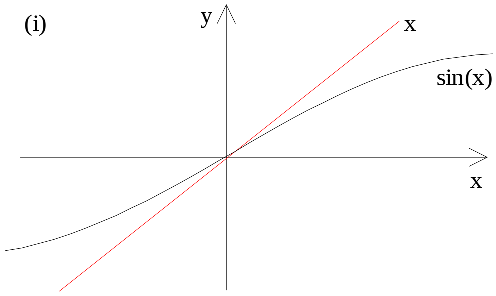
$= (1)\left(\frac{\sqrt{x+1}+2}{\sqrt{x+6}+3}\right) = (1)\left(\frac{\sqrt{4}+2}{\sqrt{9}+3}\right) = \frac{2+2}{3+3} = \frac{4}{6} = \frac{2}{3}$

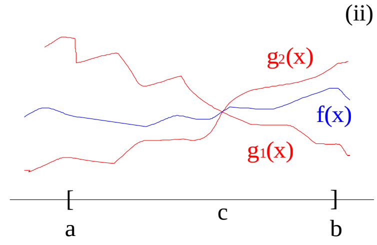
Method (1): *multiply* by 1² and use (a-b)(a+b) = a²-b² to get rid of the square root. Exercise. As shown in the **frame**. Method (2): We can use the *fact* that $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ under **certain** circumstances. So for $\lim_{x\to 1}\left(\frac{\sqrt{3x+1}-2}{\sqrt{8x+1}-3}\right)$, it is equal to $\lim_{x\to 1}\left(\frac{(3x+1)^{-1/2}(1/2)(3)}{(8x+1)^{-1/2}(1/2)(8)}\right) = \frac{(4)^{-1/2}(3/2)}{(9)^{-1/2}(4)} = \frac{(1/2)(3/2)}{(1/3)(4)} = \frac{9}{16}$.

5th February 1999

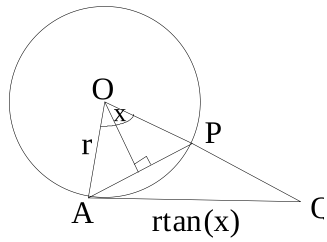
Limit of sin(x) / x and x / sin(x)

Result: $\lim_{x\to 0}\frac{\sin x}{x}=1$. The **squeeze rule**: (Diagram (ii)) Suppose that g_1 & g_2 are functions, continuous on some **interval** [a,b]. Then $g_1(x) \le g_2(x)$ for all $x \in [a,h]$, and $g_1(c) = g_2(c)$ for some $c \in [a,b]$. Further, f(x) is defined on [a,b] (except at c), and $g_1(x) \le f(x) \le g_2(x)\ \forall x \in [a,b]$, **except** at x = c. So $\lim_{x\to c} f(x) = g_1(x)\ (= g_2(c))$.





Area of Triangle OAP ≤ Area of Sector OAP ≤ Area of triangle OAQ. $r^2\sin(x/2)\cos(x/2) \le r^2\times x/2 \le r^2\times \tan(x)/2$. **Multiply** through by $2/r^2$, giving $\sin(x) \le x \le \tan(x)$. **Multiply** through by $(1/\sin(x))$ to give $1 \le x/\sin(x) \le 1/\cos(x)$. Here, $g_1(x) = 1$, $f(x) = x/\sin(x)$ and $g_2(x) = 1/\cos(x)$. [a,b] = [-1,1], say; c = 0 and $g_1(c) = g_2(c) = 1$. **Conclusion:** $\lim_{x\to 0}\frac{x}{\sin x} = 1$.



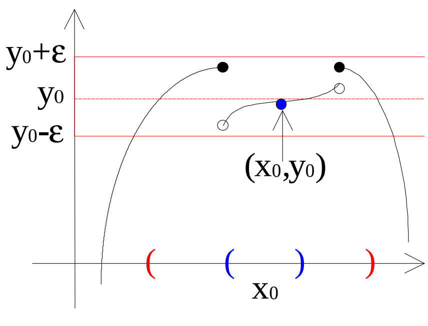
Example: $\lim_{x\to 0}(\tan(4x))(\cot(5x)) = \lim_{x\to 0}(\frac{\cos 5x}{\sin 5x})(\frac{\sin 4x}{\cos 4x}) = \lim_{x\to 0}(\frac{\cos 5x}{\sin 5x})(\frac{\sin 4x}{\cos 4x})(\frac{5x}{5x})(\frac{4x}{4x}) = \lim_{x\to 0}(\frac{5x}{\sin 5x})(\frac{\sin 4x}{4x})(\frac{\cos 5x}{\cos 4x})(\frac{4x}{5x}) = \lim_{x\to 0}(\frac{5x}{\sin 5x})(\frac{\sin 4x}{4x})(\frac{\cos 5x}{\cos 4x})(\frac{4}{5}) = 1\times1\times1\times 4/5 = 4/5$. Or, use de l'Hopital's rule (Differentiate top and bottom **for** $0/0$ problems): $\lim_{x\to 0}\frac{\tan 4x}{\tan 5x} = \lim_{x\to 0}\frac{4\sec^2 4x}{5\sec^2 5x} = 4/5$.

Rules: Where the **limit** exists, (i) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x)+\lim_{x\to a} g(x)$. (ii) $\lim_{x\to a}[f(x)g(x)] = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$. (iii) $\lim_{x\to a}(f(x)/g(x)) = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x))$ **provided** $\lim_{x\to a} g(x) \ne 0$. (iv) The **squeeze** rule: If f is a *polynomial* function, (*sin, cos, exp*, etc.), then $\lim_{x\to a} f(x) = f(a)$.

8th February 1999

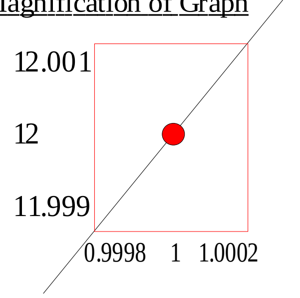
Target Values

General problem: Given (i) $y_0 \in$ image of a function f; (ii) ε small and (sometimes) (iii) x_0 with $f(x_0) = y_0$, find an **interval** contained in the domain of f, for which these *equivelant* conditions hold: $f(x) \in (y_0-\varepsilon, y_0+\varepsilon)$; $|f(x)-y_0| < \varepsilon$. Looking at the graph, The brackets indicate a possible answer.

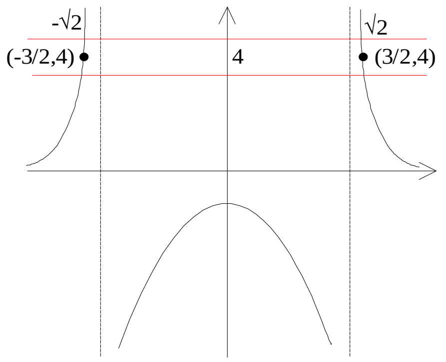


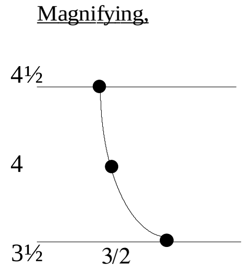
Example (1): f(x) = 5x+7, y_0 = 12; $\varepsilon = 1/1000$. **Solving**, 5x+7 = 12, *implies* x_0 = 1. Looking at the **diagram**, which is a magnification of the graph of the function, the width of the rectangle is $0.002/5 = 0.0004$. So a **possible** solution is {x | |x-1| < 0.0002} = (0.9998, 1.0002).

Magnification of Graph



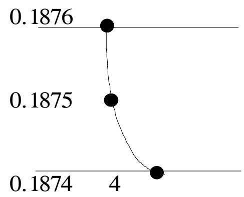
Example (2): $f(x) = {}^1/_{x^2-2}$, $y_0 = 4$, $\varepsilon = {}^1/_2$. **Solving** with ${}^1/_{x^2-2} = 4$, $¼ = x^2-2$, $x^2 = {}^9/_4$; $x = \pm{}^3/_2$. Looking at the *diagrams*, because the curve is steeper to the left of $x = {}^3/_2$, we solve ${}^1/_{x^2-2} = 4{}^1/_2$, $x^2 = 2{}^2/_9$, x » 1.4907 (taking the **+ve** square root).





At $y = 3{}^1/_2$, $2{}^2/_7 = x^2$, and $x \approx 1.51185$. Choose **any** interval inside this e.g. (1.491, 1.509), i.e. 1.5 ± 0.009 or (1.495, 1.505).

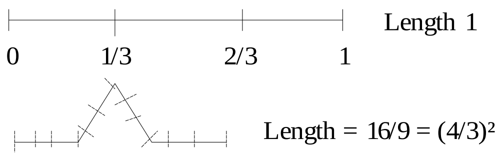
Exercise. If $W = {}^1/_R - {}^1/_{16}$, then W = 0.1875 when R = 4. How **close** to 4 must R be maintained if W is to *differ* from 0.1875 by at most 0.0001? A: $f(x) = {}^1/_x - {}^1/_{16}$, $x_0 = 4$, $y_0 = 0.1875$, $\varepsilon = 0.0001$. W must be in the **interval** 0.1874 to 0.1876. At y = 0.1876, x ≈ 3.99840064. At y = 0.1874, x ≈ 4.00160064. (The **above** uses $y = {}^1/_x - {}^1/_{16}$; $y + {}^1/_{16} = {}^1/_x$; $x = {}^1/_{(y+1/16)}$). **Possible** interval: Wrong Answer = (3.9984, 4.0018); Correct Answer = (3.9985, 4.0015).



10th February 1999

Limit Curves: The Limit of a Sequence of Curves

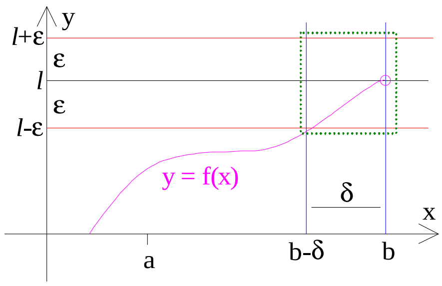
Snowflake curve: Mark ${}^1/_3$d's along each segment, repeat *producing* 4 new equilateral triangles. As we go on, the length of the curve is $({}^4/_3)^n$. **Iterate**, and we get the length of the limit curve as $\lim_{n\to\infty}(\frac{4}{3})^n = \infty$. (Note: we really start with a *triangle* and apply the process to each side).



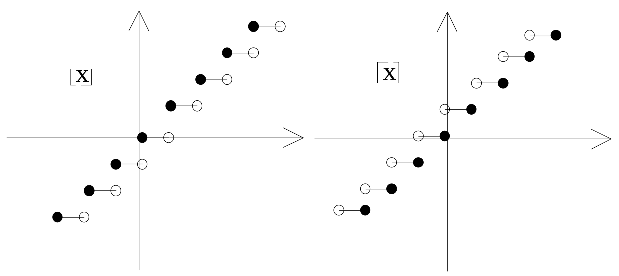
12th February 1999

Left/Right Hand Limits

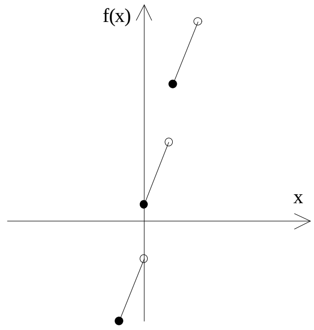
Let (a,b) ⊆ **dom** f. Suppose that *given an* ε, we can find a δ > 0 and an *l*, the limit, such that $|f(x)-l| < \varepsilon$ for all $x \in (b-\delta, b)$. Then *l* is called the **left hand limit** of f at b, and we write $\lim_{x\to b-} f(x) = l$.



Floor & Ceiling. The floor of x (also called the "integer part") *is defined by* $\lfloor x \rfloor$ = greatest integer ≤ x. The **ceiling** of x is defined by $\lceil x \rceil$ = least integer ≥ x. The floor of 5 = the ceiling of 5 = 5. The *floor* of $4{}^1/_3$ is 4, the *ceiling* of $4{}^1/_3$ is 5. The *floor* of -π is -4 and the *ceiling* of -π is -3.

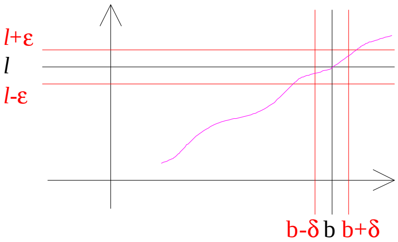


Example: $f(x) = 1+2x+3\lfloor x\rfloor$. *Problem*: find $\lim_{x\to 1-} f(x)$. "Game": one chooses ε, **another** then chooses l and δ. Choose ε = 1. *Suggestion*: l = 3, δ = π. Now x = 0 ∈ (1-π, 1), but f(0) = 1 is outside the ε <u>horizontal</u> strip. Now choose l = 3, δ = $\frac{1}{2}$, and all points are now **inside** the strip. The first person then *chooses* ε = 0.01. 2nd chooses l = 3, δ = 10^{-9}. This carries *on*. **Purpose**: the first person says the limit *doesn't* exist, then the 2nd says *it does* by defining l and δ.



<u>Definition 2:</u> Let (b,c) ⊆ dom f. *Suppose* that given an ε > 0, we can find an l and a δ such that $|f(x)-l| < \varepsilon\ \forall\ x \in (b,b+\delta)$. Then l is called the **right hand limit** of f at b, written as $\lim_{x\to b+} f(x) = l$. *Example*: $\lim_{x\to 1+} 1+2x+3\lfloor x\rfloor = 6$. **<u>Definition 3:</u>** If (a,b)∪(b,c) ⊆ dom f, (Note: b does **not** have to be in the domain), and if $\lim_{x\to b-} f(x) = \lim_{x\to b+} f(x) = l$, we say that *$l$ is the limit of f at x*, and we write $\lim_{x\to b} f(x) = l$.

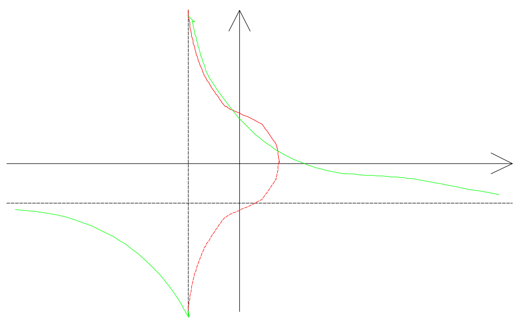
<u>15th February 1999</u>



Alternative Definition of a limit: suppose that, given ε, we can find a δ>0 of l such that $|f(x)-l| < \varepsilon\ \forall\ x \in (b-\delta, b+\delta)\setminus\{b\}$, then f has *limit l* at b.

Example: f(x) = 1.142+x² if x<1; f(x) = π-x² if x>1. <u>Find</u> $\lim_{x\to 1}$ f(x), or show that **no limit** exists. He chooses ε = 0.01. I choose l = 2.142, so we want 2.042<f(x)<0.242. **Slopes** are ±2x = ±2. I choose δ = 0.001. The condition is satisfied. He now chooses ε = 0.0001. If l = -2.1420, **however** small δ is, there is some f(x) ≈ 2.1416. If l = 2.1416, there is some f(x) ≈ 2.1420. In other words, the <u>left</u> limit is not equal to the <u>right</u> limit. In fact, $\lim_{x\to 1-} f(x) = 2.142 \neq \lim_{x\to 1+} f(x) = \pi-1$.

Example: $f(x) = \frac{1-x}{\sqrt{(1-x^2)}}$. Domain of f = (-1,1). Q: **Find** $\lim_{x\to 1} f(x)$ and δ as a *function* of ε. A: For l, $\frac{1-x}{\sqrt{(1-x^2)}} = \frac{\sqrt{(1-x)^2}}{\sqrt{(1-x)}\sqrt{(1+x)}} = \frac{\sqrt{(1-x)}}{\sqrt{(1+x)}} \to 0$ as $x \to 1$ ($x \neq 1$). So l = 0. Now the *definition* requires $\left|\frac{1-x}{\sqrt{(1-x^2)}} - 0\right| < \varepsilon$ for $x \in (1-\delta,1)$. This **implies that** $(1-x) < \varepsilon\sqrt{(1-x^2)}$ since 1-x > 0. This implies that $1-2x+x^2 < \varepsilon^2(1-x^2)$; $(1-x)(1-x-\varepsilon^2(1+x)) < 0$; $(1-\varepsilon^2)-(1+\varepsilon^2)x < 0$; $\frac{1-\varepsilon^2}{1+\varepsilon^2} < x$; $1-\left(\frac{2\varepsilon^2}{1+\varepsilon^2}\right) < x$. So **choose** $\delta \leq \frac{2\varepsilon^2}{1+\varepsilon^2}$. A *sketch* of the graph is shown (Green = $f^2(x) = (f(x))^2 = \frac{(1-x)^2}{1-x^2} = \frac{1-x}{1+x}$). We can **write** $\lim_{x\to(-1)+} f(x) = +\infty$.



<u>17th February 1999</u>

<u>Tutorial</u>

Q: If $A = \pi(\frac{x}{2})^2$, how much **deviation** is required so that the *area* comes within 0.01in² of the desired 9in? i.e. we <u>require</u> |A-9| ≤ 0.01. A: **Lower** limit of area = 8.99, for which x = 3.38325... The **upper** limit of the area is 9.01, for which x = 3.38701.... (x worked out *using* $x = (\sqrt{(\frac{y}{\pi})})\times 2$. So x must be in the **interval** (say) 3.3833 to 3.3870 i.e. 3.3833 < x < 3.3870.

Q: Find $L = \lim_{x\to\infty} \frac{3x^2+4}{6x^2-13}$. Then find N ∈ [4, ∞) such that $|f(x)-L| < 10^{-3}$ for <u>all</u> x > N. A: *<u>Divide</u>* through by x² giving $\lim_{x\to\infty} \frac{3+(4/x^2)}{6-(13/x^2)} = \frac{3+0}{6-0} = \frac{1}{2}$.

We need to **solve** for the *2nd* part $^1/_{10^3} = {}^{3x^2+4}/_{6x^2-13} - {}^1/_2$. So $1 = {}^{3000x^2+4000}/_{6x^2-13} - {}^{1000}/_2 = {}^{3000x^2+4000}/_{6x^2-13} - 500$. <u>Now</u> $501(6x^2-13) = 3000x^2+4000$; $3006x^2-6513 = 3000x^2+4000$; $6x^2 = 10513$; $x = \sqrt{(^{10513}/_6)} = 41.858889935$, so $N \approx 41.86 \approx 42$. Note the following **definition** of $\lim_{x\to\infty} f(x)$: If, given ε, we can **always** find L & N such that $|f(x)-L| < \varepsilon$ for all $x > N$, then we *say* that L is the limit of f(x) as x tends to ∞, and write $f(x)\to L$ as $x \to \infty$.

<u>*19th February 1999*</u>

<u>Sequences</u>

A **real** sequence is an infinite *ordered set* of real numbers a_i, $\{a_1,a_2,...,a_i,....\} = \{a_i\}_{i\geq 1}$, where **each** number may appear *more* than once. We sometimes <u>start</u> at a_0 (or a_{11}, or...) We can think of a **sequence** as a function: $S : ‹^+ \to ‘,\ i \mapsto a_i$. For $\{u_n\}_{n\geq -3}$, where $u_n = {}^{n+3}/_{n+4} = \{0, ½, {}^2/_3, ¾, ...\}$, this sequence *converges* to 1. For $\{(-1)^n\}_{n\geq 2} = \{1,-1,1,-1,1,-1,...\}$, this is an **oscillating** sequence e.g. points on y = cosπx, etc. For $\{n^3+3n-1\}_{n\geq 0} = \{-1,3,13,...\}$, this *diverges* to +∞.

Sequences defined by <u>**Recurrence Relations.**</u> (a) $u_0 = 1$, $u_n = nu_{n-1}$, $n\geq 1$. So $\{u_n\}_{n\geq 0} = \{1,1,2,6,...\}$; $u_n = n!$. (b) $u_0 = 1$, $u_n = 2u_{n-1}$ ($n\geq 1$). So $\{u_n\}_{n\geq 0} = \{1,2,4,8,...\}$; $u_n = 2^n$. (c) [**Fibonacci**] $u_0 = 1$, $u_1 = 1$, $u_n = u_{n-1}+u_{n-2}$ ($n\geq 2$). So $\{u_n\}_{n\geq 0} = \{1,1,2,3,5,8,13,21,34,...\}$. (d) $u_0 = 1$, $u_n = \sum_{i=0}^{n-1} u_i$ ($n\geq 1$) $= u_0+u_1+...+u_{n-1}$. So $\{u_n\}_{n\geq 0} = \{1,1,2,4,8,16,32,...\}$. (e) $u_1 = -1$, $u_n = -\frac{\sqrt{e^{u_{n-1}}}}{3}$. **So** $\{u_n\}_{n\geq 1} = \{-1, -0.35, -0.48, -0.45, -0.46, ...\}$. This is an **iteration** of one root of $e^x = 3x^2$.

<u>Calculation of Limits</u>

Example: $u_n = \sqrt{(n^2-1)}-n$ (Could be thought of as ∞-∞). $\{u_n\}_{n\geq 1} = \{-1, -0.268, -0.172, -0.127, ...\}$. So $u_n = [\sqrt{(n^2-1)} - n][\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}] = \frac{(n^2-1)-n^2}{\sqrt{n^2-1}+n} = \frac{-1}{\sqrt{n^2-1}+n} = \frac{-(1/n)}{\sqrt{(1-(1/n^2))}+1}$. So as n approaches ∞, the expression *becomes* ${}^{-(0)}/_{\sqrt{(1-0)}+1} = {}^0/_2 = 0$. **Example**: $u_n = n\sin(^\pi/_n)$ (Could be thought of as ∞×0). <u>It becomes</u> ${}^{\pi\sin(\pi/n)}/_{(\pi/n)} = \pi.1 = \pi$ as n tends to ∞.

Definition: $\{u_n\}_{n\geq a}$ has limit L, if given $\varepsilon > 0$, there is an **integer** $N > a$ such *that* $|u_n-L| < \varepsilon$ for **all** $n \geq N$. We write "$u_n\to L$ as $n \to \infty$", "$\lim_{n\to\infty} u_n = L$", or "$\{u_n\}_{n\geq a}$ has limit L". We could play "the game" with this i.e. **choosing** ε and N, ε' and N', etc...

<u>*22nd February 1999*</u>

<u>Linear Recurrence Relations</u>

A *linear recurrence relation* of degree r with **constant** coefficients has the form $a_0u_n+a_1u_{n-1}+...+a_ru_{n-r} = f(n)$. **Once** $u_1,u_2,...,u_r$ are *known/chosen*, then $\{u_n\}_{n\geq 1}$ is defined. Example: **Fibonacci**: $u_n-u_{n-1}-u_{n-2} = 0$. Example: $u_n-2u_{n-1} = 7$, $u_1 = 1$ (degree 1). **Then** $u_n = 2u_{n-1}+7$; $u_2 = 2+7 = 9$; $u_3 = (2\times 9)+7 = 25$.

Or, $u_n = 2u_{n-1}+7 = 2(2u_{n-2}+7)+7 = 4(u_{n-3}+7)+7(2+1) = 2^3(2u_{n-4}+7)+7(4+2+1) = 2^{n-2}(2u_1+7)+7(2_{n-3}+...+4+2+1)$. [the red bits sum to n-1]. The term 7(....) is a **geometric** progression of the form $7(2^{n-1}-1)/(2-1) = (8\times 2^{n-1})-7$; so $u_n = 2^{n+2}-7$. Conclusion: **the solution** involves 2^n; the relation is $u_n-2u_{n-1} = 7$.

The **homogenous** recurrence $a_0u_n+a_1u_{n-1}+...+a_ru_{n-r} = 0$ has **characteristic equation** $a_0\lambda^r+a_1\lambda^{r-1}+...+a_{r-1}\lambda+a_r = 0$. If this has **distinct** solutions $\alpha_1, \alpha_2, ..., \alpha_r$, then the *general* solution is $u_n = k_1\alpha_1^n+k_2\alpha_2^n+...+k_r\alpha_r^n$, where the **constants** k_i are determined by $u_1,...,u_r$.

Example: $u_n-5u_{n-1}+6u_{n-2} = 0$. $u_1 = u_2 = 1$. C.E. $\lambda^2-5\lambda+6 = 0$; $(\lambda-2)(\lambda-3) = 0$. $u_n = k_12^n+k_23^n$. **Compare**: $\frac{d^2y}{dx^2}-(5\times\frac{dy}{dx})+6y = 0$ has **C.E.** $\lambda^2-5\lambda+6$; general *solution* $y = k_1e^{2x}+k_2e^{3x}$. Find k_1, k_2 in the **recurrence**. $n = 1 \Rightarrow 1 = 2k_1+3k_2$; $n = 2 \Rightarrow 1 = 4k_1+9k_2$. These *imply that* $k_2 = -\frac{1}{3}$ & $k_1 = 1$. *Therefore*, $u_n = 2^n-\frac{1}{3}3^n = 2^n-3^{n-1}$. But the **roots** may not be distinct: If a_i occurs (t+1) times as a root, then $u_n = k_1\alpha_1^n+...+k_{i-1}\alpha_{i-1}^n+(k_i+nk_{i+1}+...+n^tk_{i+t})\alpha_i^n$, a **polynomial** of degree t in n.

Example: $u_n+u_{n-1}-2u_{n-2}-2u_{n-3}+u_{n-4}+u_{n-5} = 0$ has C.E. $\lambda^5+\lambda^4-2\lambda^3-2\lambda^2+\lambda = 0$; $(\lambda+1)^3(\lambda-1)^2 = 0$. **General** solution: $u_n = (k_1+nk_2+n^2k_3)(-1)^n+(k_4+nk_5)(1)^n$. Compare: $y'''''+y''''-2y'''-2y''+y'+y = 0$ has **general** solution $y = (k_1+k_2x+k_3x^2)e^{-x}+(k_4+k_5x)e^x$. Example: **Fibonacci**: $u_n-u_{n-1}-u_{n-2} = 0$; C.E. $\lambda^2-\lambda-1 = 0$; $\lambda = \frac{1}{2}(1\pm\sqrt{5})$. So $u_n = k_1(\frac{1+\sqrt{5}}{2})^n+k_2(\frac{1-\sqrt{5}}{2})^n$. **For large** n, we have u_n = large + tiny. We *substitute* for n = 0 & 1 to get <u>values</u> for k_1 & k_2, and **hence** $u_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}$. For example, $u_{11} = 144.0013-0.0013 =$ **144**.

Note: a **binomial** expansion of both sides gives $2^nu_n = (n+1)+\binom{n+1}{3}5+\binom{n+1}{5}5^2+...$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. So, for example, for u_6, $2^6u_6 = 7 + \binom{7}{3}5 + \binom{7}{5}5^2 + \binom{7}{7}5^3$; ..., $u_6 = 832$.

<u>24th February 1999</u>

If $u_n+3u_{n-1}+2u_{n-2} = 0$ (n>3), and $u_1 = 1$, $u_2 = -1$, calculate up to u_6 and **solve** the recurrence. A: *Substitute* into the formula to get {1, -2, 4, -8, 16, -32}. Now we solve as follows: $u_n+3u_{n-1}+2u_{n-2} = 0$; $\lambda^2+3\lambda+2 = 0$, $(\lambda+1)(\lambda+2) = 0$; $\lambda = -1$ or -2. **Thus** $u_n = k_1(-1)^n + k_2(-2)^n$. Substitute in *values*: $n = 1 \Rightarrow 1 = k_1(-1)^1 + k_2(-2)^1$; $1 = -k_1-2k_2$. And $n = 2 \Rightarrow -2 = k_1(-1)^2+k_2(-2)^2$; $-2 = k_1+4k_2$. From these we **get** $k_1 = 0$, $k_2 = -\frac{1}{2}$; so $u_n = -\frac{1}{2}(-2)^n = (-2)^{n-1}$.

Q: Solve the **recurrence** $u_n+3u_{n-1}+2u_{n-2} = n+1$ ($u_1 = 1$, $u_2 = -2$, ($n\geq3$)). A: **The general** solution of a non-homogenous recurrence is a **particular** solution of the recurrence + the **general** solution of the associated homogenous recurrence. How do we find the *particular* solution: if the polynomial on the RHS (in n) is of degree d, there is usually a polynomial <u>solution</u> of degree d. If one of the roots of the C.E. is a root of the RHS, then a **degree of** d+1 or more is required.

Now **here**, $u_n+3u_{n-1}+2u_{n-2} = n+1$; the C.E. is $(\lambda+1)(\lambda+2) = 0$; RHS = (n+1). **And**, -1 is a root. We expect a *particular* solution of order an^2+bn+c. Try $u_n = an^2+bn+c$. So substituting, $(an^2+bn+c) + 3(a(n-1)^2+b(n-1)+c) + 2(a(n-2)^2+b(n-2)+c) = n+1$; *rearranging* gives $6an^2+6bn-14an+6c-7b+11a = n+1$, which is *satisfied* by $a = 0$, $b = \frac{1}{6}$, $c = \frac{13}{36}$. This is the **particular** solution: $u_{particular} = \frac{n}{6}+\frac{13}{36}$.

Now for the **general** solution: $u_n = k_1(-1)^n + k_2(-2)^n + (\frac{n}{6}+\frac{13}{36})$. We *get* k_1 and k_2 by substitution again: $n = 0 \Rightarrow 1 = k_1+k_2+\frac{13}{36}$; $n = 1 \Rightarrow 1 = -k_1-2k_2+\frac{13}{36}$; so $k_1 = \frac{7}{4}$ & $k_2 = -\frac{10}{9}$. Thus we **have** $u_n = \frac{7}{4}(-1)^n - \frac{10}{9}(-2)^n + (\frac{n}{6}+\frac{13}{36})$. We can **check** the formula (*advisable*) by <u>substituting</u> in e.g. n = 6, and checking by **calculating** it in the *original* formula.

26th February 1999

Matrix Method for Fibonacci (Optional)

The recurrence relation is $u_n - u_{n-1} - u_{n-2} = 0$; $u_0 = u_1 = 1$. **Rewrite** this as $\begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_{n-1} \\ u_{n-2} \end{bmatrix}$, $\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. $\mathbf{v}_n = A\mathbf{v}_{n-1}$, where $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. First order **recurrence** for 2-element vectors. So $\mathbf{v}_n = A\mathbf{v}_{n-1} = A(A\mathbf{v}_{n-2}) = A^2(v_{n-2}) = \ldots$; $v_n = A^{n-1}v_1$. How do we **calculate** A^n? Suppose that we can *find* an invertible 2×2 matrix T s.t. $T^{-1}AT = D$ and $A = TDT^{-1}$ (D is *diagonal*). Then $A^n = (TDT^{-1})(TDT^{-1})(TDT^{-1})\ldots$ n times. But we **cancel** the matching colours, leaving $A^n = TD^nT^{-1}$.

Matrix Theory Result: $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ has *characteristic equation* $x^2 - (a+d)x + (ad-bc) = 0$. If this quadratic has **different** solutions λ & μ, then $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and $A = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$; both **have** rank 1. So we find **y** & **z**, where $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\mathbf{y} = 0$ and $\begin{bmatrix} a-\mu & b \\ c & d-\mu \end{bmatrix}\mathbf{z} = 0$. And so we **take** T = [**y** | **z**] and $D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. In **our** example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ so the C.E. is $x^2-(1+0)x+(0-1) = 0$; $x^2-x-1 = 0$; $\lambda, \mu = {}^{1\pm\sqrt{5}}/_2$.

Now $A-\lambda I = \begin{bmatrix} (+1-\sqrt{5})/2 & 1 \\ 1 & (-1-\sqrt{5})/2 \end{bmatrix}\begin{bmatrix} (1+\sqrt{5})/2 \\ 1 \end{bmatrix}$; and $A-\mu I = \begin{bmatrix} (1+\sqrt{5})/2 & 1 \\ 1 & (-1+\sqrt{5})/2 \end{bmatrix}\begin{bmatrix} (1-\sqrt{5})/2 \\ 1 \end{bmatrix}$. $T = \begin{bmatrix} (1+\sqrt{5})/2 & (1-\sqrt{5})/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix}$. Now $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = {}^1/_{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, so $T^{-1} = {}^1/_{(\lambda-\mu)}\begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix} = -{}^1/_{\sqrt{5}}\begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix}$. Now $\mathbf{v}_n = A^{n-1}\mathbf{v}_1 = TD^{n-1}T^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda^{n-1} & 0 \\ 0 & \mu^{n-1} \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda^n & \mu^n \\ \lambda^{n-1} & \mu^{n-1} \end{bmatrix}\begin{bmatrix} 1 & -\mu \\ -1 & \lambda \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda^n-\mu^n & -(\mu\lambda^n-\lambda\mu^n) \\ \lambda^{n-1}-\mu^{n-1} & -(\mu\lambda^{n-1}-\lambda\mu^{n-1}) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. **But** $\lambda\mu^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda^n-\mu^n & \lambda^{n-1}-\mu^{n-1} \\ \lambda^{n-1}-\mu^{n-1} & \lambda^{n-2}-\mu^{n-2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda^n+\lambda^{n-1} & -\mu^n-\mu^{n-1} \\ \lambda^{n-1}+\lambda^{n-2} & -\mu^{n-1}-\mu^{n-2} \end{bmatrix}$.

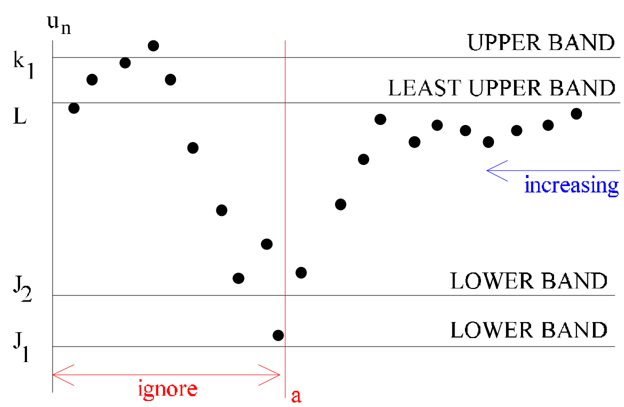
Is it Convergent / Divergent / Oscillating?

Let $u_n = (3n^4+1)/(4n^3+1)$. Think of $(3n^4+1)/(4n^3+1)$ as $3n^4/4n^3 = {}^{3n}/_4$ when n is large — this tends to ∞ for large n. Or, dividing by n^3, we get ${}^{(3n+(1/n^3))}/_{(4+(1/n^3))}$, which when n tends to ∞ is ${}^{3n+0}/_{4+0}$, which tends to ∞. So it is <u>divergent</u>. **Alternatively**, using de *l'Hopital's* rule, $\lim_{n\to\infty}\frac{3n^4+1}{4n^3+1} = \lim_{n\to\infty}\frac{12n^3}{12n^2} = \lim_{n\to\infty} n$.

1st March 1999

Upper & Lower Bounds

Consider a **sequence** $S = \{u_n\}_{n\geq a}$. If $u_{n+1} \geq u_n$ for all $n \geq a$, then S is **increasing**. If ≤, then it is *decreasing*. Replacing "≥, ≤" by ">, <" gives the definitions of <u>strictly</u> increasing and <u>strictly</u> decreasing. Note: A **constant** sequence is increasing and decreasing. Now if $u_n \leq K$ for all $n \geq a$, then S has **upper** bound K. If $u_n \geq J$ for all $n \geq a$, then S has **lower** bound J. The least upper bound for S is called the *suprenum*, and the greatest lower bound is called the *infinum*.



<u>**Axiom**</u>. A real sequence, $\{u_n\}_{n\geq a}$, which is **increasing** and which has an upper bound K, has a *limit*, L ≤ K. A real sequence, $\{u_n\}_{n\geq a}$, which is **decreasing** and which has a lower bound J, has a limit L ≥ J. Example: $\{u_n\}_{n\geq 1}$, defined by $u_1 = 1$, $u_{n+1} = {}^{3u_n+4}/_{2u_n+3}$. So $\{u_n\}_{n\geq 1} = \{1, {}^7/_5, {}^{41}/_{29}, \ldots\}$. Questions: **Prove** that (i) $u_n \geq 1$ (so 1 is a lower bound); (ii) $u_{n+1} > u_n$ (increasing sequence), (iii) $0 \leq 2-u_n^2 \leq 1/(5^{2n-2})$. Hence find an N s.t. $2-u_n^2 < 10^{-6}$ ∀ n > N. Also **prove** that $0 \leq u_n \leq {}^3/_2$.

Note: If the sequence has **limit** L, substitute in the recurrence: $L = {}^{3L+4}/_{2L+3}$; $2L^2+3L = 3L+4$; $2L^2-4 = 0$, $L = \pm\sqrt{2}$. For the last part, we want $u_n-{}^3/_2 < 0$ i.e. to be -ve. **Using** $u_{n+1}-{}^3/_2 < 0$, this gets *transformed* to be $[(3u_n+4)/(2u_n+3)]-{}^3/_2 < 0$, $(6u_n+8-6u_n-9)/(2(2u_n+3)) < 0$, $-1/(4u_n+6) < 0$. This is true because the **denominator** is +ve as long as n is +ve.

Now *show* that $(u_{n+1}-u_n) > 0$ for all n. Try $u_{n+1}-u_n = [(3u_n+4)/(2u_n+3)]-[(3u_{n-1}+4)/(2u_{n-1}+3)] = [6u_nu_{n-1}+9u_n+8u_{n-1}+12-6u_nu_{n-1}-8u_n-9u_{n-1}-12]/[(2u_n+3)(2u_{n-1}+3)] = (u_n-u_{n-1})/[(2u_n+3)(2u_{n-1}+3)]$. Numerator is +ve by **hypothesis**. Since u_n is +ve for all n, the *denominator* is +ve. So the whole thing is +ve. We have now proved that $\lim_{n\to\infty}$ exists, and that it is less than or equal to ${}^3/_2$.

To show that $\sqrt{2}$ is the limit, we work **with** $(2-u_n^2)$. Show that this has limit 0. Consider $0 \le 2-u_n^2$. When n = 1, $2-u_n^2 = 2-1 = 1 > 0$. So $2-u_{n+1}^2 = 2-[(2u_n+4)/(2u_n+3)^2] = ... = (2-u_n^2)/(2u_n+3)^2 > 0$ by hypothesis. Now $2-u_n^2 \le -1/5^{(2n-2)}$. When n = 1, $2-u_n^2 = 1 \le 1/5^0$. **Assume** that $2-u_n^2 \le 1/5^{(2n-2)}$, then $2-u_{n+1}^2 = (2-u_n^2)/(2u_n+3)^2 \le 1/5^{(2n-2)} = 1/5^{2(n-1)}$. *Now* $|2-u_0^2| < 10^{-6}$ if $5^{2n-2} > 10^{-6}$. So $2u_n-2 > \log 10^{-6}/\log 5$ is true if $u_0 > 5.29....$ *Choose* N = 6. **Conclusion**: $\lim_{n\to\infty} u_n = \sqrt{2}$.

Assignment 2: Notes

This **identity** is useful: $(a^n-b^n) = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-2} + b^{n-1})$. Be careful about the **distinction** between left hand limits, right hand limits and limits at a value 'a'. Draw *graphs* for clarification. Warning: never write "=" when using a **decimal** approximation. Use equivalence or the symbol "≈".

Solving Inequalities

Basic Properties. I1: $a < b$ *implies* $a+x < b+x$ for all real x. I2: $a < b$ and $k > 0$ *implies* $ka < kb$. And $a < b$ with $k < 0$ implies $ka > kb$. I3: $a^2 > 0$ for all $a \neq 0$. I4: $0<a<b$ and $0<x<y$ imply $0<ax<by$. Example: Solve ${}^1/_x \geq 3$. Two cases: $x>0$ and $x<0$. (i) $x>0$: ${}^1/_x \geq 3$ iff $a \geq 3x$ by I2(i). This is iff ${}^1/_3 \geq x$ and iff $x \in (-\infty, {}^1/_3]$. Since $x>0$, we have $x \in (0, {}^1/_3]$. (ii) $x<0$: ${}^1/_x \geq 3$ iff $1 \le 3x$ (by I2(ii)) iff ${}^1/_3 \le x$ iff $x \in [{}^1/_3, \infty)$. But since $x<0$, there are **no** solutions. Combining (i) & (ii), we get the *solution* set $(0, {}^1/_3]$.

Set 1: Solve $|x+1| \geq 3$. There are *2 cases to consider:* $x \le -1$ & $x \geq -1$. **Set** 2: Solve $|x+2| - 2|x+1| > -3$ (3 cases). **Example** 2: Solve $(x-\alpha)(x-\beta)(x-\chi)^2(x-\delta) \geq 0$ given that $\alpha<\beta<\chi<\delta$. When a polynomial function is **completely** factorised, it is easy to sketch (Loch-Ness monster type graph). Then read off the *solution* set: $x \in [\alpha,\beta] \cup \{\chi\} \cup [\delta, \infty)$. **Set** 3: *Solve* $x^4-x^2 > x^3-x$.

Example 3: Solve $\left|\frac{\sqrt{x+3}-2}{x-1} - \frac{1}{4}\right| < \varepsilon$. This comes from *finding* $\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right) = \frac{1}{4}$, so substitute $t = x-1$, $x = t+1$: $|(({}^{\sqrt{(t+4)}-2}/_t)-{}^1/_4| = |{}^{(4(\sqrt{(t+4)}-8-t)}/_{4t}| < \varepsilon$. So $y = 4\sqrt{(t+4)}-8 \Rightarrow \dot{y} = {}^2/_{\sqrt{(t+4)}} = 1$ at t = 0. Case $t > 0$: $-4\sqrt{(t+4)}+8+t < 4t\varepsilon \Rightarrow 8+t-4t+\varepsilon < 4\sqrt{(t+4)} \Rightarrow 64+16(t-4t\varepsilon) + (t-4t\varepsilon)^2 < 16(t+4) \Rightarrow t^2(1-4\varepsilon)^2 < 64t\varepsilon \Rightarrow t < {}^{64\varepsilon}/_{(1-4\varepsilon)^2} \Rightarrow x < 1+{}^{64\varepsilon}/_{(1-4\varepsilon)^2}$. **Set** 4: **Complete** this example by considering the *case* $t < 0$.

3rd March 1999

Worksheet: Convergence of Sequences

Q: A **sequence** $\{u_n\}_{n\ge1}$ is defined by $u_1 = 1/\sqrt{2}$; $u_{n+1} = \sqrt{(1/2+1/2u_n)}$ ($n\ge1$). Prove that (a) $u_n < 1$, (b) $u_n < u_{n+1}$, and (c) $\lim_{n\to\infty} u_n = 1$. A: (a) u_1 is **approximately** 0.7 < 1. Assume that $u_n < 1$. Then $\frac{1}{2}u_n < \frac{1}{2}$. And $1/2+1/2u_n < 1$. So $\sqrt{(1/2+1/2u_n)} = u_{n+1} < 1$. (b) We want to show that $u_{n+1}-u_n > 0$. This is *true* for u_2-u_1. *Assume true for* $u_n-u_{n-1} > 0$. Then $u_{n+1}-u_n = \sqrt{(1/2+1/2u_n)} - \sqrt{(1/2+1/2u_{n-1})} = 1/\sqrt{2}\{\sqrt{(1+u_n)}-\sqrt{(1+u_{n-1})}\}\{\sqrt{(1+u_n)}+\sqrt{(1+u_{n-1})}/\sqrt{(1+u_n)}+\sqrt{(1+u_{n-1})}\} = (1+u_n)-(1+u_{n-1})/\sqrt{2}(\sqrt{(1+u_n)}+\sqrt{(1+u_{n-1})}) = (u_n-u_{n-1})/\sqrt{2}(\sqrt{(1+u_n)}+\sqrt{(1+u_{n-1})}) >$ 0 by *assumption*.

From (a) & (b), we **know** that $\{u_n\}$ is an *increasing* sequence with upper bound 1, so $\lim_{n\to\infty}$ exists and is ≤ 1. To **find** the limit L, substitute in the recurrence: $L = \sqrt{(1/2+1/2L)}$; $L^2 = 1/2+1/2L$; $2L^2-L-1 = 0$; $(L-1)(2L+1) = 0$; $L = 1$ or $-1/2$ — we **want** L = 1 since L > 0.7. A **second** sequence $\{t_n\}_{n\ge0}$ is defined by $t_0 = 1$, $t_n = t_{n-1}.u_n$ ($n\ge1$). The **first** 5 terms are 1, u_1, u_1u_2, $u_1u_2u_3$, $u_1u_2u_3u_4$, etc. This was used to **estimate** π, as the limit of this sequence as n tends to ∞ is $2/\pi$.

Series

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$, an *infinite* series. $S_k = a_1+a_2+...+a_k$ is a **partial** sum. If $\lim_{k\to\infty} S_k$ exists, it is called the **sum** of the series. Example: $\sum_1^{\infty} 1/_{n^2+4n+3} = 1/8+1/15+1/24+...$ Now $1/_{n^2+4n+3} = 1/_{(n+1)(n+3)}$ = by **partial fractions** $^{1/2}/_{n+1} - {}^{1/2}/_{n+3} = \frac{1}{2}(1/_{n+1}-1/_{n+3})$. So $S_k = \frac{1}{2}(1/2-1/4) + \frac{1}{2}(1/3-1/5) + \frac{1}{2}(1/4-1/6) + ... + \frac{1}{2}(1/_{k+1}-1/_{k+3})$. **Cancellation**: *Everything* cancels except $\frac{1}{2}[1/2+1/3-1/_{k+2}-1/_{k+3}]$. So S_k (as k **tends** to ∞) is $\frac{1}{2}[1/2+1/3-0-0] = 5/12$. (Can use the *cover-up rule* instead of partial fractions).

5th March 1999

Reminder: **Sequence** = $\{a_i\}_{i\ge1}$; Series = $\sum_{i=1}^{\infty} a_i$. A sequence of partial sums is denoted by $\{S_n\}_{n\ge1}$, where $S_n = \sum^{k}_{i=1} a_i$. $\lim_{k\to\infty} S_k = S$ exists **iff** $\sum_{i=1}^{\infty} a_i$ is *convergent*. Proposition: if $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$. Proof: $\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0$. Note: The converse is FALSE.

Computational problem: Suppose $a_i = f(i)$. How do we compute $\sum_{i=1}^{\infty} a_i$? **Naive** method: (for a +ve decreasing sequence): *Program*: `i=1; a=f(1); SUM:=1; while (a>10^(-12)) do {i:=i+1; a:=f(1); SUM:=SUM+a;} od; return Sum;`. This is not good **because** (term < 10^{-12}) does not *imply* (error < 10^{-12}). The error is |estimate - true limit|. If you know a **function** E such that E(i) is an *upper* bound for $|S_i-S|$, then *compute* with `while (E(i) > 10^(-12)) do...`

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$ is divergent. (And so $\sum_{n=10^{-\infty}}^{\infty} 1/_n$ is *divergent*). Proof: $\sum_{n=1}^{\infty} \frac{1}{n} = 1+1/2+1/3+1/4 + ... > 1 + (1/2) + (1/4+1/4) + (1/8+1/8+1/8+1/8) + ... = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$ which is **divergent** ($1/2$ does not tend to 0). Diverges very *slowly*: Need 10^8 terms to get to 20! However, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...$ is **convergent**.

Series of Positive Terms (SPT)

When $a_n > 0$ for all n, $S_k = S_{k-1} + a_k > S_{k-1}$. So $\{S_k\}$ is increasing. So $\sum a_n$ is *convergent* provided $\{S_k\}$ has an upper bound.

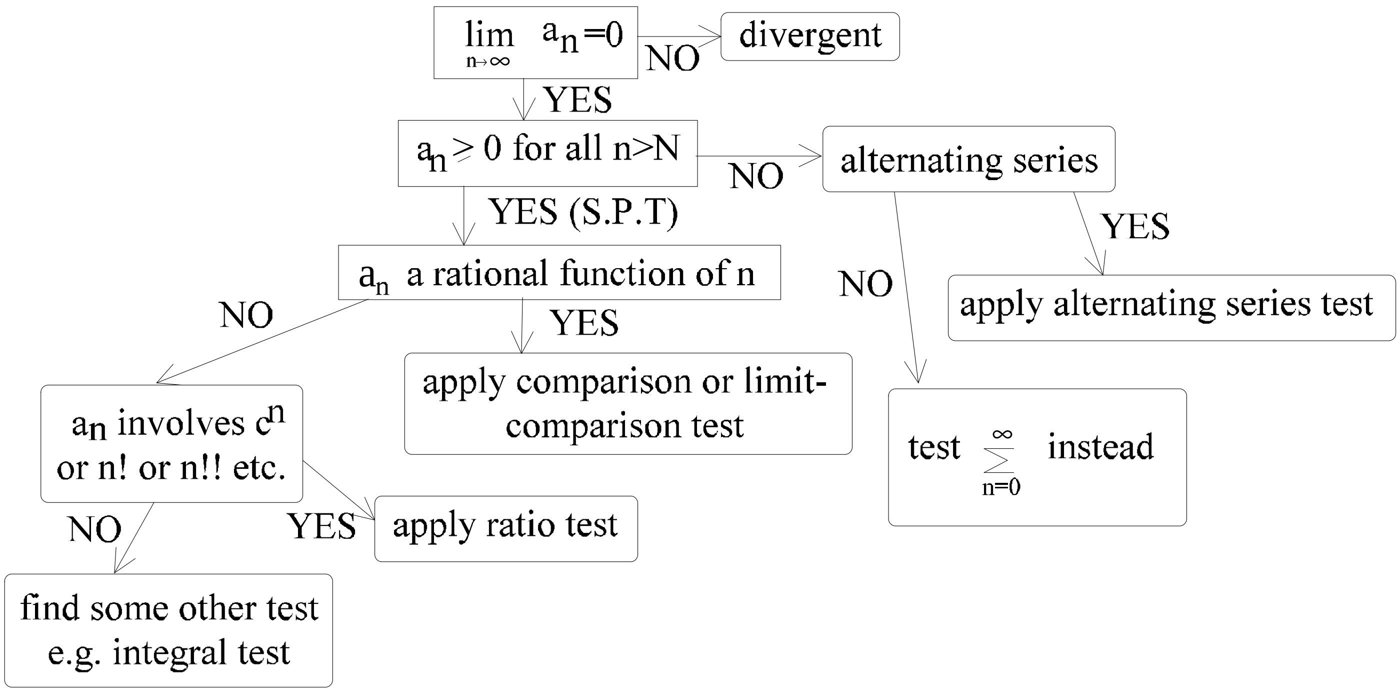
Example: Geometric Series. In $\Sigma_{n=0}^{\infty} a_n$, let $a_n = a^n$ (n > 0) so $S_k = 1+a+a^2+a^3+...+a^k = (1-a^{k+1})/(1-a)$. When 0<a<1, a^{k+1} tends to 0 as k **tends** to ∞. And S_k tends to ${}^{1-0}/_{1-a} = ({}^{1}/_{1-a})$ as k tends to ∞. When a ≥ 1, the series is <u>divergent</u>.

Example: $\Sigma_{n=1}^{\infty} \frac{1}{n^p}$, **where** p > 1. This series is *convergent*. $\Sigma_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + ...$ $< \quad 1 + (\frac{1}{2^p} + \frac{1}{2^p}) + (\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}) = 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + ... = 1 + (\frac{1}{2^{p-1}}) + (\frac{1}{2^{p-1}})^2 + ... + (\frac{1}{2^{p-1}})^n + ...$ Note: at the **start** of the inequality, (in $(\frac{1}{2^p} + \frac{1}{2^p})$), the <u>first</u> term comes from $\frac{1}{2^p}$ and the <u>second</u> term comes from $\frac{1}{3^p}$. (p > 1 so $0<\frac{1}{2^{p-1}}<1$). So this **series** has sum $\frac{1}{1-(\frac{1}{2^{p-1}})}$.

So our series has an **upper** bound and is **convergent**. (The upper bound may be a *poor* estimate). Case p = 2: $\Sigma_{n=1}^{\infty} \frac{1}{n^2} \simeq 1.65$ (In fact ${}^{\pi^2}/_6$). Case p = 4: $\Sigma_{n=1}^{\infty} \frac{1}{n^4} \simeq 1.09$ (in fact $\pi^4/90$). **Example**: Σ $1/(n^p)$ is *divergent* when 0 < p ≤ 1.

8th March 1999

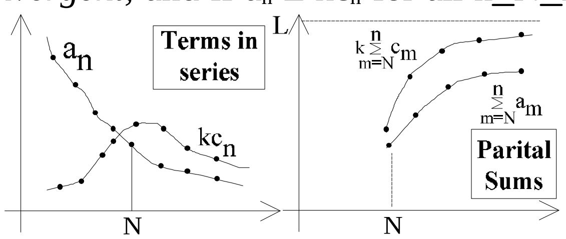
Testing the Series $\Sigma_{n=0}^{\infty} a_n$ for Convergence



For the **power series** $\Sigma_{n=0}^{\infty} a_n x^n$, find ${}^{1}/_{R} = \lim_{n\to\infty} |a_{n+1}/a_n|$. **Convergent** for x ∈ (**-R**, **R**). Test x = **R** and x = **-R** *separately*.

Series of Positive Terms

(1) **Comparison** Test. *Theorem* 1: If $\Sigma^{\infty}_{n=i} C_n$ is *convergent*, and if $a_n \le kc_n$ for all n≥N≥i, then $\Sigma^{\infty}_{n=N} a_n$ is <u>convergent</u>. **Proof**. Let $S_n = \Sigma^{n}_{m=N} c_m$. Since $\Sigma^{\infty}_{m=N} c_m$ is *convergent*, $\{S_n\}_{n\ge N}$ has an upper bound K. Then $t_n = \Sigma^{n}_{m=N} a_m \le kc_N+kc_{N+1}+...+kc_n = kS_n$, so $\{t_n\}_{n\ge N}$ has upper bound kK. **Hence** $\Sigma^{\infty}_{m=N} a_m$ is convergent, with *limit* L, say. For i < N, $\Sigma^{\infty}_{m=i}$ is convergent with limit $a_i+..+a_{N-1}+L$.



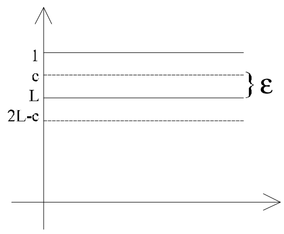
Theorem 2: If $\Sigma^{\infty}_{n=1} d_n$ is <u>divergent,</u> and $a_n \geq kd_n$ for all $n \geq N \geq i$, then $\Sigma^{\infty}_{n=N} a_n$ is <u>divergent</u>. Used for **rational** functions; *square roots* of these, etc. (e.g. $\Sigma_{n=1}^{\infty}\frac{3n^2+2}{4n^4-2n+1}, \Sigma_{n=3}^{\infty}\frac{n-9}{\sqrt{n^3+11}}$). <u>Algorithm:</u> Ignore all but the **highest** degree terms in the *numerator & the denominator*. Write the result as $\Sigma\, c/n^p$. Now use the fact that $\Sigma\, 1/n^p$ is convergent *when* $p > 1$ and divergent *when* $p \in (0,1]$.

Case $p > 1$: Show that $\Sigma a_n < \Sigma 2c/n^p$ for $n>N$. **Case** $0<p\leq 1$: Show that $\Sigma a_n > \Sigma c/2n^p$ for $n>N$. **Example**: $\Sigma a_n = \Sigma^{\infty}_{n=0}(^{12n+4}/_{n^3+1}) \sim \Sigma^{12}/_{n^2}$. Then $^{12n+4}/_{n^3+1} < {}^{24}/_{n^2}$ when $12n^3+4n^2 < 24n^3+24$, or $4n^2 < 12n^3+24$, which is **true** when $n \geq 0$. Hence the series is *convergent*. Example: $\Sigma a_n = \Sigma^{\infty}_{n=7}\, {}^{\sec(3\pi/n)}/_{2n}$. When $n \geq 7$, $0 < \theta = {}^{3\pi}/_n < {}^{\pi}/_2$, so $0 < \cos\theta < 1$ and $\sec\theta > 1$. So $^{\sec(3\pi/n)}/_{2n} > {}^1/_2.({}^1/_n)$; series is divergent.

Exercises: (i) $\Sigma^{\infty}_{n=5}(^{5n+18}/_{n^3-100}) \sim \Sigma^5/_{n^2}$. Then $^{5n+18}/_{n^3-100} < {}^{10}/_{n^2}$ when $5n^3+18n^2 < 10n^3-1000$; $18n^2 < 5n^3-1000$ which is **true** for $n > 7$. Hence the series is <u>convergent</u>. (ii) $\Sigma^{\infty}_{n=1}\, {}^{n+1}/_{\sqrt{(n^3+1)}} \sim {}^n/_{\sqrt{n^3}} \sim \Sigma^1/_{\sqrt{n}}$. Then $^{n+1}/_{\sqrt{(n^3+1)}} > {}^1/_{2\sqrt{2}}$ when $2\sqrt{[(n)(n+1)]} > \sqrt{(n^3+1)}$; $2\sqrt{[(n)(n+1)]} > \sqrt{(n^3+1)}$; $2n(n+1)^2 > n^3+1$; $4n(n^2+2n+1) > n^3+1$; $4n^3+8n^2+4n > n^3+1$, which is **true** for $n \geq 1$.

<u>D'Alembert's Ratio Test</u>

Theorem 3. **Let** $\Sigma^{\infty}_{n=i} a_n$ be a S.P.T., and **suppose that** $\lim_{n\to\infty}(\frac{a_{n+1}}{a_n}) = L$. Then when $L < 1$, Σa_n is convergent. When $L = 1$, *no conclusion* (try <u>another</u> test). $L > 1$: Σa_n is divergent. **Proof** for case $L < 1$: Let $c = {}^1/_2(1+L)$ (mid-point). From the *definition* of a limit, ∃ N s.t. $2L-c \leq a_{n+1}/a_n \leq c$ for all $n \geq N \geq i$. Then $a_n = (a_n/a_{n-1}).(a_{n-1}/a_{n-2})......(a_{N+1}/a_N) \leq c^{n-N}a_N$. Since $0<c<1$, $a_N\Sigma^{\infty}_{n=N} c^{n-N} = a_N\Sigma^{\infty}_{m=0} c^m = (a_N/1-c)$. Hence Σa_n is *convergent* by <u>comparison</u> with $a_N\Sigma c^m$.



The above is used for series where the **terms** involve 3^n, $(n+s)!$, etc. Note: *rational* functions of n, etc., have limit $a_{n+1}/a_n \to 1$, so the ratio test gives **no** information (use the *comparison* test instead). **Example**: $\Sigma^{\infty}_{n-1}\ 2^n/(n^4+3)$. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(\frac{2^{n+1}}{(n+1)^4+3})(\frac{n^4+3}{2^n}) = \lim_{n\to\infty}(\frac{2^{n+1}}{2^n})(\frac{1+(3/n^4)}{(1+(1/n))^4+(3/n^4)}) = (2)(\frac{1+0}{1+0}) = 2$. Since $2 > 1$, the series is <u>divergent</u> (Note: the rational part **always** has limit 1).

When k is an *odd integer*, we define $k!! = k(k-2)(k-4)...5.3.1. = k.(k-2)!!$, **so** $(2n-1)!! = (2n-1)(2n-3)...5.3.1 = (2n-1).(2n-3)!!$ (e.g. $3!!=3$, $5!!=15$, $7!!=105$).$\Sigma_{n=1}^{\infty}\frac{\{(2n-1)!!\}^2}{(4n-1)!!} = \frac{1}{3} + \frac{3^2}{105} + ...$. **Now** $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}[\frac{\{(2n+1)!!\}^2}{(4n+3)!!}][\frac{(4n-1)!!}{\{(2n-1)!!\}^2}] = \lim_{n\to\infty}[\frac{(2n+1).(2n-1)!!}{(2n-1)!!}]^2[\frac{(4n-1)!!}{(4n+3)(4n+1).(4n-1)!!}] = \lim_{n\to\infty}\frac{(2n+1)^2}{(4n+3)(4n+1)} = \lim_{n\to\infty}\frac{(2+(1/n))^2}{(4+(3/n))(4+(1/n))} = \frac{2^2}{4.4} = \frac{1}{4} < 1$ so the series is **convergent**.

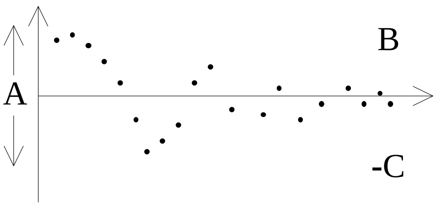
Exercise: $\Sigma^{\infty}_{n=2}(^{4n+7}/_{3\wedge n})$. $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty}(\frac{4(n+1)+7}{3^{n+1}})(\frac{3^n}{4n+7}) = \lim_{n\to\infty}(\frac{3^n}{3^{n+1}})(\frac{4n+4+7}{4n+7}) = \lim_{n\to\infty}(\frac{1}{3})(\frac{4n+11}{4n+7}) = \lim_{n\to\infty}(\frac{1}{3})(\frac{4+(11/n)}{4+(7/n)}) = (\frac{1}{3})(\frac{4+0}{4+0}) = \frac{1}{3}$. **Since** $^1/_3 > 1$, the series is *divergent*.

Q: $\Sigma_{n=1}^{\infty}\frac{(n+1)!(2n)!}{(3n+2)!}$. So $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}[\frac{(n+2)!(2n+2)!}{(3n+5)!}][\frac{(3n+2)!}{(n+1)!(2n)!}] = \lim_{n\to\infty}[\frac{(n+2)(n+1)!(2n+2)!}{(3n+5)!}][\frac{(3n+2)!}{(n+1)!(2n)!}] = \lim_{n\to\infty}[\frac{(n+1)(2n+1)(2n+1)(2n)!}{(3n+5)!}][\frac{(3n+2)!}{(2n)!}] = \lim_{n\to\infty}[\frac{(n+2)(2n+2)(2n+1)(2n+2)!}{(3n+5)(2n+4)(3n+3)(3n+2)!}] = \lim_{n\to\infty}[\frac{(n+2)(2n+2)(2n+1)}{(3n+5)(3n+4)(3n+3)}] = \lim_{n\to\infty}[\frac{(1+(2/n))(2+(2/n))(2+(1/n))}{(3+(5/n))(3+(4/n))(3+(3/n))}] = \frac{(1+0)(2+0)(2+0)}{(3+0)(3+0)(3+0)} = \frac{1.2.2}{3.3.3} = \frac{4}{27}$. As $^4/_{27} < 1$, the series is *convergent*.

Theorems for Series

If A is **the series** $\Sigma^{\infty}_{n=1}\, a_n$, and B is the series $\Sigma^{\infty}_{n=1} b_n$, then A+B = $\Sigma^{\infty}_{n=1}\,(a_n+b_n)$. Theorem: If A is **convergent** with sum L; B convergent with sum M, then A+B is convergent with *sum* L+M. Proof: A+B has the sequence of partial sums $\{u_n\}_{n\geq 1}$, where $u_n = (a_1+b_1)+(a_2+b_2)+\ldots+(a_n+b_n)$. Now $u_n = (a_1+a_2+\ldots+a_n) + (b_1+b_2+\ldots+b_n) = s_n+t_n$, where $\{s_n\}_{n\geq 1}$; $\{t_n\}_{n\geq 1}$ are sequences of **partial** sums of A & B. $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = L + M$. Theorem 2: If j & k are *real* numbers, then jA+jB is the <u>series</u> $\Sigma^{\infty}_{n=1}\,(ja_n+kb_n)$ and it has the **sum** jL+kM.

General Series. Given a series A = $a_1+a_2+\ldots+a_n+\ldots$, consider the SPT |A| = $|a_1|+|a_2|+\ldots+|a_n|+\ldots$ Proposition: If |A| is **convergent,** then A is convergent. The converse is false, e.g. A = $\Sigma\,(-1)^{n+1}/n$ is convergent, but $|A| = \Sigma^1/_n$ is *divergent*.



<u>**Proof**</u>: Put $b_n = a_n$ when $a_n \geq 0$, and $b_n = 0$ otherwise. Put $c_n = 0$ when $a_n \geq 0$, and $c_n = -a_n$ otherwise. So $b_n, c_n \geq 0$ and $a_n = b_n - c_n$ for all n (*). Put B = $\Sigma^{\infty}_1\, b_n$ and C = $\Sigma^{\infty}_1\, c_n$, both S.P.T. **And** $0 \leq b_n;\ c_n \leq |a_n|$ for all n. Hence B and C are both **convergent** by comparison with |A|. Put A = B-C by (*); this is *convergent* by Theorem (2).

<u>Terminology:</u> A is called **absolutely** convergent if |A| is convergent. Example: if A = $\Sigma^{\infty}_{n=1}\, {}^{\sin(n)}/_{n^2}$, then $|A| = \Sigma^{|\sin(n)|}/_{n^2}$ is convergent by **comparison** to $\Sigma^1/_{n^2}$.

Alternating Series

A series whose terms are **alternatively** +ve and -ve can be written as $\pm\Sigma^{\infty}_{n=i}\,(-1)^{n+i}a_i$, with $a_i > 0 = \pm(a_1-a_{i+1}+a_{i+2}-\ldots)$. **Alternative** Series **Test**: A = $\Sigma^{\infty}_{n=1}\,(-1)^{n+1}a_n$ is convergent provided (1) it **is** an **alternating** series, with $a_n \geq 0$ for all n; (2) $a_n \to 0$ as $n \to \infty$; (3) $a_n > a_{n+1}$ for all n.

<u>**Example**</u>: $\Sigma^{\infty}_1\,(-1)^{n+1}\times{}^1/_n$ clearly *satisfies* this test. Proof: Let $\{s_n\}_{n\geq 1}$ be a sequence of partial sums. **Bracket** S_{2n} in 2 different ways: $S_{2n} = (a_1-a_2)+(a_3-a_4)+\ldots+(a_{2n-1}-a_{2n})$ and $S_{2n} = a_1-(a_2-a_3)-(a_4-a_5)-\ldots-(a_{2n-2}-a_{2n-1})-a_{2n}$. Now, by (3), **every** bracket is +ve. The first expression for S_{2n} shows that $\{S_{2n}\}_{n\geq 1}$ is <u>increasing</u>. The 2nd expression shows that is has <u>upper</u> bound a. Hence $\{S_{2n}\}_{n\geq 1}$ is *convergent* with limit **L**, say. But $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n}+a_{2n+1}) = L+0 = L$. Conclusion: S_n has **limit** L.

General Example: $\Sigma^{\infty}_{n=1}\,(-1)^{n+1}\times{}^{p(n)}/_{q(n)}$. (where p & q are polynomials (or *roots* of)). Now $\deg(p) \geq \deg(q)$ implies that a_n does not tend to zero, so the series is **divergent**. If deg(p) < deg(q-1), the series is *absolutely* convergent. Finally, $0 < \deg(q)-\deg(p) \leq 1$ implies that the series is **convergent**, by the alternating series test; but not *absolutely* convergent. Example: $\Sigma\,\frac{(-1)^{n+1}\sqrt{n}}{n+1}$. Here, deg(q)-deg(p) = ½.

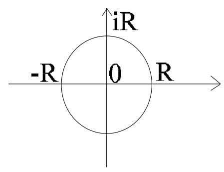
12th March 1999

Power Series

$\Sigma^{\infty}_{n=0}\, a_nx^n = a_0+a_1x+a_2x^2+\ldots+a_nx^n$. ($a_i$ are constants, x is a variable). For any **fixed** x, this is an ordinary series. It is *convergent* when x = 0 (to a_n). Otherwise, $^1/_R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, where the **limit** exists. (We say "R = ∞" when the limit is 0). *Theorem:* Where R exists, the series is convergent for |x| < R and divergent for |x| > R. (Don't know what happens when |x| = R).

Exercise: $^1/_{1-x} = 1+x+x^2+\ldots+x^n\ldots$ When x = 0, 1 = 1. When x = $^1/_2$, $2 = 1+{}^1/_2+{}^1/_4+{}^1/_8+\ldots$ When x = 1, $^1/_{1-1} \neq$ (or **is** it equal to?) 1+1+1+1+1+.... When x = -2, $^1/_3 \neq$ 1-2+4-8+16-32+... (Divergent and **oscillating**). $^1/_R = \lim_{n\to\infty}\left|\frac{1}{1}\right| = 1 \Rightarrow R = 1 \Rightarrow$ cvt when -1<x<1 and dvt when |x| > 1. R is called the *radius* of *convergence*, and the result holds for complex series $\Sigma^{\infty}_{n=0}\, C_nz^n$.

Proof: Put $b_n = |a_nx^n|$, so Σb_n is a SPT. $b_{n+1}/b_n = |a_{n+1}/a_n||x|$, which **tends** to $^{|x|}/_R$ as n → ∞. **The ratio test** implies that this is convergent when $^{|x|}/_R < 1$ i.e. |x| < R; and *divergent* when $^{|x|}/_R > 1$ i.e. |x| > R. Note: Σa_nx^n is **absolutely** convergent when |x| = R.



Example: $\Sigma^{\infty}_{n=1}\, (-1)^{n+1}x^n/n = x-{}^{x^2}/_2+{}^{x^3}/_3-(x^4/4)+\ldots$ (This is the **Taylor** series for log(1+x): see later). $^1/_R = \lim_{n\to\infty}\left|\frac{1}{n+1}/\frac{1}{n}\right| = \lim_{n\to\infty}\left(\frac{1}{1+(1/n)}\right) = 1$. 4 cases: when |x| < 1, the series is *convergent*. When |x| > 1, the series is *convergent*. When x = +1, the series is $\Sigma(-1)^{n+1}/n$, which is **known** to be convergent. When x = -1, the series is $\Sigma^1/_n$, which is known to be **divergent**. So the series is *convergent* for -1<x≤1 — the *interval* of convergence is (-1,1]. **Case** z = i: the series is $i-{}^{i^2}/_2+{}^{i^3}/_3-(i^4/4)+(i^5/5) = i+{}^1/_2-{}^i/_3-{}^i/_4+{}^i/_5+{}^1/_6\ldots. = ({}^1/_2-{}^1/_4+{}^1/_6\ldots) + i(1-{}^1/_3+{}^1/_5-{}^1/_7+\ldots)$. Both parts are **convergent** by the alternating series test; so the whole series is *convergent*.

Taylor Series

Let f be a function, **differentiable** as often as necessary. Then f has a *Maclauin* series $\Sigma^{\infty}_{k=0}\frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\ldots$ and, at x = a, Taylor Series $\Sigma^{\infty}_{k=0}\frac{f^{(k)}(a)}{k!}(x-a)^k$ = $f(a)+f'(a)(x-a)+{}^{f''(a)}/_2(x-a)^2+\ldots$ The Taylor **polynomial** of order n at x = a is the *translation* of the series when the $(x-a)^k$ (k > n) are **deleted**.

Example: f(x) = sin(x), a = $^\pi/_4$. $f(^\pi/_4) = {}^1/_{\surd 2}$. f'(x) = cos(x). $f'(^\pi/_4) = {}^1/_{\surd 2}$, f''(x) = -sin(x); $f''(^\pi/_4) = -{}^1/_{\surd 2}$, f'''(x) = -cos(x); $f'''(^\pi/_4) = -{}^1/_{\surd 2}$; f''''(x) = sin(x), etc. So, $\sin(x) = {}^1/_{\surd 2}\{1 + (x-{}^\pi/_2) - {}^{(x-(\pi/4))^2}/_2 - {}^{(x-(\pi/4))^2}/_6+\ldots\}$. The **radius** of convergence is $|a_{n+1}/a_n| = |{}^{\pm 1/(n+1)!}/_{\pm 1/n!}| = {}^1/_n$, which tends to 0 as n tends to ∞. Hence ("$^1/_R = 0$" iff "R = ∞") the series is **convergent** for all x.

Taylor Series as Polynomial Approximations to Functions

Example: f(x) = sin(x). The 1st *approximation* at x = 0 is y = x. The 2nd is $y = x-{}^{x^3}/_6$. The 3rd is $y = x-{}^{x^3}/_6+(x^5/120)$. The full series is $\sin(x) = \Sigma^{\infty}_{n=0}\, (-1)^{n-1}x^{2n-1}/(2n-1)!$ ("R=∞"). $f(x) = f(a) + {}^{f'(a)}/_1(x-a) + {}^{f''(a)}/_{2!}(x-a)^2+\ldots$ Maple: `>taylor(f(x), x=a(#terms);` e.g. `>taylor(sin(x),x=Pi/4,12);` or `>taylor(log(1+x), x=0);`. Note: `taylor` is interchangeable with `series`; Default # of terms is 8, changeable with `> order:= 20.`

15th March 1999

Methods of calculating Macaulin Series Expansions of Functions

Let $f(x) = a_0+a_1x+a_2x^2+...+a_nx^n+...$ **Method** (1): Use *Taylor's* Theorem: $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + ... + f^{(n-1)}(0)x^{n-1}/(n-1)! + R_n(x)$, where the *Remainder* $R_n(x) = f^{(n)}(c)x^n/n!$ for some $x \in [0,x]$. **Example**: $f(x) = \log(1+x)$. $f(0) = \log(1) = 0$. $f'(x) = {}^1/_{1+x}$. $f'(0) = 1$. $f''(x) = (-1)(1+x)^{-2}$. $f''(0) = -1$. $f'''(x) = 2(1+x)^{-3}$. $f'''(0) = 2$. $f^{(r)}(x) = (-1)^{r-1}(r-1)!(1+x)^{-r}$. $f^{(r)}(0) = (-1)^{r-1}(r-1)!$. **Hence** $\log(1+x) = 0 + (+1)x + {}^{(-1)}/_{2!}x^2 + {}^2/_{3!}x^3 + ... + (-1)^{r-1}(r-1)!x^r/r! + ...$ Or, $\log(1+x) = x-{}^1/_2x^2+{}^1/_3x^2+...+[(-1)^{r-1}x^r/r]+...$

The **radius** of convergence of this *infinite* series is given by ${}^1/_R = \lim_{n\to\infty}\left|-\frac{n+1}{n}\right| = 1$. Convergence for $x \in (-1,1)$. *Example*: $e^x = 1+x+{}^1/_2x^2+{}^1/_6x^3+...+{}^1/_{r!}x^r+...$ (**Real** x). Example: $(1+x)^n = 1 + nx + {}^{n(n-1)x^2}/_{2!} + {}^{n(n-1)(n-2)x^3}/_{3!} + ... + \binom{n}{r}x^r + ...$ $(x \in (-1,1))$.

Q: **Obtain** (i) $f(x) = \sin(x)$, (ii) $f(x) = \cos(x)$. A: (i) $y = \sin(x)$ so $y(0) = 0$. $y'(x) = \cos(x)$ so $y'(0) = 1$. $y''(x) = -\sin(x)$ so $y''(0) = 0$. And $y'''(x) = -\cos(x)$ so $y'''(0) = -1$. Now $y^{(4k)}(x) = \sin(x)$, $y^{(4k)}(0) = 0$. And $y^{(4k+1)}(x) = \cos(x)$, $y^{(4k+1)}(0) = 0$. So $\sin(x) = x - {}^{x^3}/_{3!} + x^5/5! - ... + (-1)^r x^{2r+1}/(2r+1)! + ...$ (ii) $y = \cos(x)$ so $y(0) = 1$. $y' = ...$, as in (i). So $\cos(x) = 1- {}^{x^2}/_{2!} + x^4/4! - ... + (-1)^r x^{2r}/(2r)!$.

Q: Obtain the **series** for ${}^1/_{(1+x)^3}$ up to the term in x^4 using the *formula* above. A: ${}^1/_{(1+x)^3} = (1+x)^{-3} = 1 + (-3)x + {}^{(-3)(-4)x^2}/_{2!} + {}^{(-3)(-4)(-5)x^3}/_{3!} + {}^{(-3)(-4)(-5)(-6)}/_{4!}\times x^4 + ... = 1-3x+6x^2-10x^3+15x^4-... = \Sigma^{\infty}_{r=0}\binom{r+2}{2}x^r(-1)^r$.

Method (2): Algebraic **operations** with known series. Example: $e^x\log(1+x) = (1+x+{}^1/_2x^2+{}^1/_3x^3+...)(x-{}^1/_2x^2+{}^1/_3x^3-{}^1/_4x^4+...)$ = see right. So $e^x\log(1+x) = x+{}^1/_2x^2+{}^1/_3x^3+0x^4+...$ $(x \in (-1,1))$. **Similarly**, $e^x+\log(1+x) = 1+2x+0x^2+{}^1/_2x^3-{}^5/_{24}x^4+...$ Q: **Obtain** the series for $\{{}^{1-x+2x^2}/_{(1+x)^3}\}$ up to the term in x^4. A: Now $\{{}^{1-x+2x^2}/_{(1+x)^3}\} = (1-x+2x^2)(1-2x+6x^2-10x^3+15x^4-...) = 1-3x+6x^2-10x^3+15x^4-....-x+3x^2-6x^3+10x^4-....+2x^2-6x^3+12x^4-.... = 1-4x+11x^2-22x^3+37x^4-...$

$$x-{}^1/_2x^2+{}^1/_3x^3-{}^1/_4x^4+...$$
$$+\ x^2-{}^1/_2x^3+{}^1/_3x^4-...$$
$$+{}^1/_2x^3-{}^1/_4x^4+...$$
$$+{}^1/_4x^4-...$$

Method (3): Assume $f(x) = \Sigma a_rx^r$ and *equate* coefficients. Example: To find the series for $f(x) = \{{}^{1+x^2}/_{1+\log(1+x)}\}$, multiply up and **use** $1+x^2 = (a_0+a_1x+a_2x^2+a_3x^3+...)(1+x-{}^1/_2x^2+{}^1/_3x^3-...)$ = see right. Therefore, **equating** coefficients, $1 = a_0$, $0 = a_0+a_1$, $1 = -{}^1/_2a_0+a_1+a_2$; $0 = {}^1/_2a_0-{}^1/_2a_1+a_2+a_1$. These 4 **equations** imply $a_0 = 1$, $a_1 = -1$, $a_2 = 2^1/_2$, $a_3 = -3^1/_3$. Thus $\{{}^{1+x^2}/_{1+\log(1+x)}\} = 1-x+{}^5/_2x^2-{}^{10}/_3x^3+...$

$$a_0+a_0x+{}^1/_2a_0x^2+{}^1/_3a_0x^3-...$$
$$+a_1x+\ a_1x^2-{}^1/_2a_1x^3+...$$
$$+a_2x^2+\ a_2x^3-...$$
$$+a_3x^3+...$$

Q: Obtain the **series** for $\tan(x)$ up to the x^5 term using $\tan(x) = a_1x+a_3x^3+a_5x^5+...$ (odd function) in $\sin(x) = \tan(x).\cos(x)$. A: The **above** imply that $x-{}^1/_6x^3+{}^1/_{120}x^5-... = \{a_1x+a_3x^3+a_5x^5+...\}\{1-{}^1/_2x^2+{}^1/_{24}x^4-...\} = a_1x+(a_3-{}^1/_2a_1)x^3+(a_5-{}^1/_2a_3+{}^1/_{24}a_1)x^5+...$ These **imply that** $1 = a_1$; $-{}^1/_6 = a_3-{}^1/_2a_1$, and ${}^1/_{120} = a_5-{}^1/_2a_3+{}^1/_{24}a_1$. So we **have** $a_1 = 1$, $a_3 = -{}^1/_6+{}^1/_2 = {}^1/_3$ and $a_5 = {}^1/_{120}+{}^1/_6-{}^1/_{24} = {}^2/_{15}$. Therefore, *tan*(x) $= x+{}^1/_3x^3+{}^2/_{15}x^5+...$

Method (4): **Differentiation**. If $f(x) = a_0+a_1x+a_2x^2+....$ ($|x|<R$), then $f'(x) = a_1+2a_1x+...$ ($|x|<R$). This is subject to certain *conditions* on f. Exercise: $e^x\log(1+x) = x+{}^1/_2x^2+{}^1/_3x^3+0x^4+...$ This implies that $e^x\{\log(1+x)+{}^1/_{1+x}\} = 1+x+x^2+0x^3+...$ Q: Obtain the **series** for $\sec^2(x)$ up to the term in x^4. A: Differentiate $\tan(x) = x+{}^1/_3x^3+{}^2/_{15}x^5+...$ to get $\sec^2(x) = 1+x^2+{}^2/_3x^4+...$ [Check using $\cos^2(x) = (1-{}^{x^2}/_2+(x^4/24)-...)^2 = 1-x^2+{}^1/_3x^4+...$].

Method (5): **Integration**. If $f(x) = a_0+a_1x+a_2x^2+...+a_rx^r+...$ ($|x|<R$), then $\int f(x)dx$ = constant$+a_0x+{}^1/_2a_1x^2+...+{}^1/_{(r+1)}a_rx^{r+1}+...$ ($|x|<R$). This is subject to certain **conditions** on f, and f(0) determines the constant. Example: ${}^1/_{1+x} = 1-x+x^2-x^3+x^4-...$ ($|x|<1$). Therefore this *implies* that $\log(1+x) = c+x-{}^1/_2x^2+{}^1/_3x^3-{}^1/_4x^4+...$ (as *before*). At x = 0, log(1) = 0 = c.

Q: Obtain the **series** for $\tan^{-1}(x)$ up to the term in x^5 using $\int {}^{dx}/_{1+x^2} = \tan^{-1}(x)$+constant. A: *Integrate* ${}^1/_{1+x^2} = 1-x^2+x^4-x^6+...$ to get $\tan^{-1}(x)$ = constant + $x - {}^{x^3}/_3 + (x^5/5) - (x^7/7) + ...$ Now $\tan^{-1}(0) = 0$ implies that the *constant* is zero, and so $\tan^{-1}(x) = x-{}^1/_3x^3+{}^1/_5x^5-...$

Method (6): Use of *Differential Equations.* Example: Let ${}^{d^2y}/_{dx^2}+y = 0$. Set $y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+...$, then $y' = a_1+2a_2x+3a_3x^2+4a_4x^3+...$ and $y'' = 2a_2+6a_3x+12a_4x^2+...$ [*Simple Harmonic Motion*]. Now $0 = y''+y$ implies that $a_0 = -2a_2$, $a_1 = -6a_3$, $a_2 = -12a_4$ and $a_r = -(r+2)(r+1)a_{r+2}$. These **all** imply that $a_2 = -{}^1/_2a_0$, $a_3 = -{}^1/_6a_1$, $a_4 = -{}^1/_{12}a_2 = {}^1/_{24} = a_0$; $a_{r+2} = {}^{\pm1}/_{(r+2)!}\{a_0$ or $a_1\}$. So $y = a_0\{1-{}^1/_2x^2+{}^1/_{4!}x^4 + ... + [(-1)^r/(2r)!]x^{2r}+...\} + a_1\{x-{}^1/_{3!}x^3+...+[(-1)^r/(2r+1)!]x^{2r+1}+...\}$.

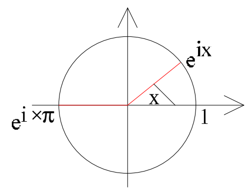
Q: Obtain the **series** for y up to x^6 when ${}^{dy}/_{dx} = xy$. Check that this is the *series* for the solution $y = Ce^{x^2/2}$ obtained by **separation** of variables. A: Here, ${}^{dy}/_{dx} = xy$. This implies the *following*: $a_1 = 0$, $2a_2 = a_0$, $3a_3 = a_1$, ..., $a_r = a_{r-2}$. These all imply that $a_1 = 0$, $a_2 = {}^1/_2a_0$, $a_3 = {}^1/_2a_1 = 0$, $a_4 = {}^1/_4a_2 = {}^1/_8a_0$; $a_5 = {}^1/_5a_3 = 0$; $a_6 = {}^1/_6a_4 = {}^1/_{48}a_0$. **Hence** $y = a_0(1+{}^1/_2x^2+{}^1/_8x^4+{}^1/_{48}x^6+...) = a_0\{1+({}^{x^2}/_2)+{}^1/_2({}^{x^2}/_2)^2+{}^1/_{3!}({}^{x^2}/_2)^3+...\} = a_0e^{x^2/2}$.

Method (7): Use *Leibnitz's Theorem* for d^ny/dx^n when y = uv. The **product** rule gives $y' = u'v+uv'$. *It also* gives $y'' = (u''v+u'v')+(u'v'+uv'') = u''v+2u'v'+uv''$. In <u>general</u>, $y^{(n)} = u^{(n)}v + nu^{(n-1)}v' + {}^nC_2\, u^{(n-2)}v'' + ... + {}^nC_r\, u^{(n-r)}v^{(r)} + ... + uv^{(n)}$. **Exercise**: find the series for $(x-1)^3e^x$. **Put** $x = (x-1)^3$; then $u' = 3(x-1)^2$, $u'' = 6(x-1)$; $u''' = 6$, $u^{(r)} = 0$ for $r > 3$. <u>And</u> $v = e^x$, $v^{(r)} = e^x$ for all r. So $y^{(n)} = (x-1)^3.e^x + n.3(x-1)^2e^x + {}^1/_2x(n-1).6(x-1)e^x + {}^{n(n-1)(n-2)}/_6.6e^x$ + *zero* terms. Now $y^{(n)}(0) = -1+3n-3n(n-1)+n(n-1)(n-2) = n^3-6n^2+8n-1$. *Therefore* $y = -1+2x-{}^1/_2x^2-{}^2/_3x^3+ ... + {}^{(r^3-6r^2+8r-1)}/_{r!}\,x^r...$

Q: Find the **coefficient** a_r in the series expansion $(1+x^2)e^{2x} = \Sigma^\infty{}_{r=0}\, a_rx^r$. A: $y = uv = (1+x^2)e^{2x}$. Now $u = (1+x^2)$, $u' = 2x$, $u'' = 2$, $u^{(r)} = 0$ **for** $r \geq 3$. And $v = e^{2x}$, $v' = 2e^{2x}$, $v'' = 4e^{2x}$, $v^{(r)} = 2^re^{2x}$. *Leibnitz's* Theorem: $y^{(r)} = (1+x^2).2^re^{2x} + r.2x.2^{r-1}e^{2x} + {}^1/_2r(r-1).2.2^{r-2}e^{2x}$ + *zero* terms. $y^{(r)}(0) = 2^r+0+r(r-1)2^{r-2}$. Hence $(1+x^2)e^{2x} = \Sigma^\infty{}_{r=0}(4-r+r^2)2^{r-2}x^r$. ($|x| < 1$).

We **know that** $\cos(x) = 1-{}^{x^2}/_2+(x^4/24)-(x^6/720)+...$, and that $\sin(x) =x-{}^{x^3}/_6+(x^5/120)-....$ What is the *radius* of convergence for cos(x)? Standard <u>formula</u>: ${}^1/_R = \lim_{n\to\infty}|a_{n+1}/a_n|$. Doesn't exists if *either* a_n or a_{n+1} is zero. Get round this by **putting** $t = x^2$. Then the series is $1-{}^t/_2+{}^{t^2}/_{24}-{}^{t^3}/_{720}$. Then ${}^1/_R = \lim_{n\to\infty}|[(-1)^{n+1}/(2n+2)!]/[(-1)^n/(2n)!]| = \lim_{n\to\infty}|{}^{(2n)!}/_{(2n+2)!}| = \lim_{n\to\infty}|{}^1/_{(2n+1)(2n+1)}| = 0$. So $R = \infty$ i.e. the series <u>converges</u> for all t.

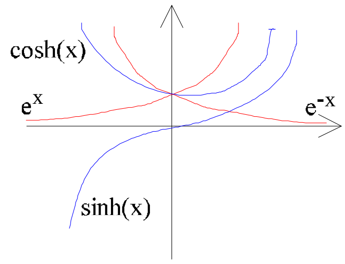
Entertaining Observation: the series for cos(x)+isin(x) is $1+ix-{}^{x^2}/_2-{}^{ix^3}/_6+(x^4/24)+(ix^5/120) = 1 + (ix) + [(ix)^2/2] + [(ix)^3/6] + [(ix)^4/24] + [(ix)^5/120] = e^{|ix|} = \cos(x)+i\sin(x)$. In *particular*, $e^{i\pi} = -1$.



17th March 1999

Hyperbolic Functions

$\cosh(x) = {}^1/_2(e^x+e^{-x})$. $\sinh(x) = {}^1/_2(e^x-e^{-x})$. Facts: cosh(-x) = cosh(x) (*even*); sinh(-x) = -sinh(x) (*odd*). Taylor series: $\cosh(x) = 1+(x^2/2)+(x^4/24)+\ldots$, $\sinh(x) = x+(x^3/6)+(x^5/120)+\ldots$. $\cosh(x)+\sinh(x) = e^x$. $\cosh(x)-\sinh(x) = e^{-x}$. A *hanging chain* takes the shape of a cosh curve (see right). Derivatives: ${}^d/_{dx}(\cosh(x)) = {}^d/_{dx}[(e^x+e^{-x})^1/_2] = {}^1/_2(e^x-e^{-x}) = \sinh(x)$. ${}^d/_{dx}\sinh(x) = \cosh(x)$. $\cosh^2x-\sinh^2x = 1$.



$\cosh(x+y) = {}^1/_2[e^{(x+y)}+e^{(-x-y)}] = {}^1/_2[e^xe^y+e^{-x}e^{-y}] = {}^1/_2[(\cosh(x)+\sinh(x))(\cosh(y)+\sinh(y)) + (\cosh(x)-\sinh(x))(\cosh(y)-\sinh(y))] = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$, etc. The *similarity* of the formulas follows from $e^x = \cosh(x)+\sinh(x)$ and $e^{ix} = \cos(x) + i\sin(x)$.

Standard Integrals. In $I = \int {}^{dx}/_{\sqrt{(x^2-a^2)}}$, Let x = acosh(u), so dx = asinh(u)du. So $I = \int {}^{a\sinh(u)du}/_{\sqrt{[(a\cosh(u))^2-a^2]}} = \int {}^{a\sinh(u)du}/_{\sqrt{[a^2(\cosh^2(u)-1)]}} = \int {}^{a\sinh(u)du}/_{\sqrt{[a^2\sinh^2u]}} = \int {}^{a\sinh(u)du}/_{a\sinh(u)} = \int du = u+c = \cosh^{-1}({}^x/_a)$. *Similarly* for $\int {}^{dx}/_{\sqrt{[x^2+a^2]}}$ using x = asinh(u) to give $\sinh^{-1}({}^x/_a)+c$.

19th March 1999

Multiplying Polynomials

				1	0	2	3	1
				1	0	2	3	1
1	0	2	3	1				
		2	0	4	6	2		
			3	0	6	9	3	
				1	0	2	3	1
1	0	4	6	6	12	13	6	1

	x^0	x^1	x^2	x^3	x^4	x^5	x^6
1/1-x	1	1	1	1	1	1	1
1/1-x	1	1	1	1	1	1	1
x^2/1-x			1	1	1	1	1
x^4/1-x					1	1	1
x^6/1-x							1
1/(1-x)(1-x^2)	1	1	2	2	3	3	4
× 1	1	1	2	2	3	3	4
× x^3				1	1	2	2
× x^6							1
	1	1	2	3	4	5	7
× 1	1	1	2	3	4	5	7
× x^4					1	1	2
	1	1	2	3	4	6	9
×1	1	1	2	3	4	6	9
× x^5						1	1
	1	1	2	3	5	7	10
×1	1	1	2	3	5	7	10
× x^6							1
	1	1	2	3	5	7	11

Example shown **on left**: $(x^4+2x^2+3x+1)^2$. Another example: $\prod_{n=1}^{\infty} \frac{1}{(1-x^n)}$. This is an *infinite* product! It is $({}^1/_{1-x})({}^1/_{1-x^2})({}^1/_{1-x^3})\ldots$ Probelm: Find its **Taylor** series up to degree six. So we want the *following* product: $(1+x+x^2+x^3+x^4+x^5+x^6+\ldots) \times (1+x^2+x^4+x^6+\ldots) \times (1+x^3+x^6+\ldots) \times (1+x^4+\ldots) \times (1+x^5+\ldots) \times (1+x^6+\ldots) \times (1+\ldots) \times (1+\ldots) \times (1+\ldots) \times \ldots$

Looking at the **table** on the left, we conclude from it that the answer is as follows: $1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6$. The coefficients of x^n correspond to the **number** of ways of writing n as an unordered sum of *positive* integers = the number of partitions of n. For n = 5, we have 5 = 4+1 = 3+2 = 2+2+1 = 1+1+1+1+1 = 3+1+1 = 2+1+1+1.

2 standard types of *power* series: (1) Either absolutely convergent for **-R**$\le$x$\le$**R** and divergent for x < **-R** and **R** < x, or (2) *Absolutely* convergent for **-R**<x<**R**; *divergent* for x < **-R** and **R** < x; *divergent* when x = **R** (or x = **-R**); and *conditionally* convergent when x = **-R** (or x = **R**).

Example 1: $\Sigma\ 3^n n^2 x^n/(n+1)^4$. Now $^1/_R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} [3^{n+1}(n+1)^2/(n+2)^4]\times[(n+1)^4/3^n n^2] = \lim_{n\to\infty} 3\times[(1+(1/n)^6/(1+(2/n))^4] = 3(1+0)^6/(1+0)^4 = 3$. So $R = {}^1/_3$. Now at x = $^1/_3$, $\Sigma\ n^2 x/(n+1)^4$ is a SPT so it is **convergent** (think of it as $^1/_{n^2}$). At x = $-^1/_3$, we *have* $\Sigma\ (-1)n^2/(n+1)^4$: alternating, so *absolutely* convergent. Conclusion: Type (1).

Example 2: $\Sigma\ 3^n n^3 x^n/(n+1)^4$. Here $^1/_R$ is 3 as before. At x = $^1/_3$, series is $\Sigma\ n^3/(n+1)^4$, (SPT — *think* of it as $^1/_n$), so **divergent**. At x = $-^1/_3$, series is $\Sigma\ (-1)^n n^3/(n+1)^4$. This is an *alternating* series $((-1)^n/n)$, so it is *convergent*. Conclusion: Type 2. It is **cvt** on $[-^1/_3, ^1/_3)$, *dvt* on $(-\infty, -^1/_3) \cup [^1/_3, \infty)$. It is *conditionally* convergent at $-^1/_3$, and *absolutely* convergent for $(-^1/_3, ^1/_3)$.

Assignment 2

Q: Determine the **character** of the following *sequences* (where the sequence is $\{u_n\}_{n\ge2}$). (a) $u_n = (2n^5+1)/(1-n^5)$. (b) $u_n = (1-n)^4/(1+n^2)(1-3n)$. (c) $u_n = (-\sin(n\pi))^n$. A: (a) $u_n = (2n^5+1)/(1-n^5) = [1+(1/n^5)]/[(1/n^5)-1]$ which tends to $^{2+0}/_{0-1} = -2$ as n *tends* to ∞.

(b) $u_n = (1-n)^4/(1+n^2)(1-3n) = (1-n)(^{(1/n-1)^3}/_{((1/n^2)+1)(((1/n)-3))})$. The **second** bracket has limit $^{(0-1)^3}/_{(1)(-3)} = {}^1/_3$ as n tends to ∞. So u_n behaves like $^1/_3(1-n)$ and tends to $-\infty$ as n **tends** to $+\infty$. (c) $\sin(n\pi) = 0$ for all n. So $\{u_n\}_{n\ge2} = \{0\}_{n\ge2}$, which has limit 0.

Q: A **sequence** is defined by $u_1 = 4$, $u_{n+1} = {}^1/_3u_n+1$. Prove that $u_n \ge {}^3/_2$ for all n and that $u_{n+1} \le u_n$ for all n. Explain why these *results* show that the sequence is convergent, and find its limit, L. Prove that $|u_n-L| < 1/3^{n-2}$ for all n. A: $\{u_n\}_{n\ge1} = \{4, 2^1/_3, 1^7/_9, 1^{16}/_{27}, 1^{43}/_{81}, ...\}$. We want to prove that $u_n-{}^3/_2 \ge 0$ for all n. Case n = 1: $u_1-{}^3/_2 = 4-{}^3/_2 = {}^5/_2 > 0$. **Assume** true for n, then $u_{n+1}-{}^3/_2 = (^1/_3u_{n+1})-{}^3/_2 = {}^1/_3(u_n-{}^3/_2) \ge 0$ by *assumption*. So $u_{n+1}-u_n \le 0$ **for** all n.

Case n = 1: $u_2-u_1 = 2^1/_3-4 = -1^2/_3 \ge 0$. Assume **true** for n. Then $u_{n+1}-u_n = (^1/_3u_n+1)-(^1/_3u_{n-1}+1) = {}^1/_3(u_n-u_{n-1}) \ge 0$ by assumption. So $\{u_n\}$ is **decreasing** and has lower **bound** $^3/_2$, hence the sequence has a *limit*. The limit is given by $L = {}^1/_3L+1$; $L = 1^1/_2$. Now again **consider** case n = 1: $4-{}^3/_2 = 2^1/_2 < 3 = 1/3^{-1}$, so true for n = 1. Assume true for **all** n. Then $|u_{n+1}-L| = |^1/_3u_n+1-1^1/_2| = |^1/_3u_n-{}^1/_2| = {}^1/_3|u_n-{}^3/_2| < {}^1/_3(1/3^{n-2})$ (by assumption) $= 1/3^{(n+1)-2}$.

Note: **Newton's** method for iteration is $x_{n+1} = x_n - {}^{f(x)}/_{f'(x)}$. Q: Use *partial fractions* to evaluate the partial sum $s_k = \Sigma^k_{n=1}\ ^1/_{n^2+6n+5}$ (which comes from $\Sigma^\infty_{n=1}\ ^1/_{n^2+6n+5}$) and *evaluate* $\lim_{k\to\infty} s_k$ to determine the sum of the series. A: $s_k = \Sigma^k_{n=1}\ ^1/_{(n^2+6n+5)} = {}^1/_4\Sigma^k_{n=1}(^1/_{n+1}-{}^1/_{n+5})$ by partial fractions.

Now the above is equal to $^1/_4[(^1/_2-{}^1/_6)+(^1/_3-{}^1/_7)+(^1/_4-{}^1/_8)+(^1/_5-{}^1/_9)+(^1/_6-{}^1/_{10}) + ... + ... + (^1/_{k-6}-{}^1/_{k-2}) + (^1/_{k-5}-{}^1/_{k-1}) + (^1/_{k-4}-{}^1/_k) + (^1/_{k-3}-{}^1/_{k+1}) + (^1/_{k-2}-{}^1/_{k+2}) + (^1/_{k-1}-{}^1/_{k+3}) + (^1/_k-{}^1/_{k+4}) + (^1/_{k+1}-{}^1/_{k+5})] = {}^1/_4[^1/_2+{}^1/_3+{}^1/_4+{}^1/_5-{}^1/_{k+2}-{}^1/_{k+3}-{}^1/_{k+4}-{}^1/_{k+5}]$. So $\lim_{k\to\infty} s_k = {}^1/_4(^1/_2+{}^1/_3+{}^1/_4+{}^1/_5-0-0-0-0) = {}^{77}/_{240}$.

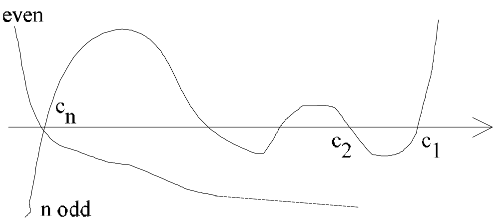
Rearrangements of a Conditionally Convergent Series

$\Sigma^{\infty}_{n=1}\ (-1)^{n+1}/n$ is a **convergent** series, but not *absolutely* convergent. Evaluating partial sums up to the 189^{th} term using Quattro, we find that the **sum** of the series lies in the interval [0.6905, 0.6958]. The series has an *alternating* pattern of signs, +-+-+-+-. Rearranging the terms to form <u>patterns</u> ++-++- and ++- - - - produces 2 new series which have sums in the **intervals** [1.0358, 1.0437] and [0.4882, 0.5015] respectively.

Polynomial and Rational Functions

An *expression* $P(x) = a_0+a_1x+a_2x^2+...+a_nx^n$, ($a_n \neq 0$, $n \geq 0$), where x is an *indeterminate* and the a_i are (real) constants, is called a real *polynomial* in x of degree n with coefficients a_i. The ring **R**[x] is the set of all real polynomials in x, with *addition* and *multiplication* defined in the usual way. (we also have **Z**[x], **C**[x],...). A polynomial P(x) has a factor F(x) if there is *another* polynomial G(x) satisfying P(x) = F(x)G(x).

<u>Example</u>: $(1-x^n) = (1-x)(1+x+x^2+...+x^{n-1})$. A function p: **R**→**R**, x→P(x) is a polynomial function. If P(x) has a factor (c-x), then c is a *root* of P(x). A polynomial equation P(x) = 0, has the roots of P as its <u>solutions</u>. If $P(x) = (x-c_1)(x-c_2)...(x-c_n)$, with $c_1>c_2>...>c_n$, then the *function* p: **R**→**R**, x→P(x) has a *graph* as shown.



Inequalities

Basic properties: (i) a < b iff a+x < b+x for all x ∈ **R**. (ii) a < b iff -a > -b. More *generally*, a < b iff ka < kb (for all +ve k); a < b iff ka > kb (for all k<0). (iii) $a^2 \geq 0$ for all Real x. (iv) 0<a<b and 0<x<y *imply that* 0<ax<by.

Algorithm for solving the *inequality* $Q(x) \geq R(x)$. [Q(x) & R(x) are rational functions]. (i) Determine when the **denominators** of Q(x) and R(x) are zero. (ii) Make both denominators into exact squares. (iii) *Cross-multiply* and transfer to one side. (iv) Take out <u>common factors</u> and rearrange in the form $P(x) \geq 0$ or $P(x) \leq 0$.

(v) Find the **remaining** factors of P(x): $P(x) = (x-c_1)(x-c_2)...(x-c_n)$ (*quadratic* factors with no real roots) such that $x_1 \leq c_2 \leq ... \leq c_n$. (vi) Write down the *solution* as a union of intervals $[c_i, c_{i+1}]$ or (c_i, c_{i+1}) or $(c_i, c_{i+1}]$ or $[c_i, c_{i+1})$, excluding all the <u>roots</u> of the original *denominators*.

<u>22nd March 1999</u>

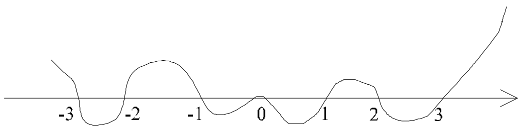
Approximations

Consider S, a set of **functions**, S: **R**→**R**, which are differentiable *infinitely often* near x = 0. We can add <u>functions</u> in S, f+g: x→f(x)+g(x); and <u>scalar</u> <u>multiply</u>, cf: x→cf(x) (c ∈ **R**).

So S is a "vector space" with these operations. There is a special **subset** $m_i \in S$ of polynomial functions, $m_i(x) = x^i$. So a *Maclaurin* expansion expresses f as a linear combination of these m_i. Think of the m_i as forming a **basis**, e.g. $\sin = m_1 - {}^1/_6 m_3 + {}^1/_{120} m_5 - \ldots$ **Another** basis: $\{1, x \to \cos(nx), x \to \sin(nx)\ \forall\ n > 0\}$. The theory *that* shows how to express f in S as a sum of these is called "Fourier Series". We get equations like $x = \sin(x) - {}^1/_3 \sin(3x) + {}^1/_5 \sin(5x) - \ldots$

Inequalities

Solve the **following** inequalities, giving each solution set as a *union* of intervals. (i) $x^2(x^2-1)(x^2-4)(x^2-9) \geq 0$. Write the factors in order: $(x+3)(x+2)(x+1)x^2(x-1)(x-2)(x-3) \geq 0$. The *idealised* sketch of the graph is as shown on the right. The **lead** term is x^8, which means that it is +ve for *large* x. The solution set is $(-\infty, -3] \cup [-2,-1] \cup \{0\} \cup [1,2] \cup [3,\infty)$.



(ii) ${}^{x^2+9}/_{x^2-9} \leq {}^{x+4}/_{x^2-4}$. (---(1)) There is a problem with the *denominators* at $x = \pm 2, \pm 3$ (it is not defined). (1) $\Leftrightarrow {}^{(x+9)(x^2-9)}/_{(x^2-9)^2} \leq {}^{(x+4)(x^2-4)}/_{(x^2-4)^2}$ $[x^2 \neq 4,9] \Leftrightarrow (x+9)(x^2-9)(x^2-4)^2 \leq (x+4)(x^2-4)(x^2-9)^2$ $[x^2 \neq 4,9] \Leftrightarrow (x+9)(x^2-9)(x^2-4)^2 - (x+4)(x^2-4)(x^2-9)^2 \leq 0$ $[x^2 \neq 4,9] \Leftrightarrow (x^2-4)(x^2-9)[(x+9)(x^2-4)-(x+4)(x^2-9)] \leq 0$ $[x^2 \neq 4,9] \Leftrightarrow (x^2-4)(x^2-9)[x^3-4x+9x^2-36-(x^3-9x+4x^2-36)] \leq 0$ $[x^2 \neq 4,9] \Leftrightarrow (x^2-4)(x^2-9)(5(x^2+x)) \leq 0$ $[x^2 \neq 4,9] \Leftrightarrow 5(x+3)(x+2)(x+1)x(x-2)(x-3) \leq 0$ $[x^2 \neq 4,9]$. *Solution* set: $(-3, -2) \cup [-1,0] \cup (2,3)$. Note: the solution set does **not** include the points that are not *defined* i.e. $x = \pm 2, \pm 3$.

24th March 1999

Q: Express **the solution** of ${}^{x+1}/_{(x-1)^3} \leq {}^{(x+2)}/_{(x-2)^3}$ as a *union* of intervals. A: The Denominators are zero when x is 1 or 2. So these points are **excluded**. So $\Leftrightarrow (x+1)(x-1)/(x-1)^4 \leq (x+2)(x-2)/(x-2)^4$ $[x \neq 1,2] \Leftrightarrow (x+1)(x-1)(x-2)^4 \leq (x+2)(x-2)(x-1)^4$ $[x \neq 1,2] \Leftrightarrow (x-1)(x-2)[(x+1)(x-2)^3-(x+2)(x-1)^3] \leq 0$ $[x \neq 1,2] \Leftrightarrow \ldots\ldots \Leftrightarrow (x-1)(x-2)(4x^3-9x^2+x+6) \geq 0$ $[x \neq 1,2]$. The **cubic** has 1 real root and 2 complex roots. So $\Leftrightarrow (x+\alpha)(x-1)(x-2)$(*quadratic* with complex roots) ≥ 0. So our solution set is $x \in [-\alpha, 1) \cup (2, \infty)$.

26th March 1999

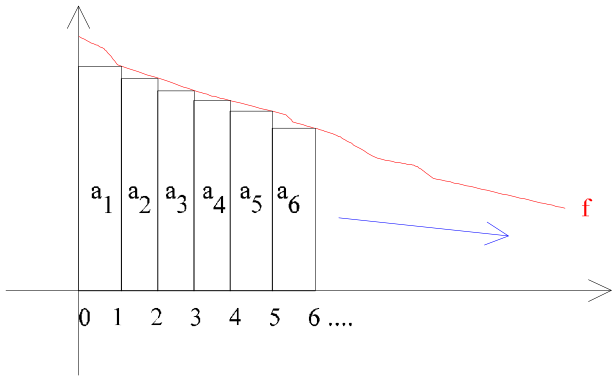
Limit Comparison Test

This is a **better** comparison test for a SPT. Consider $\sum_{n=2}^{\infty} {}^1/_{n^2+1}$, $\sum_{n=2}^{\infty} {}^1/_{n^2}$ (which is *known to be* convergent), and $\sum_{n=2}^{\infty} {}^1/_{n^2-1}$. The first series is convergent by comparison with the middle one: ${}^1/_{n^2+1} < {}^1/_{n^2}$ for all $n \geq 2$. Now $\Sigma\ {}^1/_{n^2-1}$ is convergent by *comparison* with $\Sigma\ {}^2/_{n^2}$: ${}^2/_{n^2} - {}^1/_{(n^2-1)} = {}^{n^2-2}/_{n^2(n^2-1)} > 0$ for all $n \geq 2$.

Suppose that $\Sigma\ a_n$ is to be *investigated*, and that $\Sigma\ b_n$ is known to be convergent or divergent. Then if $\lim_{n\to\infty} a_n/b_n$ exists and is not **zero,** then the two series are both convergent or divergent. Example: $\Sigma a_n = \sum_{n=2}^{\infty} {}^1/_{n^2-1}$, $\Sigma b_n = \sum_{n=2}^{\infty} {}^1/_{n^2}$. Then $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} {}^{n^2}/_{n^2-1} = \lim_{n\to\infty} {}^1/_{1-(1/n^2)} = {}^1/_{1-0} = 1$, which is non zero. Conclusion: Σa_n is convergent *since* Σb_n is.

More **generally**, $\sum_{n=k}^{\infty} \frac{1}{an^2+bn+c}$ is convergent provided that an^2+bn+c is *not* zero for all $n \geq k$ (because the limit is ${}^1/_a$).

Integral Test



Consider the SPT $\sum_{n=1}^{\infty} a_n$ with $\lim_{n\to\infty} = 0$. Here, $a_n = f(n)$, where f is a *real* function, *positive, continuous*, and *decreasing* on $[0, \infty)$ (Note: 0 can be replaced by any +ve integer). The series is convergent provided that the area **under** the curve exists i.e. $\int_0^{\infty} f(x)dx$ defined by $\lim_{b\to\infty} \int_0^b f(x)dx$ exists.

Example: $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$. The series *diverges* if the limit does not exist. Consider $\int_2^b {}^{\log x}/_{x^2}dx = [{}^{(-1)\log x}/_x]^b_2 - \int_2^b {}^{(-1)}/_x {}^1/_x dx = [{}^{(-1)\log x}/_x - {}^1/_x]^b_2 = {}^{-\log b}/_b - {}^1/_b + {}^1/_2\log 2 + {}^1/_2 \rightarrow \lim_{n\to\infty} {}^{(-1/n)}/_b + 0 + {}^1/_2\log 2 + {}^1/_2$ as $b\rightarrow\infty$; $= 0 + {}^1/_2\log 2 + {}^1/_2 = {}^1/_2\log 2 + {}^1/_2$. The integral is **finite,** so the series is *convergent*.

Padé Approximations

The **Maclaurin/Taylor** series for f(x) at x = 0 provides *polynomial approximations* to f at x = 0. The Padé approximations are **rational** functions, whose Taylor expansions agree up to a given degree. Example: e^x is approximately $({}^{2+x}/_{2-x})$. Degree of *numerator* + degree of *denominator* = 2, so expect agreement up to the x^2 term. Now $({}^{2+x}/_{2-x}) = (1+{}^x/_2)(1-{}^x/_2)^{-1} = (1+{}^x/_2)(1+{}^x/_2+{}^{x^2}/_2+{}^{x^3}/_8) = 1+x+{}^1/_2x+{}^1/_4x^3+\ldots$ And $e^x = 1+x+{}^1/_2x^2+{}^1/_6x^3$. So we *do* have *agreement* up to the x^2 term.

Method: Take *functions* $P_{m,n} = Q_{m,n}/R_{m,n}$ for f(x), **where** $Q_{m,n}$ is a polynomial in x of degree $\leq m$; $R_{m,n}$ is a *polynomial* in x of degree $\leq x$. The Maclaurin series for $P_{m,n}$ agrees with that for f up to the x^{m+n} term. $Q_{m,n+1}(x) = \lambda x Q_{m,n}(x) + \mu Q_{m+1,n}(x)$. *And* $R_{m,n+1}(x) = \lambda x R_{m,n}(x) + \mu R_{m+1,n}(x)$, with λ and μ chosen *so that* $Q_{m,n+1}$ has **no** x^{n+1} term.

$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	$P_{3,0}$
$P_{0,1}$	$P_{1,1}$	$P_{2,1}$	....
$P_{0,2}$	$P_{1,2}$	....	

The **first** row contains elements which are polynomials. The **2nd** row are poly/linear; the **3rd** row are poly/quadratic, etc. For e^x, we have ${}^1/_1$, ${}^{1+x}/_1$, ${}^{2+2x+x^2}/_2$, ...

Now $Q_{0,1} = \lambda x Q_{0,0} + \mu Q_{1,0} = \lambda x+\mu(1+x) = \mu+x(\lambda+\mu)$. [We **want** to make $x(\lambda+\mu)$ *zero*]. Further, $R_{0,1} = \lambda x R_{0,0} + \mu R_{1,0} = \lambda x+\mu = \mu+\lambda x$. *Choose* $\mu = 1$ and $\lambda = -1$ so that $P_{0,1} = {}^1/_{1-x}$.

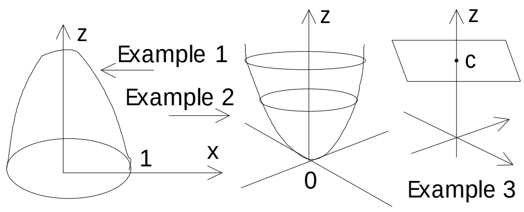
Now for $P_{1,1}$ (*where* m = 1, n = 0): $Q_{1,1} = \lambda x(1+x)+\mu(2+2x+x^2) = 2\mu+x(2\mu+\lambda)+x^2(\lambda+\mu)$. [We **want** to make $x^2(\lambda+\mu)$ *zero*]. Further, $R_{1,1} = 2\mu+\lambda x$. *Choose* $\mu = 1$ and $\lambda = -1$ so that $P_{1,1} = {}^{2+x}/_{2-x}$.

19th April 1999

Functions of 2 Variables

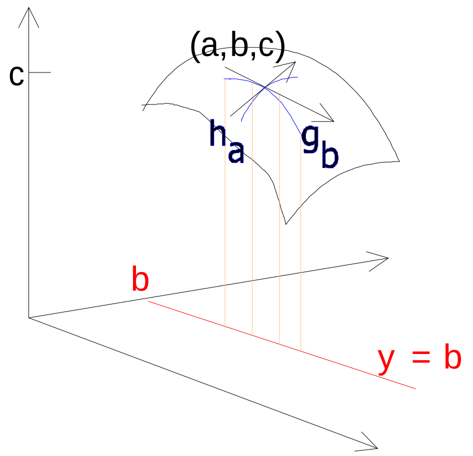
Definition: $\mathbf{R}^2 = \{(x,y) \mid x,y \in \mathbf{R}\}$. Elements of $\mathbf{R}^2$ are represented by *points* in a plane. $\mathbf{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbf{R}\}$. A function of 2 variables is a *function* f with domain*dom* $\subseteq$ ‘ ^{2} and image *im* $\subseteq$ ‘ ^{2}. We write f: $(x,y) \mapsto z = f(x,y)$. The graph of f is the **set** $\{(x,y,z) \in \mathbf{R}^3 \mid z = f(x,y)\}$. Geometrically, we take a *system* of perpendicular co-ordinates in 3 dimensional space, and represent the graph of f by a surface.

Example 1: $f(x,y) = \sqrt{(1-x^2-y^2)}$. Surface is a *hemisphere*, a subset of the sphere $x^2+y^2+z^2 = 1$. $dom_f = \{(x,y) \mid x^2+y^2 \leq 1\}$. $im_f = [0,1]$. **Example** 2: $f(x,y) = x^2+y^2$ Surface is a *paraboloid*. **Example** 3: $f(x,y) = ax+by+c = z$. Surface is a **plane**. **Example** 4: $f(x,y) = {}^{xy}/_{x+y^2}$. Here, $dom_f = \mathbf{R}^2\backslash$(parabola $x = -y^2$).



Tangent Planes

When f: $\mathbf{R}^2 \to \mathbf{R}^2$ has a "*smooth*" graph near (a,b,c = f(a,b)), the graph may be approximated by a plane. To define the plane at (a,b,c), we need 2 directions. Cut the surface by the plane **parallel** to Oxz to give a *curve* through (a,b,c). Let g_b: $\mathbf{R} \mapsto \mathbf{R}$; $x \mapsto f(x,b)$, be the 1-variable function obtained by *keeping* y fixed. This has derivative g_b' where $g_b'(a) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = m_1$, say. The graph of g_b has tangent *lines*: $y = b$; $z-c = m_1(x-d)$ at (a,b,c). This gives us **one** of the directions we need.



Combining the *derivatives* for each y, we obtain a partial derivative f_x (aD_1f): $\mathbf{R}^2 \to \mathbf{R}$; $(x,y) \mapsto g'_y(x)$. When $z = f(x,y)$, we write ${}^{\partial z}/_{\partial x}$ for $f_x(x,y)_x$. **Example**: if $z = x^4+3x^2y+2xy^4+y^7$, then ${}^{\partial z}/_{\partial x} = 4x^3+6xy+2y^4+0$ (Differentiating with **respect** to x, keeping y fixed).

Similarly, cutting the *surface* by a plane parallel to Oyz, we get another curve, h_a, where h_a: $(y) \mapsto f(x,y)$ with tangent line $x = a$; $z-c = m_2(y-b)$. Derivatives h'_x combine to form $f_y = (aD_2f)$: $\mathbf{R} \to \mathbf{R}$; $(x,y) \mapsto h'x(y) = {}^{\partial z}/_{\partial y}$. In our *example*, ${}^{\partial z}/_{\partial x} = 0+3x^2+2x4y^3+7y^6$.

21st April 1999

Example

$z = 3x^2+4xy+5y^2$. ${}^{\partial z}/_{\partial x} = 6x+4y+0$. ${}^{\partial z}/_{\partial y} = 0+4x+10y$. These are **both** functions from $\mathbf{R}^2$ to $\mathbf{R}$. At a point (a, b, c = f(a,b)) on the *surface*, the tangent plane has equation $(z-c) = m_1(x-a)+m_2(y-b)$, where $m_1 = {}^{\partial z}/_{\partial x}(a,b)$; $m_2 = {}^{\partial z}/_{\partial y}(a,b)$. At $x = 2$, $y = -1$, $z = 12-8+5 = 9$, ${}^{\partial z}/_{\partial x} = 8 = m_1$; ${}^{\partial z}/_{\partial y} = -2 = m_2$. *Tangent* plane: $z-9 = 8(x-2)-2(y+1)$; $z = 8x-2y-9$. When $z = f(x,y)$, ${}^{\partial z}/_{\partial x} = f_x(x,y)$, ${}^{\partial z}/_{\partial y} = f_y(x,y)$.

Higher Derivatives

Define $f_{xx} = (f_x)_x$ and write $f_{xx}(x,y) = {}^{\partial^2 z}/_{\partial x^2}$. *Similarly* $f_{yy} = (f_y)_y = {}^{\partial^2 z}/_{\partial y^2}$; $f_{xy} = (f_x)_y = {}^{\partial^2 z}/_{\partial x \partial y}$; $f_{yx} = (f_y)_x = {}^{\partial^2 z}/_{\partial y \partial x}$. Note: f_{xy} and f_{yx} are equal to each other for "*nice*" functions. In our example, $f_{xx} = 6$, $f_{yx} = f_{xy} = 4$, and $f_{yy} = 10$.

Stationary Points

f: **R**→**R** has a *stationary point* at (a,b,c) if (a) f is differentiable on (a,b) — "tangent plane **well** defined"; (b) $f_x(a,b) = 0 = f_y(a,b)$. So the *tangent* plane is z = c. Exercise: Find the 3 stationary points when $f(x,y) = 3x^4-6x^2-12xy+2y^2$. So $f_x(x,y) = 12x^3-12x-12y$; $f_y(x,y) = -12x+4y$. Setting to **zero**, we have $0 = 12(x^3-x-y)$ and $0 = 4(-3x+y)$. We have *2 sim. equations* which we solve to get x = 0 or 2 or -2. We get y = 0 or 6 or -6 by **back** substitution, and the z values by substitution into the *original* equation. So the three points are (0,0,0), (2,6,-48), (-2,-6,-48).

Testing: $A = f_{xx}(a,b)$; $B = f_{xy}(a,b)$; $C = f_{yy}(a,b)$. If $AC > B^2$ and $A > 0$, then we have a local **min**. If $AC > B^2$ & $A < 0$, then we have a local **max**. If $AC < B^2$ then we have a *saddle* point. If $AC = B^2$ then there is no conclusion. In our example, $f_{xx}(a,b) = 36x^2-12$. $f_{xy}(a,b) = -12$. And $f_{yy}(a,b) = 4$. From these we **construct** the following table:

(a,b,c)	A	B	C	AC-B²	conclusion
0	-12	-12	4	-192	saddle
(2,6,-48)	132	-12	4	+ve	min
(-2,-6,-48)	132	-12	4	+ve	min

23rd April 1999

Taylor Series for f(x,y) at (0,0)

This should be a **power** series in 2 variables: $f(x,y) = k+(ax+by)+{}^1/_2(Ax^2+2Bxy+Cy^2)+\dots$, and should be *convergent* for all (x,y) near (0,0). Then (i) $f(0,0) = k$; (ii) $f_x(x,y) = (a+0)+{}^1/_2(2Ax+2By+0)+\dots = a+(Ax+By)+\dots$. So $f_x(0,0) = a$, $f_y(x,y) = b+(Bx+Cy)+\dots$ So $f_y(0,0) = b$. (iii) $f_{xx}(x,y) = A+\dots$; $f_{xy}(x,y) = f_{yx}(x,y) = B+\dots$. And $f_{yy}(x,y) = C+\dots$, etc.

So $f(x,y) = f(0,0) + [f_x(0,0)x+f_y(0,0)y] + {}^1/_{2!}[f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2] + \dots + \frac{1}{n!}\sum_{i,k=0;i+j=n}(\frac{n!}{i!j!})[\partial^n f/\partial x^i \partial y^j](0,0)x^i y^j + \dots$ Use this to **derive** a test for stationery points: if (0,0,k) is a stationary point, then $f_x(0,0,k) = 0 = f_y(0,0,k)$. So the Taylor series is $f(x,y) = k + (0x+0y) + {}^1/_2[Ax^2+2Bxy+Cy^2] + \dots$ So the **values** of f(x,y)-k near (0,0) depend on $Q = Ax^2+2Bxy+Cy^2$. If $Q > 0$, then (0,0,k) is a *minimum*. If $Q < 0$, then (0,0,k) is a *maximum*.

If Q takes +ve and -ve values, then (0,0,k) is called a *saddle* point. Suppose that $A \neq 0$. Then $Q = A[x^2+{}^{2B}/_A xy+{}^C/_A y^2] = A[(x+{}^B/_A y)^2+({}^{AC-B^2}/_{A^2})y^2]$. So if $A > 0$ and $AC > B^2$, then $Q \geq 0$ implies that we have a **minimum**. If $A < 0$ and $AC > B^2$, then $Q \leq 0$ implies that we have a *maximum*.

If $AC < B^2$, then (i) $x = B\varepsilon$; $y = -A\varepsilon$ *imply that* $Q = A[0+(AC-B^2)\varepsilon^2]$, which has **opposite** sign to A. (ii) $x = \varepsilon$, $y = 0 \Rightarrow Q = A\varepsilon^2$ which has the **same** sign as A. These imply that we have a saddle point. If $A = 0$, $C \neq 0$, use $Q = C[\dots]$. If $A = 0$ and $B \neq 0$, use $Q = Bxy$ which is clearly a **saddle** point.

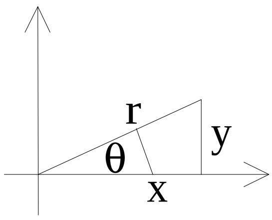
26th April 1999

Differentials

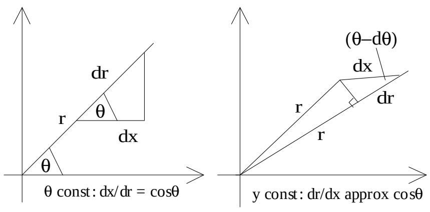
dx, dy are *small* increments in x & y. Suppose that f: $\mathbf{R}^2 \to \mathbf{R}$ is a differentiable function. Define the differential $dz = {}^{\partial z}/_{\partial x}dx + {}^{\partial z}/_{\partial y}dy$ so that (x+dx, y+dy, z+dz) is a point on the tangent plane to the *graph* of f at (x,y,z). In general, dz is not equal to the increment $\Delta z = f(x+dx,y+dx)-f(x,y)$.

Example: $z = x^3-xy$, x = 1, y = -1, z = 2, dx = 0.1, dy = -0.1. So we have $\Delta z = \{(1.1)^3-(1.1)(-1.1)\}-2 = 0.541$. This is the **exact** increment in z. But $dz = (3x^2-y)dx-xdy = 4dx-dy$. At (1,-1,2), this is 0.4+0.1 = 0.5. This is the **approximate** change in z. Exercise: *Calculate* Δz and dz at (1,-1,2) when dx = 0.01 and dy = -0.01. So $\Delta z = \{(1.01)^3-(1.01)(-1.01)\}-\{1^3-(1)(-1)\} = 0.050401$. And $dz = (3x^2-y)dx-xdy = 4dx-dy$. At (1,-1,2), dz = 0.05.

Note: Since dx and dy are *arbitrary increments*, whenever there are functions λ, μ: $\mathbf{R}^2 \to \mathbf{R}$ such that $dz = \lambda dx+\mu dy$, it *follows* that $\lambda = {}^{\partial z}/_{\partial x}$ and $\mu = {}^{\partial z}/_{\partial y}$. Example: Cartesian to Polar co-ordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$ (x and y *functions* of r and θ). So $dx = \cos\theta d\theta - r\sin\theta d\theta$ and $dy = \sin\theta d\theta + r\cos\theta d\theta$. Solve for dr, dθ: $\cos\theta dx+\sin\theta dy = dr$, $-{}^{\sin\theta}/_r dx+{}^{\cos\theta}/_r dy = d\theta$. By the *previous* note, when r and θ are considered as functions of x & y, the **partial** derivatives are ${}^{\partial r}/_{\partial x} = \cos\theta$, ${}^{\partial r}/_{\partial y} = \sin\theta$, ${}^{\partial\theta}/_{\partial x} = -{}^{\sin\theta}/_r$, ${}^{\partial\theta}/_{\partial y} = {}^{\cos\theta}/_r$.

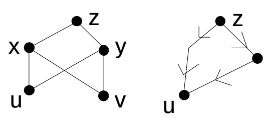


Or, $x = r\cos\theta$ and $y = r\sin\theta$ imply that $r = \sqrt{(x^2+y^2)}$ and $\theta = \tan^{-1}({}^y/_x)$. So ${}^{\partial r}/_{\partial x} = {}^{½.2x}/_{\sqrt{(x^2+y^2)}} = {}^x/_r = \cos\theta$, etc. Note: ${}^{\partial z}/_{\partial x}$ is not *equal* to $1/({}^{\partial x}/_{\partial r})$. In fact, ${}^{\partial z}/_{\partial x} = {}^{\partial x}/_{\partial r} = \cos\theta$. **Exercise**: Let $x = u^2+v^2$ and $y = 2uv$. Considering u & v as *functions* of x & y, when x is not equal to ±y, show that ${}^{\partial u}/_{\partial x} = {}^{\partial v}/_{\partial y}$ and ${}^{\partial u}/_{\partial y} = {}^{\partial v}/_{\partial x}$. A: $x = u^2+v^2$ and $y = 2uv$ **imply that** $dx = 2u.du+2v.dv$ and $dy = 2v.du+2u.dv$ So $vdx-udy = 2(v^2-u^2)dv$, and $udx-vdy = 2(u^2-v^2)du$. So ${}^{\partial u}/_{\partial x} = {}^{\partial v}/_{\partial y} = {}^u/_{2(u^2-v^2)}$; ${}^{\partial u}/_{\partial y} = {}^{\partial v}/_{\partial x} = -{}^v/_{2(u^2-v^2)}$.

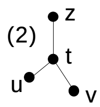


Composition of Functions

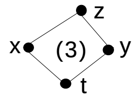
Let z = z(x,y), x = x(u,v), y = y(u,v). Then z is a **function** of u and v Find ${}^{\partial z}/_{\partial u}$ and ${}^{\partial z}/_{\partial v}$. By differentials, $dz = {}^{\partial z}/_{\partial x}dx+{}^{\partial z}/_{\partial y}dy = {}^{\partial z}/_{\partial x}({}^{\partial x}/_{\partial u}du + {}^{\partial x}/_{\partial v}dv) + {}^{\partial z}/_{\partial y}({}^{\partial y}/_{\partial u}du + {}^{\partial y}/_{\partial y}dv) = ({}^{\partial z}/_{\partial x}{}^{\partial x}/_{\partial u} + {}^{\partial z}/_{\partial y}{}^{\partial y}/_{\partial u})du + ({}^{\partial z}/_{\partial x}{}^{\partial x}/_{\partial v} + {}^{\partial z}/_{\partial y}{}^{\partial y}/_{\partial v})dv$. Hence ${}^{\partial z}/_{\partial u} = {}^{\partial z}/_{\partial x}{}^{\partial x}/_{\partial u} + {}^{\partial z}/_{\partial y}{}^{\partial y}/_{\partial u}$ and ${}^{\partial z}/_{\partial v} = {}^{\partial z}/_{\partial x}{}^{\partial x}/_{\partial v} + {}^{\partial z}/_{\partial y}{}^{\partial y}/_{\partial v}$. We may represent the *dependency* of the variables by a graph as shown.



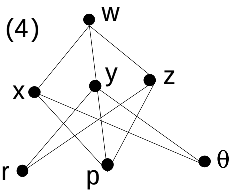
Each term in ${}^{\partial z}/_{\partial u}$ corresponds to a **descending** path in the graph from z to u. Each factor in one of these two terms corresponds to an edge in the graph. **Example:** Find ${}^{\partial}/_{\partial u}(\sin^2(u^3+v^3)+e^{8uv})$. Put $x = u^3+v^3$, $y = 8uv$, $z = \sin^2x+e^y$. Then ${}^{\partial x}/_{\partial u} = (2\sin x\cos x)3u^2+e^y8v = 3u^2\sin(2u^3+2v^3)+8ve^{8uv}$.



(2) When $z = z(t)$ and $t = t(u,v)$, the **graph** is as shown. $\partial z/\partial u = \partial z/\partial t\,\partial t/\partial u = dz/dt\,\partial t/\partial u$. And $\partial z/\partial v = dz/dt\,\partial t/\partial y$. *Exercise*: Let $z = z(u^2+v^2)$ be an *arbitrary* function of (u^2+v^2). Show that $u\,\partial z/\partial v = v\,\partial z/\partial u$ (= $2uv\,dz/dt$ when $t = u^2+v^2$). A: $z = z(u^2+v^2) = z(t)$, where $t = u^2+v^2$. $\partial z/\partial u = dz/dt.\,\partial t/\partial u = 2u\,\partial z/\partial t$. And $\partial z/\partial v = dz/dt\,\partial t/\partial v = 2v\,\partial z/\partial t$. Hence $v\,\partial z/\partial u = u\,\partial z/\partial v$ (= $2uv\,dz/dt$).



(3) **When** $z = z(x,y)$ and $x = x(t)$; $y = y(t)$, the graph is as shown and $dz/dt = \partial z/\partial x\,dx/dt + \partial z/\partial v\,dy/dt$.



(4) 3 *or more variables:* **similar** rules apply. $W = W(x,y,z)$ has *differential* $dW = \partial w/\partial x\,dx + \partial w/\partial y\,dy + \partial w/\partial z\,dz$. Example: *Spherical* polar coordinates: $x = r\cos\theta\sin\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi$, so $x^2+y^2+z^2 = r^2$. If $W = W(x,y,z)$, then $\partial w/\partial r = \partial w/\partial x\,\partial x/\partial r + \partial w/\partial y\,\partial y/\partial r + \partial w/\partial z\,\partial z/\partial r = \cos\theta\sin\varphi\,\partial w/\partial x + \sin\theta\sin\varphi\,\partial w/\partial y + \cos\varphi\,\partial w/\partial z$, etc.

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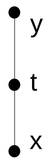
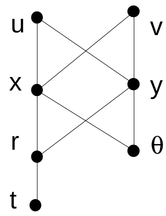
Let $Z = XY$. Cutting with plane $x = c^2$, then $xy = c^2$. Cutting with plane $z = -c^2$, then $xy = -c^2$. $\partial z/\partial x = y = 0$. $\partial z/\partial y = 0$. So *there is a point* (0,0,0). Now (A) = $\partial^2 z/\partial x^2$ = (C) = $\partial^2 z/\partial y^2 = 0$. And (B) = $\partial^2 z/\partial x\partial y = 1$. Now $AC-B^2 = -1$. This implies that we have a **saddle** point.

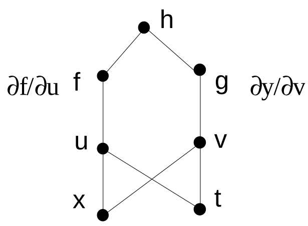
Composition Rule Example





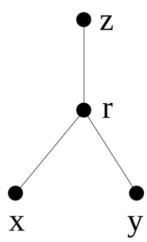
(1 **variable** case). Let $y = f(g(x))$. Put $t = g(x)$, $y = f(t)$. How many *paths* are there in the graph from y to x? Answer: 1 — with 2 edges. Conclusion: $\partial y/\partial x = \partial y/\partial t.\,\partial t/\partial x$. Now consider 1 variable functions, where $dy/dx = dy/dt.\,dt/dx$). Example: $y = \cos(x^3) = \cos t$ ($t = x^3$). $dy/dx = -\sin t.3x^2 = -3x^2\sin x^3$. Example: $u = x^2-y^2$, $v = 2xy$. $x = r\cos\theta$, $y = r\sin\theta$, $r = 2a(1-\cos t)$, a *constant*.

We want $\partial u/\partial t$ (θ constant). How many *paths* are there from x to 2? (DOWNWARD paths). Looking at the diagram above, there are 2 paths, both with 3 edges. So $\partial u/\partial t = \partial u/\partial x.\,\partial x/\partial r.\,dr/dt + \partial u/\partial y.\,\partial y/\partial r.\,dv/dt = (2x\cos\theta+2x\sin\theta)2a\sin t = 4ax(\sin\theta+\cos\theta)\sin t = 4ar\times\cos\theta(\sin\theta+\cos\theta)\sin t = 4ax/r\,(x+y)\sin t$, etc.



Exercise: Let $h = f(x-ct)+g(x+ct)$. (f and g are *arbitrary* functions; c is a constant). Prove that $c\,\partial^2 h/\partial x^2 - \partial^2 h/\partial t^2 = 0$. A: We have $h = h(f,g) = f+g$. Further, $f = f(u)$, $g = g(v)$, $u = x-ct$, and $v = x+ct$. Then $\partial h/\partial t = \partial h/\partial f.\,\partial f/\partial u.\,\partial u/\partial t + \partial h/\partial g.\,\partial g/\partial v.\,\partial v/\partial x = 1.\,\partial f/\partial u(-c) + 1.\,\partial g/\partial v(+c) = c(\partial g/\partial v - \partial f/\partial u)$. And $\partial h/\partial x = \partial h/\partial f.\,\partial f/\partial u.\,\partial u/\partial x + \partial h/\partial g.\,\partial g/\partial v.\,\partial v/\partial x = 1.\,\partial f/\partial u.1 + 1.\,\partial g/\partial v.1 = \partial g/\partial v + \partial f/\partial u$. Now $\partial^2 h/\partial x^2 = \partial/\partial x(\partial g/\partial u + \partial f/\partial u) = \partial/\partial v(\partial u/\partial v)\,\partial v/\partial x + \partial/\partial u(\partial f/\partial u)\,\partial u/\partial x = \partial^2 y/\partial v^2 + \partial^2 f/\partial x^2$. And $\partial^2 h/\partial t^2 = c\{\partial/\partial v(\partial g/\partial v)\,\partial v/\partial t - \partial/\partial u(\partial f/\partial u)\,\partial u/\partial t\} = c\{\partial^2 g/\partial v^2(c) - \partial^2 f/\partial u^2(-c)\} = c^2\{\partial^2 g/\partial v^2 + \partial^2 f/\partial u^2\}$.

More Examples



$z = \log r$, $r = x^2+y^2$. So $\partial z/\partial x = dz/dr.\,\partial r/\partial x = 2x/r$, and $\partial z/\partial y = dz/dr.\,\partial r/\partial y = 2y/r$. Thus $x\,\partial z/\partial x + y\,\partial z/\partial y = 2/r(x^2+y^2) = 2r/r = 2$. When $z = f(r)$, $\partial z/\partial x = dz/dr.2x$ and $\partial z/\partial y = dz/dr.2y$, so that $x\,\partial z/\partial x + y\,\partial z/\partial y = dz/dr\{2x^2+2y^2\} = 2r.\,dz/dr$.

$x = u^3+3uv^2$ and $y = v^3+3u^2v$. So $dx = 3(u^2+v^2)du+6uv.dv$, and $dy = 6uv.du+3(u^2+v^2)du$. *Eliminate* du: $6uv(u^2+v^2)dv = \{(u^2+v^2)dx-3(u^2+v^2)^2du\} = \{2uvdy-12u^2v^2du\}$, so that $(u^2+v^2)dx-2uvdy = 2(u^2-v^2)^2du$. Now <u>eliminate</u> du: $6uv(u^2+v^2)du = \{2uvdx-12u^2v^2du\} = \{(u^2+v^2)dy-3(u^2+v^2)^2du\}$. **And** $-2uvdx+(u^2+v^2)dy = 3(u^2-v^2)du$. *Thus* ${}^{\partial u}/_{\partial x} = {}^{\partial v}/_{\partial y} = (u^2+v^2)3(u^2-v^2)^2$ and ${}^{\partial u}/_{\partial y} = {}^{\partial u}/_{\partial x} = {}^{-2uv}/_{3(u^2-v^2)^2}$. **Then** ${}^{\partial u}/_{\partial x}+{}^{\partial u}/_{\partial y} = {}^{u^2-2uv+v^2}/_{3(u-v)^2(u+v)^2} = {}^{1}/_{3(u+v)^2}$.

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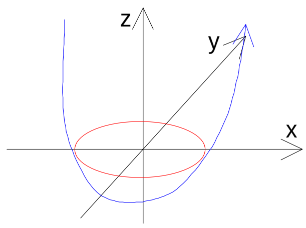


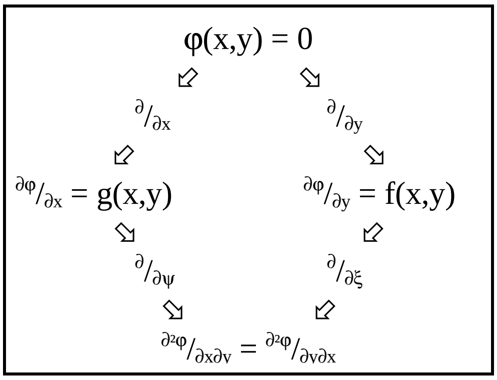
Consider a curve in $\mathbf{R}^2$ with *equation* $\varphi(x,y) = 0$, e.g. $3x^2+4xy+5y^2-12 = 0$. Differentiate out x: $6x+4(y+x{}^{dy}/_{dx})+10y{}^{dy}/_{dx} = 0$. $(6x+4y)+(4x+10y){}^{dy}/_{dx} = 0$; ${}^{dy}/_{dx} = {}^{-(3x+2y)}/_{2x+5y}$. At point (1,1), ${}^{dy}/_{dx} = -{}^{5}/_{7}$; tangent is $(y-1) = -{}^{5}/_{7}(x-1)$; $5x+7y = 12$. Now consider the **function** $z = \varphi(x,y)$. The section with the xy plane is $\varphi(x,y) = 0$. *Differential*, $dz = {}^{\partial\varphi}/_{\partial x}dx+{}^{\partial\varphi}/_{\partial y}dy$. In our *example*, $dz = (6x+4y)dx+(4x+10y)dy$.

We **force** this to lie in the x-y plane, so that dz = 0. This *gives* $0 = (6x+4y)+(4x+10y){}^{dy}/_{dx}$ as before. Tangent plane at (1,1,0) is $(z-0) = (6+4)(x-1)+(4+10)(y-1)$; $z = 10x+14y-24$, which intersects the x-y plane in the **line** $0 = 5x+7y-12$.

Surface in $\mathbf{R}^3$, $\varphi(x,y,z) = 0$. Example: sphere $x^2+y^2+z^2-3 = 0$ (centre (0,0,0), radius $\sqrt{3}$). Consider this as the <u>section</u> of the function $\mathbf{R}^3\rightarrow\mathbf{R}$, $W = \varphi(x,y,z)$, by the 3-space W = 0. *Differentials*: $dW = {}^{\partial\varphi}/_{\partial x}dx+{}^{\partial\varphi}/_{\partial y}dy+{}^{\partial\varphi}/_{\partial z}dz$. If we stay in the 3 space W = 0, i.e. force dW = 0, **then** $dz = -({}^{\partial\varphi/\partial x}/_{\partial\varphi/\partial z})dx - ({}^{\partial\varphi/\partial y}/_{\partial\varphi/\partial z})dy$. Note: Red = ${}^{\partial z}/_{\partial x}$ and Blue = ${}^{\partial z}/_{\partial y}$. Note: In $\mathbf{R}^2$, ${}^{dy}/_{dx} = -({}^{\partial\varphi/\partial x}/_{\partial\varphi/\partial y})$.

In our example, $W = x^2+y^2+z^2-3$, so ${}^{\partial W}/_{\partial x} = 2x = 2$ at (1,1,1,0); ${}^{\partial W}/_{\partial y} = 2y = 2$ at (1,1,1,0); ${}^{\partial W}/_{\partial z} = 2x = 2$ at (1,1,1,0). The tangent **hyperplane** at (1,1,1,0) is $(W-0) = 2(x-1)+2(y-1)+2(z-1)$; $W = 2x+2y+2z-6$. The solution with the 2 space W = 0 gives the plane $x+y+z = 3$, which is the *tangent* plane to the sphere at (1,1,1).

<u>Definition</u>: The D.E. $f(x,y){}^{dy}/_{dx}+g(x,y) = 0$ is <u>exact</u> if ${}^{\partial f}/_{\partial x} = {}^{\partial g}/_{\partial y}$. *Example*: $(2x+5y){}^{dy}/_{dx}+(3x+2y) = 0$ is exact because ${}^{\partial}/_{\partial x}(2x+5y) = 2$ and ${}^{\partial}/_{\partial y}(3x+2y) = 2$. <u>Motivation</u>: shown in the box. And so $dz = 0 = g(x,y)dx + f(x,y)dy$.



Method: if $f(x,y)dy + g(x,y)dx = 0$, then $f(x,y) = {}^{\partial\varphi}/_{\partial y}$ and $g(x,y) = {}^{\partial\varphi}/_{\partial x}$ for some unknown φ. Solve by integrating *partially*: $\int f(x,y)\partial y$ and $\int g(x,y)\partial x$. But **what** is partial integration w.r.t. x? $\int g(x,y)\partial x$ means: *keep* y constant, *integrate* w.r.t. x; the *constant* of integration is an *arbitrary* function of y.

Example: $(2x+5y)dy+(3x+2y)dx = 0$. Now $\int(2x+5y)\partial y = 2xy+{}^{5}/_{2}y^2+\alpha(x)$; and $\int(3x+2y)\partial x = {}^{3}/_{2}x^2 + 2xy + \beta(y)$. These two expressions must **both** be $\varphi(x,y)$. So we can *say that* $\varphi(x,y) = {}^{3}/_{2}x^2+2xy+{}^{5}/_{2}y^2+c = 0$, where we **noticed that** $\alpha(x) = {}^{3}/_{2}x^2$ and $\beta(y) = {}^{5}/_{2}y^2$. The **boundary** condition y = 1 when x = 1 gives ${}^{3}/_{2}+5+{}^{5}/_{2}+c = 0$; $c = -6$.

Assignment 3

Q: Test for *convergence* the following series: (a) $\Sigma^{\infty}_{n=1}\ {}^{n^3-2}/_{n^3+3}$; (b) $\Sigma^{\infty}_{n=1}\ {}^{n}/_{4n^2-3}$; (c) $\Sigma^{\infty}_{n=1}\ {}^{(2n)!}/_{(n!)^2}$; (d) $\Sigma^{\infty}_{n=3}\ {}^{\sin(5\pi/n^2)}/_{n^2}$; (e) $\Sigma^{\infty}_{n=1}\ (-1)^{n+1}/(2n-1)$. A: (a) $\Sigma^{\infty}_{n=1}\ {}^{n^3-2}/_{n^3+3} = {}^{1-(2/n^3)}/_{1+(3/n^3)}$, which tends to 1 as n tends to ∞. So this series is divergent. (b) $\Sigma^{\infty}_{n=1}\ {}^{n}/_{4n^2-3} \sim {}^{1}/_{4n}$. So **compare** with $\Sigma\ {}^{1}/_{5n}$. Now ${}^{n}/_{4n^2-3} - {}^{1}/_{5n} = {}^{5n^2-4n^2+3}/_{5n(4n^2-3)} = {}^{n^2+3}/_{5n(4n^2-3)} > 0$. So the series is **divergent** by the comparison test.

(c) Apply the *ratio* test: ${}^{(2(n+1))!}/_{((n+1)!)^2}.{}^{(n!)^2}/_{(2n)!} = {}^{(2n+1)(2n+1)(2n)!n!n!}/_{(n+1)^2n!n!(2n)!} = {}^{4(1+(1/n))(1+(1/2n))}/_{(1+(1/n))^2}$, which tends to 4 as $n\to\infty$ (& 4 > 1). The series diverges by the *ratio* test. (d) $|{}^{\sin(5\pi/n^2)}/_{n^2}| \leq {}^{1}/_{n^2}$, so the series is *absolutely* convergent. (e) **Alternating** series: ${}^{1}/_{2n-1} \to 0$ as $n\to\infty$. $({}^{1}/_{2n-1}) - ({}^{1}/_{2(n+1)-1}) = {}^{2n+1-(2n-1)}/_{(4n^2-1)} = {}^{2}/_{(4n^2-1)} > 0\ \forall\ n > 0$. So the series is *convergent* by the **alternating** series test.

Q: Find the **radius** of convergence and the *interval* of convergence for the following series. For what values of x do the series <u>converge</u> (i) absolutely; (ii) conditionally: (a) $\Sigma^{\infty}_{n=1}\ (-x)^{n+1}/(2n-1)$; (b) $\Sigma^{\infty}_{n=1}\ nx^n/4^n(n^2+1)$; (c) $\Sigma^{\infty}_{n=1}\ n!(x-4)^n$. A: (i) ${}^{1}/_{R} = \lim_{n\to\infty}\ {}^{(2n-1)}/_{(2n+1)} = 1$. When x = 1, the *series* is $\Sigma\ (-1)^{n+1}/2n-1$, which is convergent by the above. When x = -1, the series is $\Sigma{}^{1}/_{2n-1}$, which is divergent by *comparison* with $\Sigma\ 1/3^n$. So the series is cvt on (-1,1], *absolutely* cvt on (-1,1); conditionally cvt on {1}.

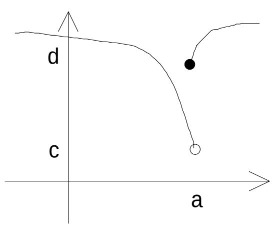
(b) ${}^{1}/_{R} = \lim_{n\to\infty}\ [(n+1)/[4^{n+1}((n+1)^2+1)]].[4^n(n^2+1)/n] = \lim_{n\to\infty}\ \frac{1}{4}({}^{1+(1/n)}/_{1})({}^{1+(1/n^2)}/_{(1+(1/n))^2+(1/n^2)}) = \frac{1}{4}$. When x = 3, the **series** is $\Sigma\ {}^{n}/_{n^2+1}$ — dvt by *comparison* with $\Sigma\ {}^{1}/_{2n}$. When x = -4, the *series* is $\Sigma\ (-1)^n n/n^2+1$ — cvt by alternating *series* test. So the series is **cvt** on [-4,4); absolutely cvt on (-4, 4); conditionally cvt on {-4}. (c) ${}^{1}/_{R} = \lim_{n\to\infty}\ {}^{(n+1)!}/_{n!} = \infty$. So R = 0 and the series *converges* only at x = 4. So the series is cvt on {4}, absolutely cvt on the empty set; **conditionally** cvt on {4}.

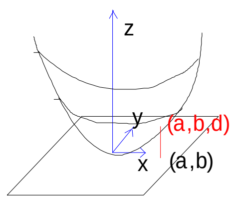
Q: Calculate the **Maclaurin** series for the following functions: (i) tan(x)+log(1+x); (ii) tan(x).log(1+x) — as far as the *term* in x^6. A: Knowing $\tan x = x+{}^{1}/_{3}x^3+{}^{2}/_{15}x^5+\ldots$, and $\log(1+x) = x-{}^{1}/_{2}x^2+{}^{1}/_{3}x^3-{}^{1}/_{4}x^4+{}^{1}/_{5}x^5-{}^{1}/_{6}x^6+\ldots$, (i) adding these polynomial *approximations* gives $\tan x+\log(1+x) = 2x-{}^{1}/_{2}x^2+{}^{2}/_{3}x^3-{}^{1}/_{4}x^4+{}^{1}/_{3}x^5-{}^{1}/_{6}x^6+\ldots$ (ii) Similarly, *multiplying* gives $x^2-{}^{1}/_{2}x^3+{}^{2}/_{3}x^4-{}^{5}/_{12}x^5+{}^{4}/_{9}x^6+\ldots$

<u>5th May 1999</u>

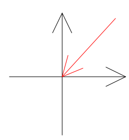
Limits

Looking at the first picture, $\lim_{x\to a-} f(x) = c$ and $\lim_{x\to a^-} f(x) = d$. There are 2 limits because there are 2 *directions* or *paths* to approach x = a. Looking at the **2nd** picture, how many paths are there to approach (a,b) in the x-y plane? Answer: there is an infinity of <u>straight</u> lines; lots of curves.



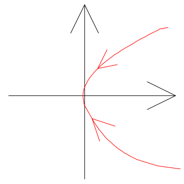


<u>Example</u>: $f(x,y) = {}^{xy}/_{x^2+y^2}$, (a,b) = (0,0). Let us choose a *few* paths to (0,0). Take the +ve y-axis: x = 0; y = t $\geq$ 0. On this **path,** $f(x,y) = {}^{0t}/_{0^2+t^2} = 0$. And so $\lim_{t\to 0+} 0 = 0$. Now take y = x in the +ve *quadrant*: x = t, y = t. So $f(x,y) = {}^{tt}/_{t^2+t^2} = \frac{1}{2}$, which tends to $\frac{1}{2}$ as t *tends* to 0+. We already have 2 **different** answers (0 and ½). So we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ does **not** exist.



Now consider the **parabola** $y = x^2$ in the +ve *quadrant*: $x = t$, $y = t^2$. Here, $f(x,y) = t.t^2/t^2+t^4 = {}^{t}/_{1+t^2}$, which tends to ${}^{0}/_{1+0} = 0$ as t *tends* to 0+. Now **take** $y = mx$: $x = t$, $y = mt$. Here, $f(x,y) = {}^{t.mt}/_{t^2+m^2t^2} = {}^{t^2m}/_{t^2+t^2m^2} = {}^{m}/_{1+m^2}$ which tends to ....

Definition: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ iff $\lim_{t\to t_0} f(x(t),y(t)) = L$ for every **possible** path $x = x(t)$, $y = y(t)$ with $x(t_0) = a$; $y(t_0) = b$. *Example*: $f(x,y) = (2x^2+y^4)/(3x^2+y^4)$ ($\sim {}^{0^2}/_{0^2}$ so expect **no** limit). Seek 2 paths giving *different* answers. Take $y = mx$: $x = t$, $y = mt$. So $f(x,y) = (2t^2+m^4t^4)/(3t^2+m^4t^4) = (2+m^4t^2)/(3+m^4t^2)$, which *tends* to ${}^{2+0}/_{3+0} = {}^{2}/_{3}$ as t *tends* to 0. Take the parabola $y^2 = x$: $x = t^2$, $y = t$. (See picture). So $f(x,y) = (2t^4+t^4)/(3t^4+t^4) = ¾$, which *tends* to ¾ as t *tends* to 0.



Assignment 4

(a) Solve the *inequality* ${}^{x}/_{3-2x^2} \le {}^{x}/_{4-3x^2}$ as a union of intervals. (b) Find the limit L as n tends to ∞ of the *sequence* $\{u_n\}_{n\ge0}$ where $u_n = {}^{2n^2+3}/_{3n^2+2}$. Determine an integer N *such that* $|u_n-L| < 10^{-12}$ for all $n > N$. A: (a) $\Leftrightarrow {}^{x(3-x^2)}/_{(3-2x^2)^2} \le {}^{x(4-3x^2)}/_{(4-3x^2)^2}$... etc., with the previously stated **method**. (b) $u_n = {}^{2n^2+3}/_{3n^2+2} = {}^{2+(3/n^2)}/_{3+(2/n^2)} \to {}^{2+0}/_{3+0} = {}^{2}/_{3} = L$ as $n\to\infty$. $|u_n-L| = |{}^{3(2n^2+3)-2(3n^2+2)}/_{3(3n^2+2)}| = {}^{5}/_{3(3n^2+2)}$. $|u_n-L| < 1/10^{12} \Leftrightarrow 5\times10^{12} < 9n^2+6$. *When* $n \ge 10^6$, $9n^2 \ge 9\times10^{12}$, and this *condition* is satisfied. So choose $N = 10^6$.

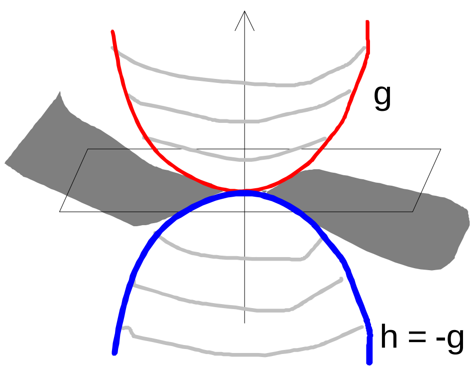
7th May 1999

Sandwich Rule

This is used to show that a **limit** does exist. Definition: (i) f: $\mathbf{R}^2\to\mathbf{R}$ is cts at (a,b) if $\lim_{(a,y)\to(a,b)} f(x,y) = L$ exists, and $f(a,b) = L$. This is similar to the *1-variable* case, i.e. continuity if left and right hand limits at $x = a$ equals $f(a)$. Also from 1 variable functions, f is cts on S (S is a **subset** of dom_f) if f is cts at every $(x,y) \in S$, and f is cts in its domain.

The Rule: If *functions* h, g: $\mathbf{R}^2\to\mathbf{R}$ are cts at (a,b), and $h(a,b) = g(a,b) = L$, and $h(x,y) \le f(x,y) \le g(x,y)$ ∀ (x,y) in a *disk* $D = \{(x,y) \mid (x-a)^2+(y-b)^2 < r^2\}$ ($r < 0$), then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Case $(a,b) = (0,0)$; $L = 0$. Note: When $L = 0$, we *usually* choose $h = -g$ and we then have (with $g(x,y) \ge 0$ for all (x,y)) $0 \le |f(x,y)| \le g(x,y)$. Some typical *functions* g in this case: $g(x,y) = x^2+y^2$, or $x^{2n}+y^{2n}$, or $|x|+|y|$, or $|x|^n+|y|^n$, or $(x^2+y^2)^n$, etc.



Example: $f(x,y) = {}^{xy^2}/_{x^2+y^2}$ ($\sim {}^{0^3}/_{0^2} \sim 0$). On a path $y = mx$, $y = t$, $x = mt$. Then $f(x,y) = {}^{mt^3}/_{m^2t^2+t^2} = ({}^{m^2}/_{1+m^2})t \to 0$ as $t\to0$. Attempt to use the *sandwich* rule: $|f(x,y)| = {}^{|xy^2|}/_{|x^2+y^2|} = {}^{|x||y^2|}/_{|x^2+y^2|} = {}^{|x|y^2}/_{x^2+y^2} \le {}^{|x|(x^2+y^2)}/_{(x^2+y^2)} = |x|$ (cts function). *Conclusion*: $\lim_{(x,y)\to(0,0)} {}^{xy^2}/_{x^2+y^2} = 0$ using $g(x,y) = |x|$ **and** $h(x,y) = -|x|$.

Example: $f(x,y) = (2x^5+3y^5)/[(x^2+y^2)(|x|+|y|)^2] \sim 0^5/0^4 \sim 0$. **Expect** $L = 0$. $|f(x,y)| \le 2|x|^5+3|y|^5/(x^2+y^2)(|x|+|y|)^2$ by the *triangle* inequality. Now $|x|^5 = |x||x|^2x^2 \le |x|(|x|+|y|)^2(x^2+y^2)$. And $|y|^5 = |y||y|^2y^2 \le |y|(|y|+|x|)^2(x^2+y^2)$. So $|f(x,y)| \le 2|x|+3|y|$ (cts *f*ⁿ on RHS). $L = 0$ at (0,0). So we take *this* to be $g(x,y)$ and $h = -g$. More examples: $f(x,y) = {}^{x^3+y^3}/_{x^2+y^2}$, $x^2y^2/(x^4+2y^3)$, $xy^3/(x^2+y^4)$, $x^3y/(x^3+y^4)$.

Q: Solve the **differential** equation $(2x^2y+5y^6)^{dy}/_{dx} = (7x^2+2xy^2) = 0$. A: First check to see if it is *exact* (i.e. in Pdx+Qdy = 0, is $^{\partial P}/_{\partial y} = {}^{\partial Q}/_{\partial x}$?). So **here** we have $(2x^2y+5y^6)dy + (7x^2+2xy^2)dx = 0$. $^{\partial Q}/_{\partial x} = 4xy$ and $^{\partial P}/_{\partial y} = 4xy$, so it **is** exact. Now compare $\int Pdx$ with $\int Qdy$.

$\int P\,dx = \int 7x^2+2xy^2dx = {}^{7x^3}/_3 + {}^{2y^2x^2}/_2 + c_1(y)$. <u>And</u> $\int Q\,dy = \int 2x^2y+5y^6dy = {}^{2x^2y^2}/_2 + (5y^7/7) + c_2(x)$. *Comparing*, $c_1(y)$ must be $5y^7/7$ and $c_2(x)$ must be $^{7x^3}/_3$. So the **general** solution is $^{7x^3}/_3+x^2y^2+(5y^7/7)+C = 0$. To find a *particular* solution, substitute in values. For example, Q: Find the solution such that y = 1 when x = 1. A: Substituting into the <u>general</u> solution, $^7/_3+1+{}^5/_7+C = 0$; $C = -{}^{85}/_{21}$.

Q: Locate the *stationary* points of the function $f(x,y) = x^3+2x^2+2xy-y^2-11x+2y+7$, and determine whether each is a **maximum**, **minimum** or **saddle** point. A: *Differentiate* with respect to x giving $3x^2+4x+2y-11 = 0$. With respect to y, we get $2x-2y+2 = 0$. Solving these <u>simultaneously</u>, we get x = -3, y = -2; and x = 1, y = 2.

Now differentiate f(x,y) with respect to x (**twice)** to get 6x+4. With respect to y twice gets us 2; and one of *each* gets us -2. Let our "Disc" be $A*C-B^2$, where A = 6x+4; B = 2 and C = -2. So the Disc = -12x-12.

First <u>solution</u> at (-3,-2): substitute into the disc to get 24; into A to get -14. We have a **maximum** because 24 > 0 and -14 < 0. <u>Second</u> solution: (1,2). Substitute into the *disc* to get -24. This is negative so we have a **saddle point**.

(Now read Exercise Sheet P and its solutions).

Exam Paper: May 1999

SECTION 1 (Compulsory)

(1) (a) Evaluate (i) $\lim_{x\to 0}\frac{(e^x-1)}{x}$, (ii) $\lim_{x\to 3}\frac{\sqrt{5x+1}-4}{\sqrt{7x+4}-5}$. **[6 marks]**

(b) Determine whether the following series are convergent or divergent, stating clearly the results used. (i) $\sum_{n=1}^{\infty}\frac{2n+3}{(n+1)^3}$, (ii) $\sum_{n=0}^{\infty}\frac{(3n+1)2^n}{(2n+1)3^n}$. **[8 marks]**

(c) Solve the differential equation $(x+y^2)\,{}^{dy}/_{dx} + 2(xy+y^3) = 0$ with boundary condition y = 1 when x = 1. **[6 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Solve the second-order, linear, recurrence relation $u_0 = -1$, $u_1 = 0$, $u_n = 5u_{n-1} - 6u_{n-2}$ $(n \geq 2)$. **[6 marks]**

(b) A sequence $S = \{u_n\}_{n\geq 1}$ is defined by $u_1 = 3$, $u_n = \sqrt{2+u_{n-1}}$ $(n \geq 2)$.
Prove that $1 \leq u_n \leq 2$ for all $n \geq 1$ and that S is increasing. Find $\lim_{n\to\infty} u_n$.
[9 marks]

(3) (a) Solve the following inequality, giving the solution set as a union of intervals:
$\frac{x^2-3}{x^2-4} \leq \frac{x^2-6}{x^2-9}$. **[8 marks]**

(b) Find the limit L as $n\to\infty$ of the sequence $\{u_n\}_{n\geq 1}$ when $u_n = \frac{n(3n+1)}{(n+1)(2n+1)}$.
Determine an integer N such that $|u_n - L| < 10^{-10}$ for all $n > N$. **[7 marks]**

(4) (a) Prove, from first principles, that $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series. **[5 marks]**

(b) Using partial fractions, obtain expressions for the k-th partial sums s_k (the sum of terms from n = 2 to n = k) of the following series, and hence find the sum of each series: (i) $\sum_{n=2}^{\infty}\frac{1}{n^2+n}$, (ii) $\sum_{n=2}^{\infty}\frac{1}{n^2+n-2}$. **[8 marks]**

What can you say about the series $\sum_{n=2}^{\infty}\frac{1}{n^2+n-1}$? **[2 marks]**

(5) Find all the stationary points of the function $f(x, y) = 2x^3-3x^2y+2y^3-y^2-8y$. **[8 marks]**
Classify these points as maxima, minima or saddle points. **[7 marks]**

(Questions done: 1, 3, 5)