Functions

The idea of a function is **basic** in calculus. It is a rule or correspondence whereby the elements of one set of objects are made to *correspond to unique* elements of another set. Illustrate a function by an arrow diagram from the 1st set A to the 2nd set B. Notice that to each element of A there is one and only one element of B corresponding to it under the function f. However, **several** elements of A may correspond to one and the same element of B.

There are *several ways of* defining a function. (i) By a **formula** e.g. $y = mx+c$, $y = 4x^2+2$ etc.; (ii) **Table**; (iii) **Graph**; (iv) A **rule of correspondence**. Examples have the form $y = f(x)$, where x is an element of the 1st set A ($x \in A$) and is called the argument or independent variable, and y is called the dependable variable.

The set of **allowable** values of x is called the domain of the function, and the corresponding values of y is called the range of the function. Domain of f: D_f; Range of f: R_f. If the domain consists of all the **real** numbers lying between certain limits, the variable x is said to be *continuous*. For example, if $10 \le x < 20$, then x is a *continuous variable* lying between 10 and 20, including 10 but excluding 20.

Use **Cartesian** graphs to illustrate functions such as $y = 4x^2+2$ $(-\infty \le x \le \infty)$. An example of a function which does not possess a graph is: $f(x) = \begin{cases} 1 \text{ when x is rational} \\ 0 \text{ when x is irrational} \end{cases}$.

Even & Odd Functions

If $f(x) = f(-x)$, for all elements of the **domain**, then the function is said to be even. If $f(x) = -f(x)$, then the function is said to be odd e.g. $y = \sin(x)$, $y=x$, $y=3x^3$. It is remarkable that any function can be expressed as a **sum** of an even function and an odd function:

$$f(x) = \underbrace{\tfrac{1}{2}(f(x)+f(-x))} + \underbrace{\tfrac{1}{2}(f(x)-f(-x))} = \text{even} + \text{odd}.$$

If a function has *symmetry* then this can be exploited in calculations.

Monotonic Functions

If $f(x_2) \ge f(x_1)$ for all x_1, x_2, such that $a \le x_1 < x_2 \le b$, **then** f is said to be a monotonically *increasing function*, in the interval $a \le x \le b$ ($x \in [a,b]$). [Note on *notation*: for $a<x\le b$, we write $x \in (a,b]$]. Similarly, if $f(x_2) \le f(x_1)$ for all x_1, x_2, *such that* $a \le x_1 < x_2 \le b$, then f is said to be a *monotonically decreasing function*, in the interval $a \le x \le b$.

Limits & Continuity

Consider $f(x) = \cos(x)$. When x **decreases towards zero** we see that cos(x) approaches +1 which is the value of cos(0). We say that *as x tends to zero* ($x \to 0$) then cos(x) tends to cos(0) [$\cos(x) \to \cos(0)$]. More precisely, we can choose a value of cos(x) as near as we like to cos(0), provided we choose x sufficiently close to zero. The value cos(0) is said to be the **limit** of cos(x) as x approaches zero.

More **generally,** we say that f(a) is the limit of f(x) as x approaches a, provided f(x) tends to f(a) as x tends to a. If x increases towards the value a *we write* x → a from the left, then use the notation f(x) → f(a-0) as x → a_-, or $\lim_{x\to a-} f(x) = f(a-0)$. If x **decreases** towards the value a or we have x → a from the right, we use the notation f(x) → f(a+0) as x → a_+, or we equivalently write $\lim_{x\to a+} f(x) = f(a+0)$.

In the **previous** example, +1 = cos(0) is the *limit* as x → 0_- and as x → 0_+ of f(x) = cos(x). in this case cos(x) is said to be continuous at x=0. More generally, f(x) is said to be continuous at x=a if f(a-0)=f(a)=f(a+0). Otherwise, f(x) is said to be discontinuous at x=a. Example: Suppose we **define a function** as follows: y = f(x) = |x| for (-1≤x≤0, 0≤x≤1); f(x) = 1 when (x=0). Now $\lim_{x\to 0-} f(x) = 0 = \lim_{x\to 0+} f(x)$. But as f(0) is *not equal to zero* (=1) then the function is discontinuous at x=0.

Inverse Functions

Given y = f(x), can we **solve** for x in terms of y? If yes, then x = fi(y) is called the *inverse function* of y = f(x). Now consider y = f(x) = cos(x), -π≤x≤π. If x = $^{\pi}/_{3}$, then y = ½. if x = -$^{\pi}/_{3}$, then y = ½. Therefore in the *interval* -π≤x≤π, y = +½ corresponds to *2 values of x*. So we cannot find an **inverse** function in the specified interval. However, if we restrict the function to the interval 0≤x≤π, there is only 1 value of x corresponding to each value of y lying between -1 and 1. So we can invert the *function* y=cos(x) in the interval 0≤x≤π. This shows the importance of defining the **domain** of values of the function. In general, given y = f(x), a≤x≤b, the inverse function y=fi(y) exists if to each *value of y*, there corresponds **exactly 1** value of x. Example: y = sin(x) for -$^{\pi}/_{2}$≤x≤$^{\pi}/_{2}$. An **inverse** function x = arcsin(y) or $\sin^{-1}$(y) *exists* for -1≤y≤1.

Examples 1

Q: Find the **domain** and **range** of f(x) = 1-√x. The domain, D_f, is 0 ≤ x < ∞, [0,∞), and the **Range**, R_f, is -∞ < y ≤ 1, (-∞, 1]. Q: Do the *same* for g(z) = √(4-z^2). Here, we want 4-z^2 to be positive. So 4 ≥ z^2, implying -2≤z≤2. So the **domain** is D_g: -2≤z≤2, [-2,2]; and the **range** is R_g: 0≤y≤2, [0,2]. When doing *questions like these*, always sketch graphs and try out values. Tip: when sketching e.g. y = sin(x) + cos(x), sketch cos(x) and sin(x) on the same graph and then **add them** up (As in SHM).

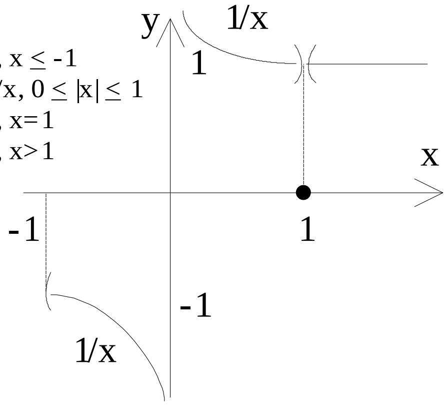
Q: Is y=x^3 odd or even? A: y(-x) = $(-x)^3$ = -x^3 = -y(x) so ODD. Q: What about y = 5cos(x)? A: y(-x) = 5cos(-x) = 5cos(x) = y(x) so **EVEN**. Q: Are the following functions monotonically *increasing, decreasing, or neither*? Q: y = x^2 A: Neither. Q: y=x^2 for [0,1]. A: Monotonically increasing.

In the following **equations**, solve for y as a function of x and determine the *domain* of the function(s). (i) xy+4x+3y=0; y(x+3) = -4x; y = -$^{4x}/_{x+3}$; x not equal to -3. **Therefore** D_f = (-∞, -3) ∪ (-3, ∞). (ii) $^{x^2}/_{a^2}$+$^{y^2}/_{b^2}$=1; y = ±b√[1-($^{x}/_{a}$)²]. For y = +b√[a-($^{x}/_{a}$)²], D_f=[-a,a]. **For** y = -b√[1-($^{x}/_{a}$)²], D_f=[-a,a].

Show that x = acos(t), y = bsin(t) are parametric equations for the two functions of x given above. Determine the *appropriate intervals* of t for these two functions. A: We have $(^x/_a)^2+(^y/_b)^2 = \cos^2 t+\sin^2 t = 1$. $y = +b\sqrt{[1-(^x/_a)^2]}$, [-a,a] *implies* (x,y) = (acost, bsint), $(0\le t\le\pi)$. For $y = -b\sqrt{[1-(^x/_a)^2]}$, [-a,a] implies (x,y) = (acost, bsint), $(-\pi\le t\le 0)$.

Q: **For the function** shown, discuss *limits, one-sided limits, continuity and one-sided continuity* for the points x = -1, 0 and 1. For x=-1, $f(-1_-)=0$, f(-1)=0, $f(-1_+) = -1$. It is **left** continuous but not **right** continuous at x=-1. Both *one sided limits* exist but are not equal so the limit does not exist. At x=0, $f(0_-) = -\infty$, $f(0_+) = +\infty$, f(0) is **not** defined. So it is neither left continuous nor right continuous at x=0. Both one-sided limits exist but are not equal. This implies that *limits does not* exist. At x=1, $f(1_-)=1$, f(1)=0, $f(1_+)=1$. So neither **left** continuous nor **right** continuous at x=1. Both one-sided limits exist and are equal, therefore limits exist at x=1. Note: The discontinuity is *removable* at x=1: by *redefining* f(1)=1, then $f(1_-) = f(1) = f(1_+) = 1$.

$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 \le |x| \le 1 \\ 0, & x=1 \\ 1, & x>1 \end{cases}$$



Q: **Analyse** $y = {}^{x^2}/_{x^2-1}$. A: We know x^2 cannot be 1, so x *cannot be* 1 or -1. When x=0, y=1, and as x tends to 1_+, y tends to $+\infty$. As x tends to $+\infty$, y tends to 1. Using **information** like this (and more) we can build up a picture of the graph of the function, and establish the function is even. Solving for x, $x = \pm\sqrt{(^y/_{y-1})}$. For the +ve sign, $0\le x\le 1$, $1<x<\infty$ and $-\infty<y<0$, $1<y<\infty$. For the **-ve** sign, $1<y<\infty$, $-\infty<x<-1$; $0\ge y>-\infty$, $-1<x\le 0$.

Differentiation

Let y = f(a) be **defined** and continuous in the interval $a\le x\le b$. We define the *left derivative* $f'_-(x)$ and the right derivative $f'_+(x)$ as follows:

$$f'_-(x) = \lim_{h\to 0-}\left[\frac{f(x+h)-f(x)}{h}\right];\ f'_+(x) = \lim_{h\to 0+}\left[\frac{f(x+h)-f(x)}{h}\right].$$

Where the **limit** exists for both formulae, $f'_-(x)$ is *defined* on $a<x\le b$, whereas $f'_+(x)$ is *defined* on $a\le x<b$. At points inside the interval i.e. $a<x<b$, if we have $f'_-(x) = f'_+(x)$ (=f'(x)) then f is said to be **differentiable** at x and f'(x) is called the **derivative** at x. Given $f'_-(x_0) = f'_+(x_0)$, where $a<x_0<b$, then $\lim_{x\to x_{0-}} f(x) = f(x_0) = \lim_{x\to x_{0+}}$ (See the **notes** for proof).

By definition, f'(x) is a function and the *domain of values of x* is included in the domain of f(x). If f'(x) is itself differentiable, its derivative is denoted by f''(x) and so on. **Alternative** notation: $f'(x) = {}^{dy}/_{dx}$ or ${}^{df}/_{dx}$, $\lim_{\delta x\to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\} = \lim_{\delta x\to 0}\left\{\frac{\delta y}{\delta x}\right\} = \frac{dy}{dx}$.

Rules of Differentiation

Let f(x), g(x) be *differentiable* in the open interval a<x<b. (i) **If** $\phi(x) = \alpha f(x) + \beta g(x)$ [α,β constants), then $\phi'(x) = \alpha f'(x)+\beta g'(x)$. (ii) If $\phi(x) = f(x)g(x)$, *then* $\phi'(x) = f(x)g'(x) + f'(x)g(x)$. This is the **product** rule, $d\phi/dx = d(uv)/dx = u(dv/dx) + (du/dx)v$. (iii) **Let** $\phi(x) = 1/g(x)$, g(x) not zero. Then $\phi'(x) = -g'(x)/[g(x)]^2$, g(x) not zero. (iv) **If** $\phi(x) = f(x)/g(x)$, g(x) not zero, *then* $\phi'(x) = g(x)f'(x)-f(x)g'(x)/[g(x)]^2$. This is the **quotient** rule. (v) **If** $\phi(x) = f(g(x))$, then $\phi'(x) = f'(g(x))\times g'(x)$. **(Provided** f is *differentiable* at g(x) and g is *differentiable* at x). Proofs of rules 1 and 2 are in the notes.

Standard Derivations: **If** f(x) = c then f'(x) = 0. **If** $f(x) = x^n$ (n +ve integer) then $f'(x) = nx^{n-1}$. If f(x) = sin(x), **then** f'(x) = cos(x). If f(x) = cos(x) *then* f'(x) = -sin(x).

Examples 2

Q: Using the **definition**, calculate the *derivative* of the following function: $F(x) = (x-1)^2+1$. Then find the **value** of the derivatives F'(-1), F'(0) and F'(2). A: Using $F(x+h)-F(x)/h = ((x+h-1)^2+1)-((x-1)^2+1)/h = (x+h-1)^2-(x-1)^2/h = ((x-1)+h)^2-(x-1)^2/h = (x-1)^2+2(x-1)h+h^2-(x-1)^2/h = 2(x-1)+h$. Here h < 0 or h > 0. Now $F'_-(x) = \lim_{h\to 0-}(F(x+h)-F(x)/h) = \lim_{h\to 0-}(2(x-1)+h) = 2(x-1)$. And $F'_+(x) = \lim_{h\to 0+}(F(x+h)-F(x)/h) = \lim_{h\to 0+}(2(x-1)+h) = 2(x-1)$. **So** $F'_-(x) = F'_+(x) = F'(x) = 2(x-1)$. Putting in *values*, F'(-1) = 2(-1-1) = -4; F'(0) = 2(0-1) = -2; F'(2) = 2(2-1) = 2.

Q: **Differentiate** the following Trig functions: (i) $d/dx(\sin^2(x)) = 2\sin(x)\cos(x)$. (ii) $\frac{d}{dx}(x^{\sqrt{2}} - x^{-\sqrt{2}}) = \sqrt{2}\,x^{\sqrt{2}-1} - (-\sqrt{2}\,x^{-\sqrt{2}-1}) = \sqrt{2}\,(x^{\sqrt{2}-1} + x^{-\sqrt{2}-1})$. (iii) $\arcsin(3+x^2)$. Let $y = \arcsin(3+x^2)$ so $\sin(y) = 3+x^2$. Now *differentiate implicitly*, $\cos(y)dy/dx = 2x$. So $dy/dx = 2x/\cos y = 2x/\sqrt{(1-\sin^2 y)} = 2x/\sqrt{[1-(3+x^2)^2]}$.

When given a **function** in terms of a *parameter t*, remember that $dy/dx = (dy/dt)/(dx/dt)$. Q: Find dy/dx and d^2y/dx^2 **when** $x^2+xy+y^2=7$. A: Differentiating *implicitly*, $2x+(x\,dy/dx+1.y)+2y\,dy/dx=0$. This implies $dy/dx(x+2y)+2x+y=0$. Therefore $dy/dx = -(2x+y)/(x+2y)$. Now $d^2y/dx^2 = d/dx(-(2x+y)/(x+2y)) = -\{(x+2y).(2+dy/dx)-(2x+y).(1+2dy/dx)/(x+2y)^2\}$ = after *simplification* = $-6(x^2+xy+y^2)/(x+2y)^3$. But as $x^2+xy+y^2=7$, then $d^2y/dx^2 = -6.7/(x+2y)^3 = -42/(x+2y)^3$.

Q: Do the **same** for $x^3+y^3=a^3$. *Differentiating* gives $3x^2+3y^2dy/dx = 0$. So $dy/dx = -(x/y)^2$, $y\neq 0$. [NOTE: Always **specify regions which** are not defined at every part of your answer]. Now $d^2y/dx^2 = -2(x/y)\times y-x(dy/dx)/y^2 = -2x/y^3(y-x-(x/y)^2) = -2x/y^3(y+x^3/y^2) = -2x/y^5(x^3+y^3) = -2a^3x/y^5$, $y\neq 0$.

Stationary points occur when $dy/dx=0$. If you arrive at *something* like $dy/dx = y-2x/2y-x$, so $0 = y-2x/2y-x$, and so y-2x=0, y=2x, then **stick** y=2x into your original equation and solve for x and y to get the points where the gradient is zero.

Q: The **height** of a body moving vertically is given by $S = -\frac{1}{2}gt^2+v_0t+s_0$, g>0, with s in metres and t in seconds. Find the body's **maximum** height. A: $dS/dt = -\frac{1}{2}.g.2t+v_0 = -gt+v_0$. When $dS/dt=0$, $-gt+v_0=0$; $t = +v_0/g$. <u>And</u> $dS^2/dt^2 = -g$, which is **less** than zero. Therefore at $t=v_0/g$, S reaches a maximum. S (with $t=v_0/g$) $= -\frac{1}{2}g.(v_0/g)^2+v_0.(v_0/g)+S_0 = -(v_0^2/2g)+(v_0^2/g)+S_0 = (v_0^2/2g)+S_0$. Therefore, *maximum* height = $S_0+(v_0^2/2g)$.

Q: For the **function** $f(x) = {}^{ax+b}/_{x^2-1}$, find **values** for a and b such that f(x) has a local extreme value of a at x=3. Is this a *max or min*? A: Differentiate, obtaining ${}^{dy}/_{dx} = ax^2+2bx+a$. So when ${}^{dy}/_{dx}=0$, $0=ax^2+2bx+a$. Put x=3 in, **then** 0=9a+6a, or 10a+6b=0. But, at x=3, y=1. Substituting x=3, y=1 into f(x), $1={}^{3a+b}/_{9-1}$; 3a+b=8. We now have *2 sim. equations* which we solve to get a = 6, b = -10. **Therefore** $f(x) = -6x-10/x^2-1$ has a <u>stationary value</u> of 1 at x=3. Now, with (a,b) = (6,-10), $f'(x) = {}^{(x^2-1)6-(6x-10)2x}/_{(x^2-1)^2} = -{}^{1}/_{(x^2-1)^2}[6x^2-20x+6]$.

Since f'(3) = 0, this implies $6x^2-20x+6$ has *one root:* α=3. Now use "αβ=c/a", then the other <u>root</u> is β = (1/α).(c/a) = (1/3).(6/6) = ${}^{1}/_{3}$. Therefore $6x^2-20x+6 = 6(x-{}^{1}/_{3})(x-3)$. **When** x<3, $f'(x) = -{}^{1}/_{(x^2-1)^2}[6x^2-20x+6] > 0$ and **when** x>3, f'(x) < 0. *Therefore* f'(x) changes sign from +ve to -ve as x increases through <u>stationary</u> point x=3. f(x) has a local maximum at x=3.

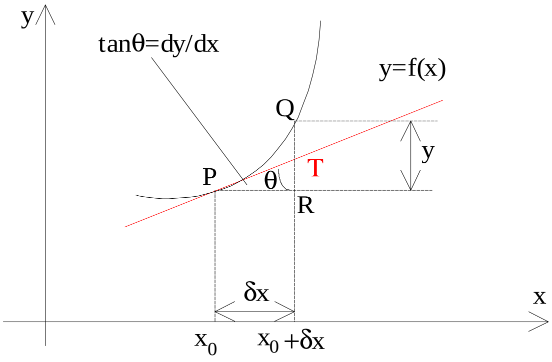
<u>Further Rules of Differentiation</u>

(For proofs and further explanation, see the notes)

${}^{dy}/_{dx} = {}^{1}/_{dx/dy}$. **BUT**, ${}^{d^2x}/_{dy^2}$ is not *equal* to ${}^{1}/_{d^2y/dx^2}$. **Parametric** differentiation: ${}^{dy}/_{dx} = {}^{dy/dt}/_{dx/dt}$, with dx/dt **not** zero. To find the *2nd* derivative (Parametric differentiation), proceed as follows: We <u>already</u> know ${}^{dy}/_{dx} = {}^{dy/dt}/_{dx/dt}$, so ${}^{d^2y}/_{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{dx/dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)$

<u>Increment of a Function</u>

Geometric interpretation of a derivative. Let Q be a point near to P on the curve y = f(x). $f'_+(x) = \lim_{\delta x\to 0+}\left[\frac{f(x_0+\delta x)-f(x_0)}{\delta x}\right] = \lim_{\delta x\to 0+}\left(\frac{\delta y}{\delta x}\right)$. Similarly, $f'_-(x) = \lim_{\delta x\to 0-}\left(\frac{\delta y}{\delta x}\right)$. If f is *differentiable* at x_0, then $f'_+(x_0) = f'_-(x_0) = f'(x_0) = {}^{dy}/_{dx}$. ${}^{dy}/_{dx}$ is the **slope** of the chord PQ and as δx→0+, the point Q→P and the slope of PQ *tends to the slope* of the tangent PT at P. **Similarly**, for δx<0, the δy is called the *increment* in y following from the increment δy in x. **Finally**, if f is differentiable at x_0, then $\lim_{\delta x\to 0}({}^{\delta y}/_{\delta x}) = {}^{dy}/_{dx}$. *So* ${}^{\delta y}/_{\delta x} = {}^{dy}/_{dx}+\varepsilon$, where ε>0 as δx→0. Therefore, $\delta y = ({}^{dy}/_{dx})\delta x+\varepsilon\delta x$. In the **picture**, δx = PR, $({}^{dy}/_{dx})\delta x$ = TR, εδx = QT; δy=QR. **Therefore** when δx is small, ε is small, *so that* εδx is very much **smaller** again. Indeed, we write that δy is approximately $({}^{dy}/_{dx})\delta x$ when dx is small, or δy is *approximately* $f'(x_0)\delta x$.



<u>Applications of Implicit Differentiation</u>

Implicit differentiation can be used to prove that ${}^{d}/_{dx}(x^a)$ is always defined for rational a and x>0 [SEE NOTES]. It can also be used to *derive formulae* for differentiating inverse circular functions. <u>Consider</u> $y = \sin^{-1}(x)$, then sin(y) = x; $\cos(y){}^{dy}/_{dx}=1$; ${}^{dy}/_{dx} = {}^{1}/_{\cos(y)} = {}^{1}/_{\pm\sqrt{(1-\sin^2 y)}} = {}^{1}/_{\pm\sqrt{(1-x^2)}}$. But cos(y)>0 **when** $-{}^{\pi}/_{2}<y<{}^{\pi}/_{2}$, so ${}^{dy}/_{dx} = {}^{1}/_{\sqrt{(1-x^2)}}$, -1<x<1. **Similarly** for $y = \cos^{-1}(x)$, ${}^{dy}/_{dx} = -{}^{1}/_{\sqrt{(1-x^2)}}$, |x|<1.

Maxima & Mimima

A function f(x) is said to reach the **global** (or *absolute*) maximum f(c) on its interval I if there exists a $c \in I$ such that $f(x)<f(c)$ for all $x \in I$ except x=c. The *function* is said to reach a local (or relative) maximum f(c) at x=c if there **exists** a positive number h such that for $0<|h|<H$, then $f(c)>f(c+h)$. There are similar definitions for *minimum*'s. If f(x) has a local max or min at x=c, then c is called a turning point (or critical point), and f(c) is called a turning value (or critical value). It can be **proved** (NOTES) that f'(c)=0.

f'(c)=0 is a necessary condition for f(c) be a *turning value*. It is not a sufficient condition for a turning value. Points C when f'(c)=0 are called **stationary** points (or critical points) and f(c) is called a stationary value (or critical value). To find which stationary values are turning values, examine the *sign change* of f'(x) as x increases through x=c.

If $f'(x) > 0$ when $x<c$ and $f'(x) < 0$ **when** $x > c$, then f'(c) is a local maximum. Note: $f''(c)<0$ is sufficient for a *maximum, not* necessary. Now if $f'(x) < 0$ when $x < c$ and $f'(x) > 0$ when $x > c$, then f'(c) is a local minimum. [Note...] A point of **inflexion** c is one where f''(c)=0 and f''(x) changes sign as x *increases through c*. The **concavity** changes at an inflexion point.

The Logarithmic and Exponential Functions

$\frac{d}{dx}(\log(x)) = \frac{1}{x}$. See notes for proof of this and **all subsequent results** quoted below. Theorem: log(ab) = log(a)+log(b). Theorem: $\log(a^n) = n\log(a)$. Theorem: $\log(a) = -\log(\frac{1}{a})$. Theorem: $\log(a^{1/n}) = \frac{1}{n}\log(a)$. Theorem: $(\frac{m}{n})\log(a) = \log(a^{m/n})$. Theorem: $\log(a^0) = \log 1$. Theorem: rlog(a) = log(ar) for any rational r. By **definition**, log(x) is differentiable on $(0<x<\infty)$ therefore it is *continuous* on $0<x<\infty$. Since log(x) is a monotonic function, it possesses an inverse, $y = e^x$. **Detailed notes** are available for the proof of this. Let $y = e^x$, the exponential function. $\frac{d}{dx}(e^x) = e^x$.

General Exponential Function

Define a^x by $a^x = e^{x\log(a)}$, where x is *any real number*, $x \in \mathbf{R}$. We know $\log(a^x) = x\log(a)$. Now $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log(a)}) = e^{x\log(a)} \times \log(a)$. **Therefore** $\frac{d}{dx}(a^x) = a^x\log(a)$. The **inverse** function of $y = a^x$ is called the logarithm to the base a, denoted by $x = \log_a(y)$. Now $\log(y) = \log(a^x) = x\log(a) = \log_a(y)\log(a)$. So $\log_a(y) = \log(y)/\log(a) = \log_e(y)/\log_e(a)$.

More **generally**, consider $\log_b(y)/\log_b(a) = [\log_e(y)/\log_e(b)]/[\log_e(a)/\log_e(b)] = \log_e(y)/\log_e(a) = \log_a(y)$. Therefore, $\log_a(y) = \log_b(y)/\log_b(a)$.

$\frac{d}{dy}(\log_a(y)) = \frac{d}{dy}(\log_e(y)/\log_e(a)) = [1/\log_e(a)] \times \frac{1}{y}$. Finally *consider* x^α, $\alpha \in \mathbf{R}$ $(x>0)$. Then $x^\alpha = e^{\alpha\log x}$. $\frac{d}{dx}(x^\alpha) = \frac{d}{dx}(e^{\alpha\log x}) = e^{\alpha\log x}.\alpha.\frac{1}{x} = \alpha x^\alpha/x = \alpha x^{\alpha-1}$. **Therefore,** $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$, $\alpha \in \mathbf{R}$, $x>0$.

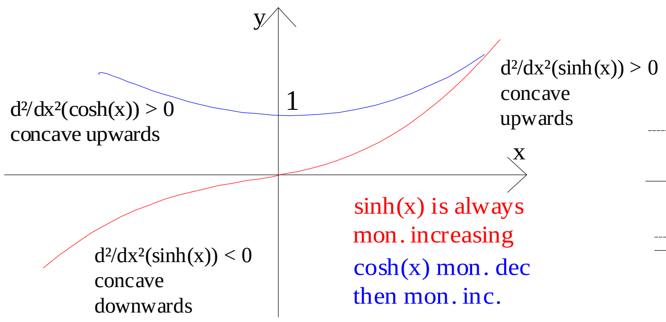
Hyperbolic Functions

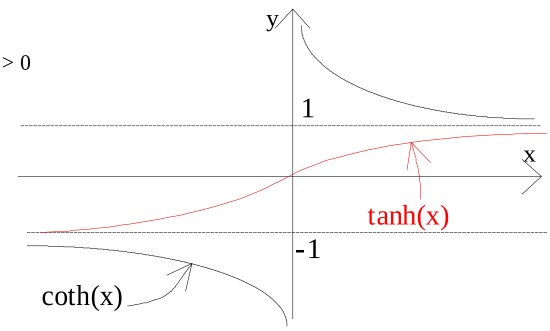
Definitions: $\cosh(x) = e^x+e^{-x}/2$, $\sinh(x) = e^x-e^{-x}/2$; $\tanh(x) = \sinh(x)/\cosh(x)$, $\coth(x) = \cosh(x)/\sinh(x)$, $\text{sech}(x) = 1/\cosh(x)$, $\text{cosech}(x) = 1/\sinh(x)$. Example of *manipulation*: $\cosh^2(x)-\sinh^2(x) = (½(e^x+e^{-x})^2-(½(e^x-e^{-x})^2)) = (¼(e^{2x}+2+e^{-2x})-¼(e^{2x}-2+e^{-2x})) = 1$.

Properties:

Hyperbolic Functions	*Circular Functions*
$\cosh^2(x)-\sin^2(h)=1$	$\cos^2(x)+\sin^2(x)=1$
$1-\tanh^2(x)=\text{sech}^2(x)$	$1+\tan^2(x)=\sec^2(x)$
$\coth^2(x)-1=\text{cosech}^2(x)$	$\cot^2(x)+1=\text{cosec}^2(x)$
$\sinh(x\pm y)=\sinh(x)\cosh(y)\pm\cosh(x)\sinh(y)$	$\sin(x\pm y)=\sin(x)\cos(y)\pm\cos(x)\sin(y)$
$\cosh(x\pm y)=\cosh(x)\cosh(y)\pm\sinh(x)\sinh(y)$	$\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$
$\sinh(2x)=2\sinh(x)\cosh(x)$	$\sin(2x)=2\sin(x)\cos(x)$
$\cosh(2x)=\cosh^2(x)+\sinh^2(x)$	$\cos(2x)=\cos^2(x)-\sin^2(x)$

Differentiation Formulae: $y = \sinh(x) = ½(e^x-e^{-x})$. ${}^{dy}/_{dx} = {}^{d}/_{dx}(\sinh(x)) = ½(e^x-(-e^{-x})) = ½(e^x+e^{-x})$, so ${}^{d}/_{dx}(\sinh(x)) = \cosh(x)$. And ${}^{d}/_{dx}(\cosh(x)) = {}^{d}/_{dx}[½(e^x+e^{-x})] = ½(e^x-e^{-x}) = \sinh(x)$.





Inverse Hyperbolic Functions

$y = \sinh(x)$ is a **monotonically increasing** function of x from $-\infty<x<\infty$, therefore the *inverse function* exists. $x = \sinh^{-1}(y)$ or $x = \arg\sinh(y)$, for $-\infty<y<\infty$. $y = \cosh(x)$ is not **invertible** for $-\infty<x<\infty$ since for each value of $y>1$ there are 2 values of x such that $y = \cosh(x)$, *namely* x and -x. However, for $0<x<\infty$, cosh(x) is invertible as there, cosh(x) is *monotonically increasing*. And $x = \cosh^{-1}(y)$, or $x = \arg\cosh(y)$, $(y\geq 1)$. $y = \tanh(x)$ is monotonically **increasing** for $-\infty<x<\infty$, therefore it is *invertible* and $x = \tanh^{-1}(y)$, or $x = \arg\tanh(y)$ $(-1<y<+1)$.

Logarithmic form of the *inverse hyperbolic function*. Interchanging x,y in $x = \sinh^{-1}(y)$, we get $y = \sinh^{-1}(x)$. This implies $\sinh(y) = x$, but $\sinh(y) = ½(e^y-e^{-y})$, so $\sinh(y) = ½(e^y-e^{-y})$. Rearrange to give a **quadratic** equation in e^y, which implies $e^{2y}-2xe^y-1=0$; $e^y=x\pm\sqrt{(x^2+1)}$. But $e^y>0$, so $e^y=x+\sqrt{(x^2+1)}$, $y = \log(x+\sqrt{(x^2+1)})$; $\sinh^{-1}(x) = \log(\sqrt{(x^2+1)})$, $(-\infty<x<\infty)$.

Similarly, $\cosh^{-1}(x) = \log(x+\sqrt{(x^2-1)})$, $(x \geq 1)$. (Note: ± *possibilities* as before; the '-' option gives the **other** branch of $\cosh^{-1}(x)$. And $\tanh^{-1}(x) = \frac{1}{2}\log({}^{1+x}/_{1-x})$, $|x|<1$. And $\coth^{-1}(x) = \frac{1}{2}\log({}^{x+1}/_{x-1})$, $|x|>1$.

Derivatives of Inverse Hyperbolic Functions

$y = \sinh^{-1}(x)$ **implies** $x = \sinh(y)$. *Differentiate implicitly* w.r.t. x. So $1 = {}^{dx}/_{dx} = {}^{d}/_{dx}(\sinh(y)) = \cosh(y){}^{dy}/_{dx}$. This <u>implies</u> ${}^{dy}/_{dx} = {}^{1}/_{\cosh(y)} = {}^{1}/_{\pm\sqrt{(1+\sinh^2 y)}}$ (Using $\cosh^2(y)-\sinh^2(y)=1$). So **as** $\cosh(y)>0$, ${}^{dy}/_{dx} = {}^{1}/_{\pm\sqrt{(1+x^2)}}$, $(-\infty<x<\infty)$. Similarly, ${}^{d}/_{dx}(\cosh^{-1}(x)) = {}^{1}/_{\sqrt{(x^2-1)}}$, $(x>1)$; ${}^{d}/_{dx}(\tanh^{-1}(x)) = {}^{1}/_{1-x^2}$, $|x|<1$, and ${}^{d}/_{dx}(\coth^{-1}(x)) = {}^{1}/_{1-x^3}$, $|x|>1$.

Examples

(1) $y = \exp(x^2)$. ${}^{dy}/_{dx} = \exp(x^2).2x$.

(2) $y = 10^{\log(\sin(x))}$ $[\sin(x)>0]$. **Remember** that $a^{\alpha}=e^{\alpha\log(a)}$; *therefore* $y = e^{\log(\sin(x)).\log(10)}$; ${}^{dy}/_{dx} = [e^{\log(\sin(x)).\log(10)}]x = 10^{\log(\sin(x))}.\cot(x).\log(10)$.

Assignment 2

Q: If $y = t-t^3$ and $x = t-t^2$, **find** ${}^{dy}/_{dx}$ and ${}^{d^2y}/_{dx^2}$. A: **Using** ${}^{dy}/_{dx} = {}^{dy/dt}/_{dx/dt}$, ${}^{dy}/_{dx} = {}^{1-3t^2}/_{1-2t}$, (t not a half). **Further** more, ${}^{d^2y}/_{dx^2} = [{}^{d}/_{dt}(dy/dx)]/(dx/dt) = [{}^{(1-2t)(-6t)-(1-3t^2)(-2)}/_{(1-2t)^2}]/(1-2t) = {}^{2(3t^2-3t+1)}/_{(1-2t)^3}$. Q: Find local *max, min and points of inflexion* of $y = {}^{1}/_{3}x^3+2x^2+3x+{}^{1}/_{3}$. A: Find y', equate y'=0 so we find that x=-1 or -3. Then **substitute** your *values* into y'' and find whether they are min/max/inflexion. For points of inflexion, look for *turning* values of y'=f'(x) [=g(x), say]. Therefore $g(x) = x^2+4x+3$ and $0 = g'(x) = 2x+4$, <u>implying</u> x = -2. g''(x) = 2 **which** is +ve so MIN of g(x). Therefore g(x) reaches a *turning value* at x=2 which is a local <u>minimum</u>. So g'(x) = f''(x) changes sign from -ve to +ve as x *increases* through x = -2. Therefore x = -2 is a point of <u>inflexion</u>.

Q: Find the **derivative** of coth(x) and verify that it is *negative when x* is not zero. Express f(x) = coth(x)-1 in terms of e^x and **deduce** that f(x)>0 if x>0. Show *similarly* that coth(x)<-1 if x<0. A: y = coth(x) = cosh(x)/sinh(x), ${}^{dy}/_{dx} = {}^{(\sinh(x))(\sinh(x))-(\cosh(x))(\cosh(x))}/_{\sinh^2(x)} = {}^{\sinh^2(x)-\cosh^2(x)}/_{\sinh^2(x)} = -{}^{1}/_{\sinh^2(x)} = -\mathrm{cosech}^2(x)$. As **sinh**(x) > 0 for x not equal to zero, then ${}^{dy}/_{dx} = -\mathrm{cosech}^2(x)$ implies ${}^{dy}/_{dx} < 0$ when we have *non zero* x.

Now y = f(x) = **coth**(x)-1 $= \left(\frac{(e^x+e^{-x})/2}{(e^x-e^{-x})/2}\right)-1 = \ldots = 2/e^{2x}-1$. Now $\infty > e^t > 1$ if t > 0. Therefore f(x) > 0 if 2x > 0 (or x>0). If x < 0, $0 < e^{2x} < 1$. Therefore $f(x) = 2/e^{2x}-1 < -2$. Therefore coth(x) = 1+f(x) < 1+(-2) = -1.

Q: Starting from the **definition** for sinh(x), solve the equation y=sinh(x) to *write* x=ln(f(y)) for some function f(y). Hence <u>calculate</u> ${}^{dx}/_{dy}$. A: $y = \sinh(x) = \frac{1}{2}(e^x-e^{-x})$. So $2y = e^x-e^{-x}$; $e^{2x}-2ye^x-1=0$. This **implies** $e^x = \frac{-(-2y)\pm\sqrt{(-2y)^2-4.1-1}}{2}$; $e^x = y\pm\sqrt{(y^2+1)}$. Take **+ve** sign as $e^x>0\ \forall$ x. Therefore, $x = \ln\{y+\sqrt{(y^2+1)}\}$. So ${}^{dx}/_{dy} = {}^{1}/_{y+\sqrt{(y^2+1)}}.(1+({}^{1}/_{2\sqrt{(y^2+1)}}.2y)) = {}^{1}/_{(y+\sqrt{(y^2+1)})}.{}^{(\sqrt{(y^2+1)}+y)}/_{\sqrt{(y^2+1)}} = {}^{1}/_{\sqrt{(y^2+1)}}$.

Examples 3

Q: **Differentiate** $x\tan^{-1}(x)-\ln(\sqrt{(1+x^2)})$; ${}^{dy}/_{dx} = \tan^{-1}(x)+x.{}^1/_{1+x^2} = -{}^1/_{\sqrt{(1+x^2)}}.{}^1/_{2\sqrt{(1+x^2)}}.2x = \tan^{-1}(x)$. Note: when differentiating logs, **separate** the part you differentiate and the part that stays the same (i.e. don't create a *fraction*). Q: **Differentiate** with respect to x: (i) $y = x^x$, so $\log(y) = \log(x^x) = x\log x$. So ${}^1/_y{}^{dy}/_{dx} = \log(x)+{}^x/_x$; ${}^{dy}/_{dx} = (x^x)(1+\log(x))$. (ii) $y = (\sin(x))^x$; $\log(y) = x\log(\sin(x))$. ${}^1/_y{}^{dy}/_{dx} = \log(\sin(x))+x{}^1/_{\sin(x)}.\cos(x)$; *therefore* ${}^{dy}/_{dx} = (\sin(x))^x\{\log(\sin(x))+x\cot(c)\}$. (iii) $y = (\sqrt{x})^x = x^{x/2}$. So $\log(y) = {}^x/_2\log(x)$; ${}^1/_y{}^{dy}/_{dx} = ½\log(x)+{}^x/_2.{}^1/_x$; ${}^{dy}/_{dx} = (\sqrt{x})^x/2\{\log(x)+1\}$. (iv) $y = x^{\sqrt{x}}$; $\log(y) = \sqrt{x}\times\log(x)$. Now ${}^1/_y{}^{dy}/_{dx} = {}^1/_{2\sqrt{x}}.\log(x)+\sqrt{x}{}^1/_x$; ${}^{dy}/_{dx} = x^{\sqrt{x}}/2\sqrt{x}\{\log(x)+2\}$. (v) $y = 10^{\ln(\sin(x))} = e^{\ln(\sin(x)).\log(10)}$. So ${}^{dy}/_{dx} = e^{\ln(\sin(x)).\log(10)}.{}^1/_{\sin(x)}.\cos(x)\log(10)$.

Q: **If** $y = \sin^{-1}(e^{-x})$, *show that* $\cos^2(y){}^{d^2y}/_{dx^2}+\sin^2(y){}^{dy}/_{dx} = \sin(y)\cos(y)$. A: $y = \sin^{-1}(e^{-x})$. So $\sin(y) = e^{-x}$; $\cos(y){}^{dy}/_{dx} = -e^{-x} = -\sin(y)$. **Therefore** ${}^{dy}/_{dx} = -\tan(y)$. And ${}^{d^2y}/_{dx^2} = -\sec^2(y){}^{dy}/_{dx} = -\sec^2(y)(-\tan(y)) = -\sec^2(y)\tan(y)$. **Therefore** $\cos^2(y){}^{d^2y}/_{dx^2}+\sin^2(y){}^{dy}/_{dx} = \tan(y)+\sin^2(y)(-\tan(y)) = \tan(y)(1-\sin^2(y)) = \tan(y)\cos^2(y) = \sin(y)\cos(y)$ as *required*.

Q: **Prove** that if $y = \tan^{-1}(\sinh(x))$, then ${}^{d^2y}/_{dx^2}+\tan(y)({}^{dy}/_{dx})^2=0$. A: **If** $y = \tan^{-1}(\sinh(x))$, then ${}^{dy}/_{dx} = {}^1/_{1+\sinh^2(x)}.\cosh(x) = {}^1/_{\cosh(x)}$. Now ${}^{d^2y}/_{dx^2} = -{}^1/_{\cosh^2(x)}.\sinh(x) = -({}^{dy}/_{dx})^2.\tan(y)$. This implies ${}^{d^2y}/_{dx^2}+\tan(y)({}^{dy}/_{dx})^2=0$.

Q: Using the **definitions** of cosh(x) and sinh(x), solve the *sim. equations* $\cosh(x)+\cosh(y)=4$, $\sinh(x)-\sinh(y)=2$. A: **We'll** use $\cosh(x) = ½(e^x+e^{-x})$ and $\sinh(x) = ½(e^x-e^{-x})$. So the **first** equation becomes $½(e^x+e^{-x})+½(e^y+e^{-y})=4$; $e^x+e^y+e^{-x}+e^{-y}=8$. The *2nd equation* becomes $e^x-e^{-x}-(e^y-e^{-y})=4$. If we add these *equations* we get $2e^x+2e^{-y}=12$; $e^x+e^{-y}=6$. If we **subtract** we obtain $2e^y+2e^{-x}=4$; $e^y+e^{-x}=2$. So *combining these two* equations, we obtain $e^x+(2-e^{-x})^{-1} = 6$; $e^x(2-e^{-x})+1 = 6(2-e^{-x})$. This **implies** that $2e^{2x}-e^x+e^x=12e^x-6$; $e^{2x}-6e^x+3=0$. **This** is a quadratic equation and we *use the formula* to get $e^x = 3\pm\sqrt{6}$. From this, we can find *pairs* (x,y) of solutions by <u>substitution</u>.

Q: Solve the **equation** $16\cosh(x)-4\sinh(x)=19$ to 3 d.p. A: $½(e^x+e^{-x})-4.½(e^x-e^{-x})=19$. *Therefore* $8e^x+8e^{-x}-2e^x+2e^{-x}=19$; $6e^x+10e^{-x}=19$. **Therefore** $6e^{2x}-19e^x+10=0$. Using the quadratic **formula**, $e^x={}^{19\pm11}/_{12} = {}^{30}/_{12}$ or ${}^8/_{12} = {}^5/_2$ or ${}^2/_3$. *Therefore* $x = \log({}^5/_2) = 0.916$ to 3 <u>sig. fig</u>.; or $x = \log({}^2/_3) = -0.405$ (3 sig. fig.).

Integration

Integration is the **reverse** of differentiation. Given ${}^{dy}/_{dx} = f(x)$, <u>find</u> y ($=\int f(x)dx$). $\int f(x)dx$ = indefinite integral. When integrating between **limits**, $\int^b_a f(x)dx = [F(x)]^b_a = F(b)-F(a)$, the definite **integral**, where ${}^{dF}/_{dx} = f(x)$. *Geometric* Interpretation: See notes, but basically it is the sum of the <u>area</u> strips beneath the curve of the function on a graph. **Standard form** for integrals are derived from standard form for *derivatives* e.g. ${}^d/_{dx}(\sin(x)) = \cos(x)$, so $\int\cos(x)dx = \sin(x)+c$. See the <u>handout</u> for other standard forms e.g. as ${}^d/_{dx}(a^x) = a^x\log a$ ($a>0$), then $\int a^x dx = (a^x/\log a)+c$ ($a\neq1$).

From the **rules** of differentiation, we *have the following*: (i) $\int(af(x)+bg(x))dx = a\int f(x)dx + b\int g(x)dx$. (ii) $\int f(ax+b)dx = {}^1/_a F(ax+b)+c$ (non zero a), where ${}^d/_{dx}[F(x)] = f(x)$. (iii) When n is not -1, then $\int\{f(x)\}^n f'(x)dx = {}^1/_{(n+1)}\{f(x)\}^{n+1}+c$. When n=1, $\int {}^{f'(x)}/_{f(x)}dx = \log(|f(x)|)+c$. Note: we have ${}^d/_{dx}(\log(x)) = {}^1/_x$, for x>0. But also (see <u>*notes*</u>) $\int {}^1/_x = \log(|x|)+c$.

Consider integrals of the type $I = \int {}^{dx}/_{x^2+px+q}$, where p and q are constants. Applying *completing the square*, $x^2+px+q = (x+{}^p/_2)^2+q-{}^{p^2}/_4$. (1) Consider $p^2=4q$, **then** $I = \int {}^{dx}/_{(x+p/2)^2} = -{}^1/_{(x+p/2)^2} + c$. (2) **Consider** $p^2<4q$. Then $I = \int {}^{dx}/_{(x+p/2)^2+\beta^2}$ (Where $\beta^2=q-{}^{p^2}/_4$ [>0], β real) $= {}^1/_\beta \tan^{-1}({}^{x+p/2}/_b)+c$. Here we are using the **standard from** $\int {}^{dy}/_{y^2+a^2} = {}^1/_a \tan^{-1}({}^y/_a)+c$. (3) Consider $p^2>4q$. Therefore x^2+px+q has **real** factors, $(x-\alpha)(x-\beta)$ i.e. $x^2+px+q = (x-\alpha)(x-\beta)$. Now, use *partial fractions* to express ${}^1/_{x^2+px+q} = {}^1/_{(x-\alpha)(x-\beta)} = {}^{1/(\alpha-\beta)}/_{(x-\alpha)} - {}^{1/(\alpha-\beta)}/_{(x-b)}$. Now $I = {}^1/_{(\alpha-\beta)}\int {}^{dx}/_{(x-\alpha)} - ({}^1/_{(\alpha-\beta)})\int {}^{dx}/_{(x-\beta)} = ({}^1/_{(\alpha-\beta)})\log|{}^{x-\alpha}/_{x-\beta}|+c$. **Example**: $I = \int {}^{dx}/_{x^2+4x+7}$. Here p=4, q=7; $p^2=16$, 4q = 28, so $p^2<4q$. **Changing** to $\int {}^{dx}/_{(x+2)^2+3}$ (*Completing the square*), we apply the **standard** result to obtain $I = {}^1/_{\surd 3} \tan^{-1}({}^{(x+2)}/_{\surd 3})+c$.

<u>Examples 4</u>

Q: **Integrate** these: (i) $\int\coth^2(x)dx = \int(1+\text{cosech}^2(x))dx = x-\coth(x)+c$. Watch out for *top heavy fractions*. (ii) $\int\sin(3x)\cos(5x)dx = ½\int(\sin(8x)\sin(2x))dx = [-\cos(8x)/16]+[\cos(2x)/4]+c$. (iii) $\int {}^{2x-1}/_{x^2+4x}dx = \int {}^{(2x+4)}/_{x^2+4x} - {}^5/_{(x^2+4x)}dx = I_1-I_2$, where $I_1 = \ln(x^2+4x)$; $I_2 = \int {}^5/_{x^2+4x}dx$ = use *partial fractions* with $\int {}^5/_{x(x+4)}dx$. (iii) $\int {}^{2x-1}/_{\surd(x^2+4x+8)}dx = \int {}^{2x+4}/_{\surd(x^2+4x+8)}dx - 5\int {}^{dx}/_{\surd(x^2+4x+8)}dx = 2\surd(x^2+4x+8) - 5\int {}^{d(x+2)}/_{\surd[(x+2)^2+2^2]} =$ **”**$-5\int {}^{2\sec^2\theta d\theta}/_{\surd[(2\tan\theta)^2+2^2]}$ (*where* $x+2=2\tan\theta$, $d(x+2) = 2\sec^2\theta d\theta$. So = **”**$-5\int\sec\theta d\theta$ = **”**$-5\ln|\sec\theta+\tan\theta|$ = **”**$-5\ln|{}^{x+2}/_2+½\surd(x^2+4x+8)|$. Therefore, $I = 2\surd(x^2+4x+8) - 5\ln|(x+2)+\surd(x^2+4x+8)|+c$. (iv) $\int[e^x/1+e^{2x}]dx$. Let $u = e^x$ so $du = e^x dx$. **So** $I = \int {}^{du}/_{1+u^2} = \tan^{-1}(u) = \tan^{-1}(e^x)+c$.

Q: Integrate this definite **integral**: $\int^{-1}_{-2} {}^{dx}/_{(2x+1)^2} = [½\times -{}^1/_{(2x+1)}]^{-1}_{-2} = (-½)\{{}^1/_{(-2+1)} - {}^1/_{(-4+1)}\} = -½\{-1-{}^1/_{-3}\} = -½ \cdot -{}^2/_3 = {}^1/_3$. Q: **Evaluate** the following integral: $\int^2_1 x-2\ln(1+x)dx$. Let $u = \ln(1+x)$ and $dv = {}^{dx}/_{x^2}$, So $du = {}^1/_{1+x}dx$ and $v = -{}^1/_x$. Therefore $I = uv-\int vdu = [-{}^1/_x.\ln(1+x)]^2_1 - \int^2_1 -{}^1/_x.{}^1/_{1+u}dx$. ${}^1/_{x(1+x)} = {}^A/_x + {}^B/_{1+x}$; $1 = A(1+x)+Bx$. Solving the partial fractions and substituting back in gives an *easy integral* to solve in terms of <u>logs</u>.

Q: $I = \int^{\pi/2}_0 x^2\sin(x)dx$. By **parts**, where $u=x^2$ and $dv=\sin(x)$, $= [x^2(-\cos(x))]^{\pi/2}_0 - \int^{\pi/2}_0(-\cos(x)2x)dx = [-x^2\cos(x)]^{\pi/2}_0 + [2x\sin(x)]^{\pi/2}_0 - \int^{\pi/2}_0\sin(x).2dx = [-x^2\cos(x)+2x\sin(x)+2\cos(x)]^{\pi/2}_0 = ... = \pi-2$. Q: Find a **reduction** formula for I_n in terms of I_{n-1} given that $I_n = \int e^x x^n dx$. By parts, where $dv=e^x$, $u=x^n$, *then* $= x^n e^x - \int e^x n x^{n-1}dx = e^x x^n - nI_{n-1}$. **Therefore** $I_n=e^x x^n-nI_{n-1}$. This is the **reduction** formula. Now $I_0 = \int e^x x^0 dx = \int e^x dx = e^x$. And $I_1 = e^x x^1 - 1I_0 = e^x x - e^x = e^x(x-1)$. And $I_2 = e^x x^2 - 2I_1 = e^x x^2 - 2e^x(x-1) = e^x(x^2-2(x-1)) = e^x(x^2-2x+2)$. And **therefore** $I_3 = e^x x^3 - 3I_2 = e^x\{x^3-3(x^2-2x+2)\} = e^x\{x^3-3x^2+6x-6\}+c$.

Q: **Given** that $I_n = \int^\pi_0 e^x\sin^n(x)dx$, prove that for n>1, $(n^2+1)I_n = n(n-1)I_{n-2}$. Hence evaluate I_4 and I_5. A: This question involves the **application** of integration by parts several times to get a repeating expression (Remember A-Level notes?). Q: Evaluate $\int {}^{dx}/_{x\surd x^2-2}$ using $x={}^1/_t$. So $I = \int {}^{(-1/t^2)dt}/_{1/t\surd[1/t^2-2]} = \int -{}^{dt}/_{\surd[1-2t^2]} = -{}^1/_{\surd 2}\int {}^{ds}/_{\surd[1-s^2]}$, where $s=\surd[2]t$. Now let $\sin\theta = s$, so $= -{}^1/_{\surd 2}\int {}^{\cos\theta d\theta}/_{\surd[1-\sin^2\theta]} = -{}^1/_{\surd 2}\int d\theta = -[{}^1/_{\surd 2}]\theta = -({}^1/_{\surd 2}\sin^{-1}(s)) = -({}^1/_{\surd 2}\sin^{-1}(\surd(2)t) = -{}^1/_{\surd 2}\sin^{-1}({}^{\surd 2}/_x) + c$, $x > \surd 2$.

Q: **Evaluate** $\int x(3x^2-2)^9dx$ using $t = 3x^2-2$. So $dt = 6xdx$; $I = \int t^9dt/6 = (t^{10}/60)+c = [(3x^2-2)^{10}/60]+c$. Q: Evaluate $I = \int \frac{\cos(x)dx}{\sqrt{(1+\sin^2(x))}}$ using $u = \sin(x)$. So $du=\cos(x)dx$; $I = \int \frac{du}{\sqrt{(1+u^2)}} = (u=\sinh(v)) = \int \frac{\cosh(v)dv}{\sqrt{(1+\sinh^2(v))}} = (\cosh^2(v)-\sinh^2(v)=1) = \int dv = v = \sinh^{-1}u = \sinh^{-1}(\sin(x))+c$.

The **t-substitution**, $t = \tan(x/2)$. Q: Evaluate $I = \int \frac{dx}{3+5\cos x}$ using the t-substitution (see later for notes) $= \int \frac{2dt/[1+t^2]}{[3+5(1-t^2)]/(1+t^2)} = \int \frac{2dt}{3(1+t^2)+5(1-t^2)} = \int \frac{2dt}{8-2t^2} = \int \frac{dt}{2^2-t^2} = \frac{1}{4}\int \frac{dt}{(2-t)} + \frac{1}{4}\int \frac{dt}{(2+t)} = \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| = \frac{1}{4}\ln\left|\frac{2+\tan(x/2)}{2-\tan(x/2)}\right| + c$.

Types of Integration

Integrals of **type** $\int \frac{(ax+b)dx}{x^2+px+q}$, where a,b,p,q are *constants*. Check whether or not x^2+px+q has <u>real</u> factors. If yes, then express integrand as *partial* fractions. If there are no real factors, then **proceed** as follows: Example: $I = \int \frac{(2x-1)}{(x^2+4x+8)}dx$. Here x^2+4x+8; $x = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a} = \frac{-4\pm\sqrt{(16-32)}}{2} = -2\pm\sqrt{-2}$. Write the **numerator** as the derivative of the denominator, $\frac{d}{dx}(x^2+4x+8) = 2x+4$. Now $(2x-1) = (2x+4)-5$. **So** $I = \int \frac{(2x+4)}{x^2+4x+8}dx - \int \frac{5}{x^2+4x+8}dx = I_1-I_2$. Because $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$, then $I_1 = \ln|x^2+4x+8|+c$. Find I_2 as shown **before**.

Integrals of type $\int \frac{f(x)dx}{x^2+px+q}$, p,q constants. f(x) is a *polynomial* of degree 2 or more. Divide the numerator by the <u>denominator</u> to give a polynomial and a remainder with a linear numerator. Then proceed as **above**. Integrals of type $\int \frac{P(x)}{R(x)}dx$, where P and R are polynomials in x. *Express* $\frac{P(x)}{R(x)}$ as $a^nx^n+a^{n-1}x^{n-1}+...+ax+a_0+\frac{f(x)}{R(x)}$ where degree f(x) < degree of R(x). Write R(x) in terms of its *linear and quadratic factors,* and split $\frac{f(x)}{R(x)}$ into **partial** fractions.

<u>Partial Fraction rules:</u> For factor in $R(x) = (x-\alpha)$, *partial fraction* $= \frac{A}{(x-\alpha)}$. For $(x-\alpha)^n$, partial fraction $= \frac{A_1}{(x-\alpha)}+\frac{A_2}{(x-\alpha)^2} +...+\frac{A_n}{(x-\alpha)^n}$. For x^2+px+q, **it is** $\frac{Ax+B}{x^2+px+q}$. *And for* $(x^2+px+q)^m$, it is $\frac{A_1x+B_1}{x^2+px+q} + \frac{A_2x+B_2}{(x^2+px+q)^2} +...+ \frac{A_mx+B_m}{(x^2+px+q)^m}$. **Integrals** of type $\int \frac{dx}{\sqrt{(x^2+px+q)}}$, *Write* x^2+px+q as $(x+p/2)^2+q-p^2/4$.

(i) $q = p^2/4$, so $I = \frac{dx}{x+(p/2)} = \ln|x+p/2|+c$. (ii) $p^2<4q$: Let $\beta^2 = q-p^2/4$, β real. $I = \int \frac{dx}{(x+(p/2))^2+\beta^2} = \sinh^{-1}(\frac{x+(p/2)}{\beta})+c$. Using the *standard form* $\int \frac{dx}{\sqrt{(x^2+a^2)}} = \sinh^{-1}(x/a)+c$. (iii) $p^2>4q$: Let $\chi^2 = p^2/4-q$, χ real. Then $I = \int \frac{dx}{\sqrt{(x+(p/2))^2-\chi^2}} = \cosh^{-1}(\frac{x+(p/2)}{\chi})+c$ using the *standard form* $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}(x/a)+c$.

Integrals of type $\int \frac{(ax+b)dx}{\sqrt{(x^2+px+q)}}$. Example: $I = \int \frac{x-5}{\sqrt{(4x-x^2)}} dx$. The **denominator** differentiated is $\frac{d}{dx}(4x-x^2) = 4-2x$. Then <u>write</u> $(x-5) = -\frac{1}{2}(4-2x)-3$. Then $I = \int \frac{-(1/2)(4-2x)-3}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\int \frac{(4-2x)}{\sqrt{4x-x^2}}dx - 3\int \frac{dx}{\sqrt{4x-x^2}} = I_1 - I_2$. I_2: $3\int \frac{dx}{\sqrt{[2^2-(x-2)^2]}}$. But $\int \frac{dx}{a^2-x^2} = \sin^{-1}(x/a)+c$. **Therefore** $I_2 = 3\sin^{-1}(\frac{x-2}{2})+c$. I_1: $-\frac{1}{2}\int \frac{4-2x}{\sqrt{(4x-x^2)}}dx = -\frac{1}{2}(4x-x^2)^{1/2}.2+c = -\sqrt{(4x-x^2)}+c$. *Thus* $I = -\sqrt{(4x-x^2)}-3\sin^{-1}(\frac{x-2}{2})+c$.

Integration by Parts

The **product** rule is $\frac{d}{dx}(uv) = u\frac{dv}{dx}+\frac{du}{dx}v$.
So $\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int \frac{du}{dx}vdx$. So $\int u.dv = uv - \int v.du$.
Example: $\int x\sin(x)dx$, *where* $u=x$; $du=dx$, and $dv=\sin(x)dx$; $v=-\cos(x)$.
Thus $I = \int udv = uv-\int vdu = x(-\cos(x))-\int(-\cos(x))dx = -x\cos(x)+\sin(x)+c$.

Assignment 3

Q: **Evaluate** $I = \int \frac{(1-\sin(x))}{(x+\cos(x))}dx$. This is of the **form** $\int \frac{f'(x)}{f(x)}dx$ so $I = \ln|x+\cos(x)|+c$. Q: *Evaluate* $I = \int \frac{dx}{x(1-x^2)}$. This is done by partial fractions, where $\frac{1}{x(1-x^2)} = \frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$, getting A=1, $B=\frac{1}{2}$ and $C=-\frac{1}{2}$ in the *usual way*, then $I = \int\frac{dx}{x}+\int\frac{(1/2)dx}{(1-x)}+\int\frac{(-1/2)dx}{1+x} = \ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}|1+x| + c = \ln\left|\frac{x}{\sqrt{1-x^2}}\right|+c$. Q: **Evaluate** $\int_0^1 \frac{1-x^2}{1+x^2}dx$. A: Now $\frac{1-x^2}{1+x^2} = \frac{-(x^2-1)}{x^2+1} = \frac{-(x^2+1)}{(x^2+1)} + \frac{2}{(x^2+1)} = -1 + \frac{2}{x^2+1}$. So $I = \int_0^1(-1+\frac{2}{x^2+1})dx = [-x+2\tan^{-1}(x)]_0^1$ (Using $\int\frac{dx}{x^2+1} = \tan^{-1}(x)$) $= \ldots \frac{\pi}{2} -1$.

Q: **Prove** that if $I_n = \int\tan^n(x)dx$, then $I_n+I_{n-2} = \tan^{n-1}(x)/n-1$, $(n\neq1)$ and hence **evaluate** $\int_0^{\pi/4}\tan^6(x)dx$. A: By **definition**, $I_n = \int\tan^n(x)dx$, and $I_n = \int\tan^{n-2}(x)\tan^2(x)dx = \int\tan^{n-2}(x)(\sec^2(x)-1)dx = \int\tan^{n-2}(x)d(\tan(x)) - \int\tan^{n-2}(x)dx = [\tan^{n-1}(x)/(n-1)]-I_{n-2}$. **Therefore** $I_n+I_{n-2} = \tan^{n-1}(x)/(n-1)$, $(n\neq1)$.

Now from the above, $[I_n]_0^{\pi/4}+[I_{n-2}]_0^{\pi/4} = [\frac{1}{n-1}\tan^{n-1}(x)]_0^{\pi/4} = \frac{1}{n-1}$. **Now** $[I_0]_0^{\pi/4} = \int_0^{\pi/4}\tan^0(x)dx = \int_0^{\pi/4}dx = [x]_0^{\pi/4} = \frac{\pi}{4}$. Using the *formula*, $[I_2]_0^{\pi/4} = (\frac{1}{2-1})-[I_0]_0^{\pi/4} = 1-\frac{\pi}{4}$. *Similarly*, $[I_4]_0^{\pi/4} = (\frac{1}{4-1})-[I_2]_0^{\pi/2} = \frac{1}{3}-(1-\frac{\pi}{4}) = \frac{\pi}{4}-(1-\frac{1}{3})$. **And** $[I_6]_0^{\pi/4} = (\frac{1}{6-1})-[I_4]_0^{\pi/4} = \frac{1}{5}-(\frac{\pi}{4}-(1-\frac{1}{3})) = -\frac{\pi}{4}+(1-\frac{1}{3}+\frac{1}{5}) = \frac{13}{15}-\frac{\pi}{4}$. **Therefore**, $\int_0^{\pi/4}\tan^6(x)dx = \frac{13}{15}-\frac{\pi}{4}$.

Substitutions

Consider $I = \int\frac{dx}{\sqrt{(a^2-x^2)}}$, $-1\leq x\leq a$, $a \neq 0$. Let $x = a\sin\theta$ then $dx/d\theta = a\cos\theta$; $dx = a\cos\theta d\theta$. So $I = \int\frac{a\cos\theta d\theta}{\sqrt{(a^2-a^2\sin^2\theta)}} = \int\frac{a\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}} = \int\frac{a\cos\theta}{a\cos\theta d\theta} = \int d\theta = \theta+c$. Now $\theta = \sin^{-1}(\frac{x}{a})$, so $I = \theta+c = \sin^{-1}(\frac{x}{a})+c$. Now consider $I = \int\frac{dx}{a^2+x^2}$, $-\infty<x<\infty$, $a \neq 0$. Let $x = a\tan\theta$, for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. So $dx = a\sec^2\theta d\theta$. So $I = \int\frac{a\sec^2\theta d\theta}{a^2+a^2\tan^2\theta} = \int\frac{a\sec^2\theta}{a^2\sec^2\theta}d\theta = \int\frac{1}{a}d\theta = \frac{1}{a}\int d\theta$. Now as $\theta = \tan^{-1}(\frac{x}{a})$, $I = [\tan^{-1}(\frac{x}{a})/a]+c = (\frac{1}{a})\tan^{-1}(\frac{x}{a})+c$. Now **consider** $I = \frac{dx}{\sqrt{(a^2+x^2)}}$, $a \neq 0$, $-a<x<0$. Let $x = a\sinh(\theta)$, so $dx = a\cosh(\theta)d\theta$. So $I = \int\frac{a\cosh(\theta)d\theta}{\sqrt{(a^2+a^2\sinh^2(\theta))}} = \int\frac{a\cosh(\theta)d\theta}{\sqrt{a^2\cosh^2(\theta)}} = \int\frac{a\cosh(\theta)}{a\cosh(\theta)d\theta} = \int d\theta = \theta+c = \sinh^{-1}(\frac{x}{a})+c$.

General Procedure. Consider $I = \int f(x)dx$. *Substitute* $x = \theta(u)$, so $dx = \frac{d\theta}{du}du$ and $I = \int f(\theta(u))\frac{d\theta}{du}du$. Now, if θ has been suitably *chosen*, then $f(\theta(u))\frac{d\theta}{du} = \frac{df(u)}{du}$ for some function $f(u)$. Then $I = \int\frac{df}{du}(u)du = f(u)+c$. Now, if θ is **invertible**, $u = \theta^{-1}(x)$ where θ^{-1} denotes the *inverse* to θ. Then $I = f(\theta^{-1}(x))+c$.

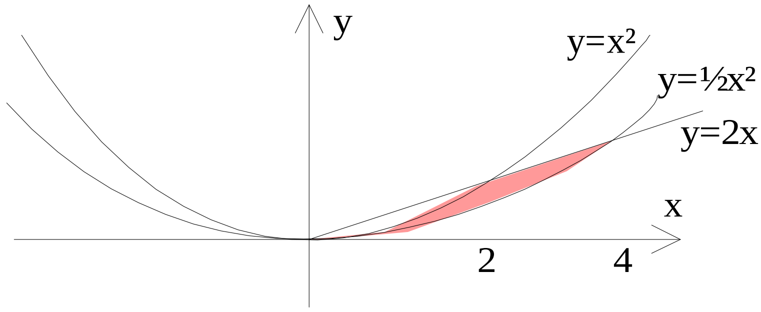
Example: Evaluate $I = \int\frac{dx}{x\sqrt{(x^2-2)}}$ using $t = \frac{1}{x}$. So $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$. So $I = \int\frac{-dt/t^2}{(1/t)\sqrt{(1/t^2)-2}} = \int\frac{-dt}{\sqrt{1-2t^2}} = \int\frac{-dt}{\sqrt{2}\sqrt{(1/2)-t^2}} = -\frac{1}{\sqrt{2}}\int\frac{dt}{(1/\sqrt{2})^2-t^2} = -\frac{1}{\sqrt{2}}\sin^{-1}(\frac{t}{1/\sqrt{2}})+c = c-\frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}t) = c-\frac{1}{\sqrt{2}}\sin^{-1}(\frac{\sqrt{2}}{x})$.

Useful Substitutions. For $\sqrt{(ax+b)}$, write $ax+b = u^2$, $u\geq0$. For a^2-x^2 or $\sqrt{(a^2-x^2)}$, write $x = a\sin\theta$ $(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2})$ or $x = a\cos\theta$ $(0\leq\theta\leq\pi)$. For a^2+x^2 or $\sqrt{(a^2+x^2)}$, write $x=a\tan\theta$ $(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2})$ or $x = a\sinh(u)$ $(-\infty<u<\infty)$. Sometimes the **substitution** $t = \tan(\frac{x}{2})$ is used. Here $\sin(x) = \frac{2t}{1+t^2}$, $\cos(x) = \frac{1-t^2}{1+t^2}$, $\frac{dx}{dt} = \frac{2}{1+t^2}$. And $t = \frac{\sin(x/2)}{\cos(x/2)} = \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} = \sin(\frac{x}{2})(1+\tan^2(\frac{x}{2})) = \sin(\frac{x}{2})(1+t^2)$. $t = \tan(\frac{x}{2})$; $\frac{dt}{dx} = \sec^2(\frac{x}{2})\times\frac{1}{2} = (1+\tan^2(\frac{x}{2}))\times\frac{1}{2}. = \frac{1+t^2}{2} = \frac{2}{1+t^2}dt$.

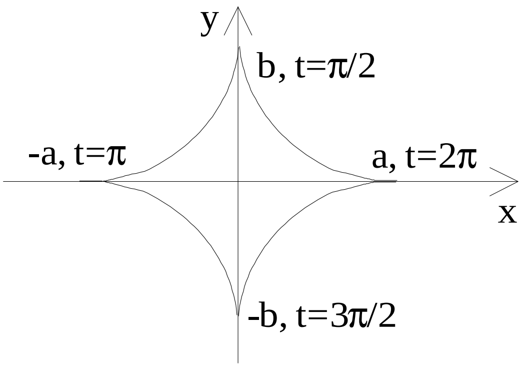
Example: Evaluate $I = \int {}^{dx}/_{3+5\cos x}$. Use $t = \tan(^x/_2)$, so $I = \int(^2/_{1+t^2})dt/3+5(^{1-t^2}/_{1+t^2}) = \int {}^{2dt}/_{3(1+t^2)+5(1-t^2)} = \int {}^{dt}/_{4-t^2}$. Using the *standard form* $\int {}^{dx}/_{a^2-x^2} = {}^1/_a\tanh^{-1}(^x/_a)$, $I = (^1/_2)\tanh^{-1}(^t/_2)+c = (^1/_2)\tanh^{-1}(^1/_2\tan(^x/_2))+c$. Or **directly** let $t = 2\tanh(\theta)$ in I, so $dt = 2\text{sech}^2(\theta)d\theta$. So $I = \int {}^{2\text{sech}^2(\theta)d\theta}/_{4-4\tanh^2(\theta)}$. using $\cosh^2\theta+\sinh^2\theta = 1$, $1-\tanh^2(\theta) = \text{sech}^2(\theta)$, So $I = ½\int d\theta = ½\theta+c = ½\tanh^{-1}(^1/_2) = {}^1/_2\tanh^{-1}(\tan(^x/_2).{}^1/_2)+c$.

Examples 5

Q: Calculate the **area** bounded by the two parabolas $y=x^2$, $y={}^1/_2x^2$ and the line $y=2x$. A: Make a **sketch** as shown. The point of *intersection* of $y=x^2$, $y=2x$ is: $x^2=2x$; $x(x-2)=0$, so $x=0$ or $x=2$. Similarly for $y={}^{x^2}/_2$ and $y=2x$, intersection is $x(x-4)$ so $x=0$ or $x=4$. The area we therefore want is $\int^2_0 (x^2-{}^1/_2x^2)dx + \int^4_2(2x-{}^1/_2x^2)dx = \int^2_0 ½x^2dx + \int^4_2(2x- ½x^2)dx = [^{x^3}/_{2\times3}]^2_0 + [x^2-{}^{x^3}/_{2\times3}]^4_2 = [(^{2^3}/_6-{}^{0^3}/_6)+(16-{}^{4^3}/_6-4+{}^{2^3}/_6)] = {}^8/_6+16-{}^{64}/_6-4+{}^8/_6 = 4$.



Q: Find the **area** contained in the asteroid $x = a\cos^3(t)$, $y = b\sin^3(t)$, $0\le t\le2\pi$. First **sketch** the curve. The *total area* we want is $|4 \times \int^{\pi/2}_0 ydx| = \int^{\pi/2}_0(b\sin^3(t))(3a\cos^2(t))(-\sin(t))dt = -3ab\int^{\pi/2}_0\sin^4(t)\cos^2(t)dt = -3ab\int^{\pi/2}_0\sin^4(t)(1-\sin^2(t))dt$. **Consider** $I_n = \int^{\pi/2}_0\sin^n(t)dt$ and $\int^{\pi/2}_0ydx\ (3ab)(I_6-I_4)$, where $I_n = \int^{\pi/2}_0\sin^n(x)dx$. Now $I_n = \int^{\pi/2}_0\sin^{(n-1)}(x)\sin(x)dx$. Let $u = \sin^{(n-1)}(x)$ so $du = (n-1)\sin^{(n-2)}(x)\cos(x)dx$; and $dv=\sin(x)dx$; $v = -\cos(x)$.

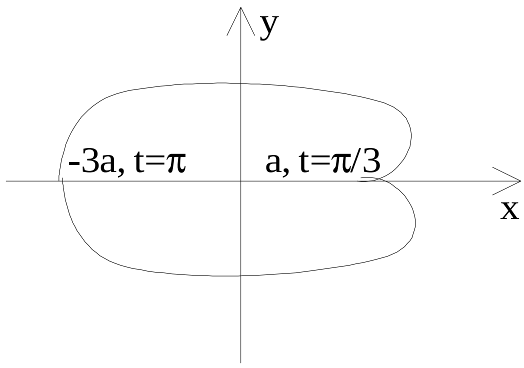


So $\int^{\pi/2}_0\sin^n(x)dx = [-\cos(x)\sin^{(n-1)}(x)]^{\pi/2}_0-\int^{\pi/2}_0[-\cos(x)](n-1)\sin^{(n-2)}(x)\cos(x)dx$. $I_n = (n-1)\int^{\pi/2}_0\sin^{(n-2)}(x)\cos^2(x)dx = (n-1)\int^{\pi/2}_0\sin^{(n-2)}(x)(1-\sin^2(x))dx = (n-1)\int^{\pi/2}_0\sin^{(n-2)}(x)dx-\int^{\pi/2}_0\sin^n(x)dx$. So $I_n = (n-1)\{I_{n-2}-I_n\}$. $I_n(1+(n-1)) = (n-1)I_{n-2}$; $nI_n=(n-1)I_{n-2}$; $I_n = (^{n-1}/_n)I_{n-2}$. So $I_0 = \int^{\pi/2}_0\sin^0(x)dx = \int^{\pi/2}_0 1.dx = [x]^{\pi/2}_0 = {}^\pi/_2$. Now $I_2 = (^{2-1}/_2)I_0 = ½I_0$ and so on to **find** I_6 and I_4. So $\int^{\pi/2}_0ydx = (3ab)(I_6-I_4) = -{}^{3ab}/_{32}\pi$. So $A = 4\times|-{}^{3ab}/_{32}\pi| = {}^{3ab}/_8\pi$.

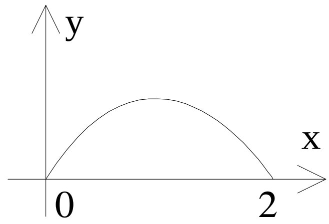
Q: Find the area *bounded by the cardioid* $x = a(2\cos(t)-\cos(2t))$, $y = a(2\sin(t)-\sin(2t))$, $0\le t\le2\pi$.

A: The **area** is $A = 2|\int^\pi_0ydx| = 2\int^\pi_0ydx =$
$2\int^\pi_0a(2\sin(t)-\sin(2t)).a(-2\sin(t)+2\sin(2t))dt =$
$(2a)^2\int^\pi_0 2\sin(t)-\sin(2t)(-\sin(t)+\sin(2t))dt =$
$(2a)^2\int^\pi_0-2\sin^2(t)+3\sin(t)\sin(2t)-\sin^2(2t)dt =$
$(2a)^2\int^\pi_0(\cos(2t)-1)+{}^3/_2(\cos(t)-\cos(3t))+{}^{\cos(4t)-1}/_2dt =$
$(2a)^2[^{\sin(2t)}/_2-t+{}^3/_2(\sin(t)-{}^{\sin(3t)}/_3)+{}^1/_2(^{\sin(4t)}/_4-t)]^\pi_0 = ... = -6\pi a^2$.

Therefore $A = |-6\pi^2| = 6\pi a^2$.



Q: Find the **volume** of the solid generated by *rotating about the x-axis* that part of the *parabola* $y = 2x-x^2$ which lies above the x-axis. A: Required **Volume** is $\int_0^2(\pi y^2)dx = \int_0^2 y^2dx = \pi\int_0^2(2x-x^2)^2dx = \pi\int_0^2 4x^2-4x^3+x^4dx = \pi[{}^4/_3x^3-x^4+(x^5/5)]_0^2 = {}^{16\pi}/_{15}$.

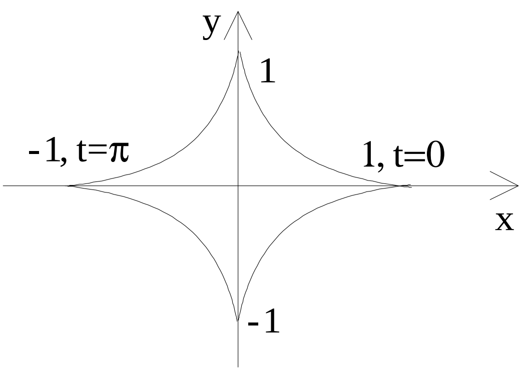


Theory

Arc Length L, from a to b (See notes for explanation) = $\int_{x=a}^{x=b}\sqrt{1+(\frac{dy}{dx})^2}\,dx$. If the curve is given **parametrically**, $(x=x(t),\ y=y(t))$ then ${}^{dy}/_{dx} = {}^{dy}/_{dt}\times{}^{dt}/_{dx} = y' = {}^1/_{x'} = {}^{y'}/_{x'}$. So $= \int_{t=t_1}^{t=t_2}\sqrt{(1+({}^{y'}/_{x'})^2)}x'dt = \int_{t=t_1}^{t=t_2}\sqrt{(x'^2+y'^2)}dt = \int_{t=t_1}^{t=t_2}\sqrt{\dot{x}^2+\dot{y}^2}\,dt$. **Example**: Find the length of the curve $y = x^4+({}^1/_{32})x^{-2}$ from $x=1$ to $x=2$. A: $L = \int_1^2\sqrt{[1+({}^d/_{dx}(x^4+({}^1/_{32})x^{-2}))^2]}dx = \int_1^2\sqrt{[1+(4x^3-{}^1/_{16}x^{-3})^2]}dx = \int_1^2\sqrt{[1+16x^6-{}^1/_2+{}^1/_{16^2}x^{-6}]}dx = \int_1^2\sqrt{[{}^1/_2+16x^6+{}^1/_{16^2}x^{-6}]}dx = \int_1^2({}^1/_2\sqrt{[2+64x^6+{}^1/_{64}x^{-6}]})dx = \int_1^2({}^1/_2\sqrt{((2x)^3+({}^1/_{2x^3}))^2} = \frac{1}{2}\int_1^2(2x^3+{}^1/_{2x^3})dx = {}^1/_2[(2x^4/4)-{}^1/_{2x^2}]_1^2 = {}^{963}/_{64}$.

Examples 5, Part II

Q: Sketch the **curve** $x=\cos^3(t)$, $y=\sin(t)$, $0\le t\le 2\pi$. Find the area enclosed by the curve and the volume of the solid formed when the **upper** half of the area is rotated about the x-axis. A: Area $= |2\int_0^\pi(ydx)| = |2\int_0^\pi(\sin(t).3\cos^2(t))(-\sin(t))dt| = (+6)|\int_0^\pi(\sin^2(t)\cos^2(t))dt| = (6)|\int_0^\pi({}^{\sin(2t)}/_2)^2dt| = {}^6/_4|\int_0^\pi({}^{1-\cos(4t)}/_2)dt = {}^6/_{2.4}|[t-{}^{\sin(4t)}/_4]_0^\pi| = {}^{6\pi}/_8 = {}^{3\pi}/_4$.



Volume $= |\int_0^\pi(\pi y^2dx)| = |\pi\int_0^\pi(\sin^2(t)3\cos^2(t)(-\sin(t))dt| = 3\pi|\int_0^\pi((\sin(t)(10\cos^2(t))\cos^2(t))dt| = 3\pi|[-{}^{\cos^3(t)}/_3+(\cos^5(t)/5)]_0^\pi| = (3\pi)|\{({}^{-(-1)^3}/_3+[(-1)^5/5])-(-{}^1/_3+{}^1/_5)\}| = (3\pi)|\{{}^1/_3-{}^1/_5+{}^1/_3-{}^1/_5\}| = (3\pi).2.({}^{5-3}/_{15}) = {}^{(3\pi).2.2}/_{15} = {}^{4\pi}/_5$. Q: Find the **length** of the curve $y = \cosh(x)$ from $x=1$ to $x=3$. A: $L = \int_1^3\sqrt{[1+({}^{dy}/_{dx})^2]}dx = \int_1^3\sqrt{[1+(\sinh(x))^2]}dx = \int_1^3(\cosh(x))dx = [\sinh(x)]_1^3 = \sinh(3)-\sinh(1)$.

Assignment 3

Q: Show that the **volume** generated with the area contained between the curve $y=x\cos(x)$ $(0\le x\le{}^\pi/_2)$ and the x-axis is ${}^{\pi^2(\pi^2-6)}/_{48}$ (as it is *rotated about the x-axis*). A: **Volume** required (values from sketch) is $V = \int_0^{\pi/2}(\pi y^2)dx = \pi\int_0^{\pi/2}(x^2\cos^2(x))dx = \pi\int_0^{\pi/2}(x^2(1+{}^{\cos(2x)}/_2)dx = \pi\int_0^{\pi/2}(x^2dx/2) + {}^\pi/_2\int_0^{\pi/2}(x^2\cos(2x))dx = {}^\pi/_2[{}^{x^3}/_3]_0^{\pi/2} + ({}^\pi/_2)I = (\pi^4/48) + ({}^\pi/_2)I$, where $I = \int_0^{\pi/2}(x^2\cos(2x))dx$.

Integrating by **parts**, $u=x^2$, $du=2xdx$; $\cos(2x)dx=dv$, $v={}^{\sin(2x)}/_2$. So $I = [uv]_0^{\pi/2}-\int_0^{\pi/2}(vdu) = [x^2\times{}^{\sin(2x)}/_2]_0^{\pi/2}-\int_0^{\pi/2}({}^{\sin(2x)}/_2)2x)dx$. Integrate the RHS by *parts* again and obtain $I = [x^2\times{}^{\sin(2x)}/_2]_0^{\pi/2} - \{[x({}^{-\cos(2x)}/_2)]]_0^{\pi/2} + \int_0^{\pi/2}({}^{-\cos(2x)}/_2)dx = [{}^{x\cos(2x)}/_2]_0^{\pi/2} + [{}^{-\sin(2x)}/_4]_0^{\pi/2} = ({}^\pi/_4(-1)-0)-(0) = -{}^\pi/_4$. So $V = (\pi^4/48) + {}^\pi/_2(-{}^\pi/_4) = -{}^{\pi^2(\pi^2-6)}/_{48}$.

Q: Show that the **length** of one arch of the cycloid $x=a(\theta-\sin(\theta))$, $y=a(1-\cos(\theta))$, $(0\leq\theta\leq2\pi)$ is **8a**. Length $= \int^{2\pi}_0\sqrt{[(^{dx}/_{d\theta})^2+(^{dy}/_{d\theta})^2]}d\theta = \int^{2\pi}_0\sqrt{[a(1-\cos(\theta))^2+(a\sin(\theta))^2]}d\theta = \int^{2\pi}_0\sqrt{[a(1-\cos(\theta))^2+(a\sin(\theta))^2]}d\theta = \int^{2\pi}_0\sqrt{[a^2(1-2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)]}d\theta = a\int^{2\pi}_0\sqrt{[2(1-\cos(\theta)]}d\theta = a\int^{2\pi}_0\sqrt{[4\sin^2(^{\theta}/_2)]}d\theta = a\int^{2\pi}_0(2\sin(^{\theta}/_2))d\theta = a[-4\cos(^{\theta}/_2)]^{2\pi}_0 = (-4a)(\cos(\pi)-\cos(0)) = (-4a)(-1-1) = 8a$. QED.

Q: Find the *general solution* of the differential equation $(1+x^2)y'+2xy=\tan(x)$. The equation implies $^{dy}/_{dx} + (^{2x}/_{1+x^2})y = {}^{\tan(x)}/_{1+x^2}$. Comparing with the *standard form* $^{dy}/_{dx} + P(x)y = Q(x)$, we see that $P(x) = {}^{2x}/_{1+x^2}$. Therefore $R(x) = \exp\int Pdx = \exp(\ln(1+x^2)) = 1+x^2$. *Multiplying* equations, $^{d}/_{dx}((1+x^2)y) = \tan(x)$. $(1+x^2)y = \int\tan(x)dx = -\ln(|\cos(x)|A)$ $(A>0)$. **Therefore** $y = -\ln(|\cos(x)|A)/(1+x^2)$.

Q: Find the *particular solution* of the **differential** equation $y'=\cos(y-x)$ which satisfies $y={}^{\pi}/_2$ when $x=0$. A: Let $z=y-x$. **So** $^{dz}/_{dx} = {}^{dy}/_{du}-1$. Substituting in the *original equation*, $^{dz}/_{dx}+1 = \cos(z)$; so $\int^{dz}/_{\cos(z)-1} = \int dx$; $\int^{dz}/_{-2\sin^2(z/2)} = x+c$; $\int-\mathrm{cosec}^2(^{z}/_2) = x+c$. **Now** $\cot(^{z}/_2) = x+c$, <u>so</u> $\cot(^{y-x}/_2) = x+c$. If $y = {}^{\pi}/_2$ when $x=0$, then we **get** $\cot(^{\pi}/_4) = c$; $c = 1$. **Therefore** $\cot(^{y-x}/_2) = x+1$. *Manipulating*, $y = x+2\cot^{-1}(x+1)$.

<u>Ordinary Differential Equations</u>

<u>Definitions</u>

Let y denote a variable which is *dependent* on an independent variable x. Suppose that the derivatives of y w.r.t. x exist. Any equation involving at least one of these derivatives is called a <u>differential equation</u>. Notation: $y' = {}^{dy}/_{dx}$, $\dot{y} = {}^{dy}/_{dt}$, $y'' = {}^{d^2y}/_{dx^2}$, $\ddot{y} = {}^{d^2y}/_{dt^2}$. **Examples**:

Example	**Power**	**Linear**	**Order**
$^{dy}/_{dx} + {}^{y}/_{x} = 0$	1	Linear in y	ONE
$^{dy}/_{dx} + xy = 0$	1	Linear in y	ONE
$^{d^2y}/_{dx^2} + x^{dy}/_{dx} + \sin(y) = x^3$	1	Not linear in y	TWO
$x(^{dy}/_{dx})^2 - 2y(^{dy}/_{dx}) + 4x = e^y$	2	Not linear in y	ONE
$[a+(^{dy}/_{dx})^2]^{4/3} = 3^{d^2y}/_{dx^2}$	3	Not linear in y	TWO
$^{\delta^2y}/_{\delta x^2} - {}^{\delta^2y}/_{\delta x^2} = 0$	1	Linear in y	TWO

Examples 1 to 5 involve **ordinary derivatives** and are called ordinary differential equations (ODE's). Example 6 is where $y = y(x,t)$ and the equation *involves partial derivatives*, and so is called a partial differential equation (PDE). The order of an ODE is the order of the **highest** derivative appearing in the equation. The degree of the ODE is the degree, or power, of the highest order *derivative when* the equation has been rationalised in the derivatives (i.e. <u>fractional</u> powers removed).

If in an equation of **1st** degree, the dependent variable itself and its derivatives occur raised only to the first power, and *do not occur in products*, then the ODE is said to be linear in the dependent variable. General form of an n^{th} order **linear** equation: $a_n(x)(d^ny/dx^n) + a_{n-1}(x)(d^{n-1}y/dx^{n-1}) + \ldots + a_1(x)(dy/dx) + a_0(x)y = b(x)$.

Consider a **first order ODE**. The roles of the *dependent* and *independent* variables may be interchanged since ${}^{dy}/_{dx} = {}^{1}/_{dx/dy}$. So examples 1 and 2 **become** ${}^{dx}/_{dy} = {}^{x}/_{y} = 0$ and ${}^{dx}/_{dy} = ({}^{1}/_{y})x^{-1} = 0$. Here, the y is regarded as the *independent* and the x *dependent* on y.

Formation of ODE's by Elimination

Consider $y(x) = c$, c constant. So ${}^{dy}/_{dx} = 0$. **Consider** $y = ax+b$ (a,b constants). **Differentiate** twice giving ${}^{dy}/_{dx}=a$ and ${}^{d^2y}/_{dx^2}=0$. *Consider* $y = ax^2+bx+c$ (a,b,c constants). Here we differentiate <u>three</u> times giving ${}^{d^3y}/_{dx^3}=0$. If $y = ax^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, where the a_i are constants (i = 0,1,2...n), then $d^{n+1}y/dx^{n+1} = 0$. Consider $y = Ae^x$. Then ${}^{dy}/_{dx} = Ae^x$, so ${}^{dy}/_{dx} = y$.

Q: **Show** that if $y = A\cos(x+\varepsilon)$ then ${}^{d^2y}/_{dx^2}+y = 0$. A: The above appears to *suggest* that to eliminate n parameters from a given function, an ODE of <u>order n</u> is required. [**Exceptions**: (1) $y = (A+B)x + D$ (A,B,D parameters), then ${}^{d^2y}/_{dx^2} = 0$ (${}^{dy}/_{dx} = A+B$). (2) Consider $y^2 = 2Axy+Bx^2$. *Differentiate* w.r.t. x giving $2y{}^{dy}/_{dx} = 2A(x{}^{dy}/_{dx}+y)+2Bx$; ${}^{dy}/_{dx}(y-Ax) = Ay+Bx$. But $y^2 = 2Axy+Bx^2$; $y^2-Axy = Axy+Bx^2$; $y(y-Ax) = x(Ay+Bx)$. We <u>know</u> ${}^{dy}/_{dx} = {}^{Ay+Bx}/_{y-Ax} = {}^{y}/_{x}$ from above. **Therefore** ${}^{dy}/_{dx} = {}^{y}/_{x}$. So we have eliminated **two** parameters A & B — is achieved with a differential equation of *order 1 only*].

Solution of ODE's: General, Particular, Singular

A function y(x) which *satisfies a d.e. is called* a solution or integral or primitive of the equation. The table suggests that the most **general** solution or complete primitive of an n^{th} order diff. equation contains n arbitrary constants. Any solution obtained from the general solution by giving particular values to the *arbitrary constants* is called a **particular** solution or particular primitive. However, some equations exist for which there are solutions which <u>cannot</u> be obtained in this way; they are called *singular* solutions.

Example of singular solution: consider $y = cx+{}^{1}/_{c}$ (c parameter). <u>Differentiating</u>, ${}^{dy}/_{dx} = c$. Eliminating c from the above, $y = {}^{dy}/_{dx}x + {}^{1}/_{dy/dx}$; $y{}^{dy}/_{dx}-({}^{dy}/_{dx})^2x = 1$. This is a **first** order d.e. of order 2. So $y = cx + {}^{1}/_{c}$ is a *general* solution to $y{}^{dy}/_{dx}-({}^{dy}/_{dx})^2x = 1$. **However**, look at $y^2 = 4x$. It is claimed that this is a *solution* of $y{}^{dy}/_{dx}-({}^{dy}/_{dx})^2x = 1$. **Proof**: Differentiate $y^2 = 4x$ giving $2y{}^{dy}/_{dx} = 4$; $y{}^{dy}/_{dx} = 2$. **Sub.** in $y{}^{dy}/_{dx}-({}^{dy}/_{dx})^2x = 1$, $2-({}^{2}/_{y})^2x = 2-{}^{4x}/_{y^2}$; $2-({}^{2}/_{y})^2x = 2-1$; $2-({}^{2}/_{y})^2x = 1$. So $y^2 = 4x$ is a *singular* solution which cannot be obtained from $y = cx + {}^{1}/_{c}$ for any **particular** value of c.

The general *expression* for an n^{th} order d.e. is $f(x,y,y',y'',y(n)) = 0$ where $y' = {}^{dy}/_{dx}$, $y'' = {}^{d^2y}/_{dx^2}$, etc. This equation can only be solved for y in terms of a **finite** number of elementary functions (like x^n, sin(x), cos(x), e^x, etc.) for certain *restricted classes* of equations. In such cases the equation is said to be **solvable** or integrable in *closed form*.

First Order Differential Equations

The **General** form is ${}^{dy}/_{dx} = f(x,y)$. In general, it is not possible to solve this equation for an arbitrary function f(x,y) in closed form. We look at *special* types where this <u>can</u> be done.

Separable Equations

If $f(x,y)$ can be **expressed** as $f(x,y) = {}^{u(x)}/_{v(y)}$ for suitable functions $u(x)$, $v(y)$ respectively, then ${}^{dy}/_{dx} = f(x,y)$ *becomes* ${}^{dy}/_{dx} = {}^{u(x)}/_{v(y)}$; $v(y)dy = u(x)dx$. Here the **LHS** involves y only and the **RHS** involves x only. So the variables are *separate*. Integrating both sides, $\int v(y)dt = \int u(x)dx$. **Example**: $x{}^{dy}/_{dx}+y^2 = 1$; $x{}^{dy}/_{dx} = 1-y^2$; ${}^{dy}/_{1-y^2} = {}^{dx}/_{x}$. **Integrating**, $\frac{1}{2}\ln|{}^{1+y}/_{1-y}| = \ln|x|+c$; $\ln|{}^{1+y}/_{1-y}| = 2\ln|x|+2c$. Let $2c = \ln(A)$, $A>0$; $\ln|{}^{1+y}/_{1-y}| = \ln|x|^2 = \ln|(A)|$; $\ln|{}^{1+y}/_{1-y}| = \ln(x^2A)$; $\ln|{}^{1+y}/_{1-y}| = Ax^2$; ${}^{1+y}/_{1-y} = Bx^2$, **where** $B = \pm A$ $(\neq 0)$.

Linear Equations

General form is ${}^{dy}/_{dx} + P(x)y = Q(x)$, where P and Q are *functions* of x only. Example: ${}^{dy}/_{dx}+{}^{y}/_{x} = 1$. Substituting, $P(x) = {}^{1}/_{x}$, $Q(x) = 1$. Our equation is *not separable*, so we multiply both sides by x: $x{}^{dy}/_{dx}+y = x$; ${}^{d(xy)}/_{dx} = x$. This equation **is** separable in xy and x, so $\int d(xy) = \int x dx$; $xy = {}^{x^2}/_{2} + c$; $y = {}^{x}/_{2} + {}^{c}/_{x}$.

Note: the *solution* of ${}^{dy}/_{dx} + P(x)y = Q(x)$ is $y = {}^{1}/_{v(x)}\int v(x)Q(x)dx$, where $v(x) = \exp(\int P(x)dx)$. Any anti derivative for v will do (for P(x)).

General method: ${}^{dy}/_{dx} + P(x)y = Q(x)$. Multiply this by an *integrating factor* R(x) to be found, $R{}^{dy}/_{dx} + Rdy = RQ$. We want to **express** the LHS of this as ${}^{d}/_{dx}(Ry)$. **Now** ${}^{d}/_{dx}(Ry) = R{}^{dy}/_{dx} + ({}^{dR}/_{dx})y$. Comparing the *last* 2 equations, ${}^{d}/_{dx}(Ry) = R{}^{dy}/_{dx}+RPy$ *provided* ${}^{dR}/_{dx} = RP$. Or, $\int {}^{dR}/_{R} = \int P.dx$; $\log(|R|) = \int P.dx$; $|R| = \exp(\int P.dx)$; $R = \pm\exp(\int Pdx)$. Take +ve sign since *only one integrating factor is required.*

Example: ${}^{1}/_{x^2}{}^{dy}/_{dx} + ({}^{2}/_{x^2})y = -4xe^{-2x}$. Put this in the **form** of ${}^{dy}/_{dx}+P(x)y = Q(x)$ by *multiplying* by x^2: ${}^{dy}/_{dx} + 2y = -4xe^{-2x}$. Comparing, $P(x) = 2$. Therefore $R(x) = +\exp(\int 2dx)$; $R(x) = e^{2x}$. **Multiplying**, $e^{2x}\times{}^{dy}/_{dx} + 2e^{2x}y = -4x^3e^{-2x}e^{2x}$; ${}^{d}/_{dx}(e^{2x}y) = -4x^3$; $\int d(e^{2x}y) = \int -4x^3dx$; $e^{2x}y = -x^4+c$; $y = (c-x^4)e^{-2x}$.

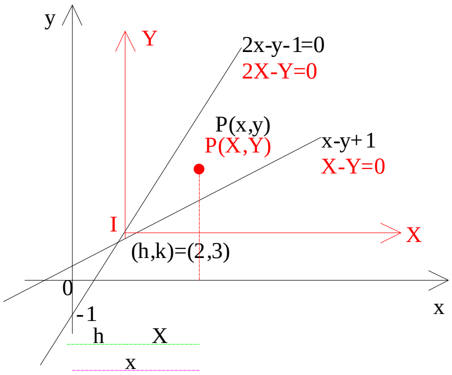
Homogenous Equations

The *general* form is ${}^{dy}/_{dx} = g({}^{y}/_{x})$. Examples of g: (i) ${}^{dy}/_{dx} = {}^{3x^2+4y^2}/_{8x^2} = {}^{3+4(y/x)^2}/_{8}$. (ii) ${}^{ax+by}/_{y} = {}^{a+b(y/x)}/_{(y/x)}$. (iii) $\log(|x|) - \log(|y|) = \log|{}^{x}/_{y}| = \log({}^{1}/_{\log(y/x)})$. In ${}^{3x^2+4y^2+x}/_{8x^2} = {}^{3+4(y/x)^2+(1/x)}/_{8}$, this is not homogeneous — it is not a **function** of $({}^{y}/_{x})$. For homogenous equations, write $v = {}^{y}/_{x}$. This *implies* ${}^{dy}/_{dx} = {}^{d}/_{dx}(xv(x)) = x{}^{dv}/_{dx}+v$. So in ${}^{dy}/_{dx} = g({}^{y}/_{x})$, ${}^{dy}/_{dx} = x{}^{dv}/_{dx}+v = g(v)$; $\int{}^{dv}/_{g(v)-v} = \int{}^{dx}/_{x}$; ${}^{dv}/_{g(v)-v} = \ln(x) + \text{constant}$.

Example: ${}^{dy}/_{dx} = {}^{3x^2+4y^2}/_{8x^2} = {}^{3+4(y/x)^2}/_{8} = {}^{3+4v^2}/_{8}$. Now $x{}^{dv}/_{dx}+v = {}^{3+4v^2}/_{8}$; $\int{}^{8dv}/_{4v^2-8v+3} = \int{}^{dx}/_{x} = \ln|x|+\ln(A)$ $[A>0]$. The LHS: ${}^{8}/_{4v^2-8v+3} = {}^{8}/_{(2v-1)(2v-3)}$. Write in terms of **partial fractions** to get ${}^{4}/_{2v-3} - {}^{4}/_{2v-1}$. So $\int{}^{4}/_{2v-3}dv - \int{}^{4}/_{2v-1}dv = \ln|x|+\ln(A)$; $2\ln|2v-3|-2\ln|2v-1| = \ln|x|+\ln(A)$; $\ln({}^{2v-3}/_{2v-1})^2 = \ln(|x|A)$; $\ln(A|x|) = \ln({}^{2(y/x)-3}/_{2(y/x)-1})^2$; $\ln(A|x|) = \ln({}^{2y-3x}/_{2y-x})^2$; $x = B({}^{2y-3x}/_{2y-x})^2$, where $B = \pm{}^{1}/_{A}$ $(A>0)$.

Equations Reducible to Homogenous Form

Consider ${}^{dy}/_{dx} = {}^{2x-y-1}/_{x-y+1}$. It is *not homogenous* but it can be made homogeneous by a transformation of the form $x=X+h$, $y=Y+k$, where h,k are constants to be found. So ${}^{dy}/_{dx} = {}^{d}/_{dx}(Y+k) = {}^{dY}/_{dx} = {}^{dY}/_{dX} \times {}^{dX}/_{dx} = {}^{dY}/_{dX}$. **Since** ${}^{dX}/_{dx} = 1$, then ${}^{dy}/_{dx} = {}^{dY}/_{dX} = {}^{2(X+h)-(Y+k)-1}/_{X+h-(Y+k)+1} = {}^{2X-Y+(2h-k-1)}/_{X-Y+(h-k+1)}$. So *choose* h and k to satisfy $2h-k-1=0$ and $h-k+1=0$. **Solve** to get $h=2$ and $k=3$. So now ${}^{dY}/_{dX} = {}^{2X-Y}/_{X-Y}$, homogenous. ${}^{dY}/_{dX} = {}^{2-(Y/X)}/_{1-(Y/X)}$; ${}^{dY}/_{dX} = {}^{2-v}/_{1-v}$, where $v = {}^{Y}/_{X}$. Solve this to give $Y^2-2YX+2X^2 = c$ with $X = x-h = x-2$ and $Y = y-k = y-3$. So $2x^2+y^2-2xy-2x-2 = D$. The **geometrical** interpretation is on the *right*.



Examples 6

	Order	Degree	Linear in y?
$y' = \cos(x)$	1	1	Yes
$y''' + 4y = 0$	3	1	Yes
$x^2(y'')^2 + 2e^xy = (x + 2)y$	2	2	No
$x^2yy'' + \ln(x) = 2xy^2$	2	1	No
$(y'')^2 + y''' + 3y = 0$	3	1	No
$(\sec(y) - x\tan(y))y' = 1$	1	1	No

Show that if y is **considered** to be the independent variable and x the dependent variable, then the *last example* in the table is linear in x. A: $(\sec(y)-x\tan(y)){}^{dy}/_{dx} = 1$. This implies ${}^{dx}/_{dy} = \sec(y)-x\tan(y)$. So ${}^{dx}/_{dy} + (\tan(y))x = \sec(y)$ — this is **linear** in x.

Q: Find the ODE's which have the following **general** solutions. (i) $y = \text{atan}(x)$. So ${}^{dy}/_{dx} = a\sec^2(x)$; ${}^{dy}/_{dx} = ({}^{y}/_{\tan(x)})\sec^2(x) = {}^{y}/_{([\sin(x)/(\cos(x).]\cos^2x)} = {}^{2y}/_{\sin(2x)}$. So $\sin(2x){}^{dy}/_{dx}-2y = 0$. (ii) $y = ae^{bx}$, so ${}^{dy}/_{dx} = abe^{bx}$. Now ${}^{dy}/_{dx}$ = [by substitution] $= {}^{(dy/dx)}/_{y} = b$. Now ${}^{d}/_{dx}({}^{y'}/_{y}) = {}^{db}/_{dx} = 0$. Therefore ${}^{yy''-(y')^2}/_{y^2} = 0$. So $y{}^{d^2y}/_{dx^2}-({}^{dy}/_{dx})^2 = 0$. (iii) $y = ax^2+({}^{b}/_{x})+\ln|x|$, $y' = 2ax-{}^{b}/_{x^2}+{}^{1}/_{x}$. So $y+xy' = 3ax^2+\ln|x|+1$; $y'' = 2a+{}^{2b}/_{x^3}-{}^{1}/_{x^3}$. So *therefore* $x^2y'' = 2ax+{}^{2b}/_{x}-1$. And $x^2y''-2y+2\ln|x|+1 = 0$. (iv) $y = \Sigma_{i=0}^{i=n} a_ix^i$ (a_i arb. const). **Therefore**, $y^{(n+1)}=0$.

Section A: Find the *solutions* to the following **Separable** ODE's: (i) $x^3y'+3y^2=xy^2$. ${}^{dy}/_{dx} = {}^{y^2(x-3)}/_{x^3}$. $\int{}^{dy}/_{y^2} = \int{}^{x-3}/_{x^3}dx$. $-{}^{1}/_{y} = -{}^{1}/_{x}+{}^{3}/_{2}.{}^{1}/_{x^2}+c$. So $y^{-1} = x^{-1}-{}^{3}/_{2}.x^{-2}$. (ii) $y\ln(|x|)\ln(|y|)+y' = 0$. ${}^{dy}/_{dx} = -y\ln(|x|)\ln(|y|)$; $\int{}^{dy}/_{y\ln(|y|)} = \int-\ln(|x|)dx$. Integrating by **parts**, $\ln(|\ln(|y|)|) = -\{x\ln|x|-\int x.{}^{1}/_{x}dx\} = -x\ln|x|+x+c$. Therefore, $\ln|\ln(|y|)| = x(1-\ln|x|)+c$.

(iii) $x(y+2)+y(x+2)y' = 0$. ${}^{dy}/_{dx} = -{}^{x(y+2)}/_{y(x+2)}$. $\int{}^{ydy}/_{y+2} = -\int{}^{xdx}/_{x+2}$. $\int({}^{y+2-2}/_{y+2})dy = -\int(x+2-2/x+2)dx$. $\int(1-{}^{2}/_{y+2})dy = -\int(1-{}^{2}/_{x+2})dx$. $y-2\ln|y+2| = -\{x-2\ln|x+2|\}+c$. $y+x = \ln\{[(y+2)^2(x+2^2)^2]A\}$, where $c = \ln(A)$. So $e^{y+x} = A(y+2)^2(x+2)^2$. (iv) $y' = \sqrt{}(4x+2y-1)$. Let $z = \sqrt{}(4x+2y-1)$. Now ${}^{dz}/_{dx} = {}^{1}/_{2\sqrt{}(4x+2y-1)}.(4+2{}^{dy}/_{dx})$. ${}^{dz}/_{dx} = {}^{1}/_{z}.(2+{}^{dy}/_{dx}) = {}^{1}/_{z}(2+z)$.$\int({}^{z}/_{z+2})dx = \int dx = x+c$. $\int({}^{z+2-2}/_{z+2})dx = x+c$. $z - 2\ln|z+2| = x+c$. $(z-x-c) = \ln(z+2)^2$. $De^{(z-x)} = (z+2)^2$, where $D = e^{-C}$. $De^{(\sqrt{}(4x+2y-1)-x)} = (\sqrt{}(4x+2y-1)+2)^2$. (v) $z'=10^{x+z}$. $z' = 10^{x+z} = e^{(x+z)\ln(10)}$. $\int{}^{dz}/_{\exp(z\ln(10))} = \int\exp(\ln(10))dx$. ${}^{\exp(-z\ln(10))}/_{-\ln(10)} = {}^{\exp(x\ln(10))}/_{\ln(10)} + c$. So $10^{-z} + 10^{-x} = d$, where $d = -c\ln(10)$.

Section B: Find the *solutions* to the following **Linear** ODE's. (i) $y'+(\frac{y}{x}) = 4x^2$. A: $xy'+y = 4x^3$. $\frac{d}{dx}(xy) = 4x^3$. $xy = x^4+c$. So $y = x^3+\frac{c}{x}$. (ii) $x\ln(|x|)y'+y = 2\ln(|x|)$, $y=0$ when $x=1$. A: $\frac{dy}{dx} + \frac{1}{x\ln|x|}.y = \frac{2}{x}$. If $R(x) = \exp\int Pdx = \exp\int\frac{dx}{x\ln(|x|)} = \exp(\ln(|\ln|x||)) = \ln(|x|)$. **Multiplying**, $\frac{d}{dx}[y\ln(|x|)] = \frac{2}{x}\ln(|x|)$. So $y\ln(|x|) = \int\frac{2\ln(|x|)}{x}dx = (\ln|x|)^2+c$. Now putting in *values*, 0=0+c so c=0. Therefore $y = \ln(|x|)$.

(iii) $y'+y\tan(x) = \sec(x)$. If $|R(x)| = \exp\int\tan(x)dx = \exp(\ln|\sec(x)|) = |\sec(x)|$. Take $R(x) = \sec(x)$. So $\frac{d}{dx}(y\sec(x)) = \sec^2(x)$. Therefore, $y\sec(x) = \int\sec^2(x)dx = \tan(x)+c$. So $y = \sin(x) + c\times\cos(x)$. (iv) $(y'-y)xe^{-x} = 1$. $\frac{dy}{dx}-y = \frac{\exp(x)}{x}$. I.F. $= \exp\int-1dx = e^{-x}$. **Therefore** $\frac{d}{dx}[e^{-x}y] = \frac{1}{x}$. So $e^{-x}y = \ln(|x|A)$, $A>0$. Therefore $y = e^x\ln(A|x|)$. (v) $(3x-y^2)y' = y$. A: $\frac{3x-y^2}{y} = \frac{dx}{dy}$. $\frac{dx}{dy} - (\frac{3}{y})x = -y$. This is *linear* in x, so I.F. $R(y) = \exp\int(-\frac{3}{y})dy = \exp(-3\ln|y|) = |y|^{-3}$. Let $R(y) = y^{-3}$. Therefore $\frac{d}{dy}(xy^{-3}) = -y^{-2}$. So $xy^{-3} = y^{-1}+c$. And $x = cy^3-y^2$; $x = y^2(cy-1)$.

Section C: Find the *solutions* to the following **Homogenous** ODE's. (i) $(x^2-xy+y^2)-xyy' = 0$. A: $\frac{dy}{dx} = \frac{x^2-xy+y^2}{xy}$, *homogenous*. Let $\frac{y}{x} = v$, so $x\frac{dv}{dx}+v = \frac{1-v+v^2}{v}$. $x\frac{dv}{dx} = \frac{1-v+v^2-v^2}{v} = \frac{1-v}{v}$. $\int\frac{vdv}{1-v} = \int\frac{dx}{x}$. $-\int(\frac{1-v-1}{1-v})dv = \ln|x|+c$. $-v-\ln|1-v| = \ln|x|+c$. $-v = \ln(|x||1-v|A)$, where $C=\ln(A)$, $(A>0)$. Now $-\frac{y}{x} = \ln(|(x-y)|A)$, $\exp(-\frac{y}{x}) = \pm(x-y)A$. $(x-y) = Be^{-y/x}$, $B = \pm\frac{1}{A}$. (ii) $xyy'+x^2+y^2 = 0$. $\frac{dy}{dx} = -\frac{(x^2+y^2)}{xy}$. Let $\frac{y}{x}=v$, so $x\frac{dv}{dx}+v = -\frac{(1+v^2)}{v}$; $x\frac{dv}{dx} = -\frac{(2v^2+1)}{v}$. $\int\frac{vdv}{2v^2+1} = -\int\frac{dx}{x} = -\ln(|x|A)$, $A>0$. $\frac{1}{4}\ln(2v^2+1) = -\ln(|x|A)$. $\ln[(2v^2+1)(x^4A^4)] = 0$. $(2V^2+1)x^4A^4 = 1$. $(2y^2+x^2)x^2A^4 = 1$. $(2y^2+x^2)x^2A^4 = 1$. $(2y^2+x^2)x^2 = C$, where $C = 1/A^4$.

(iii) $(x-y\ln(|y|)+y\ln(|x|))+x(\ln(|y|)-\ln(|x|))y' = 0$. $(x-y\ln|\frac{y}{x}|)+x\ln(|\frac{y}{x}|)\frac{dy}{dx} = 0$. $\frac{dy}{dx} = \frac{y\ln(|y/x|)-x}{x\ln|y/x|}$. $\frac{dy}{dx} = \frac{((y/x)\ln(|y/x|)-1)}{\ln|y/x|}$. Let $\frac{y}{x} = v$. So $x\frac{dv}{dx}+v = \frac{(v\ln|v|-1)}{\ln|v|} = v-\frac{1}{\ln|v|}$. So $x\frac{dv}{dx} = -\frac{1}{\ln|v|}$. $\int\ln|v|dv = -\int\frac{dx}{x}$. $v\ln|v|-v = -\ln(|x|A)$, $(A>0)$. $v(\ln|v|-\ln(e)) = -\ln(|x|A)$. So $v\ln(\frac{v}{e})+\ln(|x|A) = 0$. $\ln[(v/e)^v|x|A] = 0$. $(v/e)^v|x|A = 1$. $(y/ex)^{y/x}x = B$, where $B = \pm\frac{1}{A}$. $(y/x)^{y/x}x = Be^{y/x}$. $(y/x)^{y/x}(y/x)^{-1} = \frac{B\exp(y/x)}{y}$. $(y/x)^{(y/x -1)} = \frac{B.\exp(y/x)}{y}$.

(iv) $2y'+x = 4\sqrt{y}$. Let $\sqrt{y} = a$, then $\frac{dz}{dx} = \frac{1}{2\sqrt{y}}$, $\frac{dy}{dx} = \frac{1}{2z}\frac{dy}{dx}$. $4z\frac{dz}{dx}+x = 4z$. $\frac{dz}{dx} = \frac{4z-x}{4z}$ (homogenous). Let $\frac{z}{x} = v$. So $x\frac{dv}{dx}+v = \frac{4v-1}{4v}$. $x\frac{dv}{dx} = \frac{4v-1-4v^2}{4v}$; $\int\frac{4vdv}{4v^2-4v+1} = -\frac{dx}{x}$. $\int\frac{\frac{1}{2}(8v-4)dv}{4v^2-4v+1} + \int\frac{2dv}{4v^2-4v+1} = -\ln(A|x|)$, $(A>0)$. Therefore $\frac{1}{2}\ln|4v^2-4v+1| + \int\frac{2dv}{(2v-1)^2} = -\ln(A|x|)$. $\frac{1}{2}\ln|4v^2-4v+1| + \frac{-1}{(2v-1)} = -\ln(A|x|)$. $\ln\{A^2x^2|4v^2-4v+1|\} = \frac{2}{2v-1}$. $(2\sqrt{(y)}-x)^2 = B.\exp[\frac{2x}{2\sqrt{(y)}-x}]$.

(v) $(3x-5y)y' = x-3y+2$. A: $\frac{dy}{dx} = \frac{x-3y+2}{3x-5y}$. Let $x=X+h$, $y=Y+k$, then $\frac{dy}{dx} = \frac{dY}{dX} = \frac{X-3Y+(h-3k+2)}{3X-5Y+(3h-5k)}$. Let h,k **satisfy** $h-3k+2=0$ and $3h-5k=0$ so $h=\frac{5}{2}$, $k=\frac{3}{2}$. *Therefore* $\frac{dY}{dX} = \frac{X-3Y}{3X-5Y}$. Let $\frac{Y}{X} = V$. $X\frac{dV}{dX}+V = \frac{1-3V}{3-5V}$. Now $X\frac{dV}{dX} = \frac{1-3V-3V+5V^2}{3-5V}$; $\int(\frac{3-5V}{1-6V+5V^2})dv = \int\frac{dX}{X}$. $-\frac{1}{2}\ln|1-6V+5V^2| = \ln(|x|A)$, $(A>0)$. Now $\ln((|x|A)^2|1-6V+5V^2|) = 0$. $\pm A^2(X^2-6XY+5Y^2) = 1$. $(X-5Y)(X-Y) = B$, where $B = \pm\frac{1}{A^2}$. $((x-\frac{5}{2})-5(y-\frac{3}{2}))((x-\frac{5}{2})-(y-\frac{3}{2})) = B$. $(x-5y+5)(x-y-1) = B$.

(vi) $x-2y-5 = (3x-6y-10)y'$. A: $\frac{dy}{dx} = \frac{x-2y-5}{3x-6y-10}$. Let $z = x-2y$, so $\frac{dz}{dx} = 1-2\frac{dy}{dx}$. Now $\frac{dy}{dx} = (\frac{1}{2})(1-\frac{dz}{dx}) = \frac{z-5}{3z-10}$. So $1-\frac{dz}{dx} = \frac{2z-10}{3z-10}$. Now $\frac{dz}{dx} = 1-(\frac{2z-10}{3z-10}) = \frac{3z-10-2z+10}{3z-10} = \frac{z}{3z-10}$. $\int(\frac{3z-10}{z})dz = \int dx$. So $3z-10\ln|z| = x+c$. $3z-x = 10\ln(|z|A)$, $10\ln(A) = C$, $(A>0)$. **Now** $3x-6y-x = 10\ln(|x-2y|A)$. $x-3y = \ln(|x-2y|A)^5$. So $(x-2y)^5 = B.\exp(x-3y)$, where $B = \pm1/A^5$.

General Linear Equations

Homogenous Type

Let $D = {}^{d}/_{dx}$, where x **denotes** the independent variable. So $D^r = d^r/dx^r$, r positive integral. The general *linear equation of order* n is $a_n(x)D^n y + a_{n-1}(x)D^{n-1}y + ... + a_1(x)Dy + a_0(0)y = f(x)$. The **homogenous** type are those for which f(x) = 0. We shall *restrict ourselve*s to the above equation having **constant** coefficients i.e. $b_nD^ny+b_{n-1}D^{n-1}y+...+b_1Dy+b_0y = 0$, where b_i are all **constants**, i = 0...n.

n=2. Suppose $D^2y-5Dy+6y = 0$. *Try* $y=Ae^{mx}$ where A & m are constants, $A \neq 0$. **Substituting**, $Ae^{mx}(m^2-5m+6) = 0$. Now, A *cannot be zero*, e^{mx} *cannot be zero*, but (m^2-5m+6) can. So the **solutions** to $Ae^{mx}(m-3)(m-2) = 0$ are m=2, m=3. So $y = A_1e^{2x}$, $y = A_2e^{3x}$ are both **solutions**, and $y = A_1e^{2x}+A_2e^{3x}$ is *also* a solution.

The above is an example of the **principle of superposition** of solutions. In general, suppose $b_2D^2y+b_1Dy+b_0y = 0$. Let $y = Ae^{mx}$. Then, by substitution, $(b_2m^2+b_1m+b_0) = 0$ is the <u>**Auxiliary**</u> equation. If α_1, α_2 are the roots of this equation, then **verify that** $y = A_1e^{\alpha_1 x}+A_2e^{\alpha_2 x}$ satisfies the **general** equation. If we have *equal roots*, then it reduces to $y = (A_1+A_2)e^{\alpha_1 x}$. Later we shall show that a "*different* solution" is $y = Axe^{\alpha_1 x} = x(Ae^{\alpha_1 x})$.

Check for <u>this</u> example: $Dy = {}^{d}/_{dx}(Axe^{\alpha_1 x}) = A\{1e^{\alpha_1 x}+x\alpha_1e^{\alpha_1 x}\} = A(1+x\alpha_1)e^{\alpha_1 x}$. $D^2y = A(2\alpha_1+x(\alpha_1)^2)e^{\alpha_1 x}$. **Substituting** in the general equation, $b_2D^2y+b_1Dy+b_0y = 0$, $A\{x(b_2\alpha_1^2+b_1\alpha_1+b_2)+(2b_2\alpha_1+b_1)\}e^{\alpha_1 x} = 0$. Now **since** $b_2\alpha_1^2+b_1\alpha_1+b_0 = 0$, and $\alpha_1+\alpha_2 = -{}^{b_1}/_{b_2}$. But $(\alpha_2=\alpha_1)$ **therefore** $2\alpha_1 = -{}^{b_1}/_{b_2}$, which *implies* $2b_2\alpha_1 = b_1$, $2b_2\alpha_1-b_1 = 0$. So $Axe^{\alpha_1 x}$ is a solution of the **general** equation.

So in this *case* $\alpha_1=\alpha_2$. We have 2 independent *solutions*, $e^{\alpha_1 x}$, $xe^{\alpha_1 x}$, so that $y = (A_1+A_2x)e^{\alpha_1 x}$ is also a solution, where A_1 and A_2 are **arbitrary** constants. Let y_1 and y_2 be 2 solutions of the **general** equation $b_2D^2y+b_1Dy+b_0y = 0$. These solutions are said to be linearly <u>independent</u> if for any constants c_1, c_2, the function $c_1y_1(x)+c_2y_2(x) = 0$ implies $c_1 = 0 = c_2$.

Example: Consider $y_1=e^{\alpha_1 x}$ and $y_2=xe^{\alpha_1 x}$. Query: Linearly **independent**? Let c_1, c_2 be any constants. Then $c_1y_1c_2y_2 = 0$; $c_1e^{\alpha_1 x}+c_2xe^{\alpha_1 x} = 0$. $(c_1+c_2x)e^{\alpha_1 x} = 0$. So $c_1+c_2x = 0$ *because* $e^{\alpha_1 x}$ is not zero. So put x=0; $c_1=0$, and then $c_2x=0$. Put x not **equal** to zero — e.g. 1, so $c_2=0$, i.e. $0=c_1=c_2$ is the only solution, so we **conclude** that $y_1=e^{\alpha_1 x}$ and $y_2=xe^{\alpha_1 x}$ are linearly <u>independent</u>.

If c_1 and c_2 are not both equal to zero i.e. at **least** one of c_1 and c_2 is non-zero, then y_1 and y_2 are linearly *dependent*, thus $Ae^{\alpha_1 x}$, $Be^{\alpha_1 x}$ are linearly **dependent**. $c_1Ae^{\alpha_1 x}+c_2Be^{\alpha_1 x} = 0$. As $e^{\alpha_1 x}$ is not **zero**, then $c_1A+c_2B = 0$ tells us that $c_1/c_2 = -{}^{B}/_{A}$. So if $c_2=A$ (not *equal* to zero), then $c_1 = -{}^{B}/_{A}c_2 = -{}^{B}/_{A}A = -B$.

Theorem: The equation possesses n *linearly independent solutions* $y_1, y_2, \ldots, y_n$ such that any solution may be written as a **linear combination** of $y_1,y_2,\ldots,y_n$, $y = c_1y_1+c_2y_2+\ldots c_ny_n$, where $c_1\ldots c_n$ are constants. So, in the case n=2, there exists **two** independent solutions y_1, y_2 such that any solution is a *linear combination* of these, $y = c_1y_1+c_2y_2$. This is called the **general** solution, and c_1 and c_2 are called *arbitrary* constants.

For n=2, the **general** solution of the above is given by $b_2D^2y+b_1Dy+b_0y = 0$ if α_1 is not equal to α_2, and by $y = (A_1+A_2x)e^{\alpha_1 x}$ if $\alpha_1=\alpha_2$. If n>2, and *suppose the root* α is repeated r times, then $e^{\alpha x}$, $xe^{\alpha x}$, $x^2e^{\alpha x}$, ..., $x^{n-1}e^{\alpha x}$ are r linearly independent solutions corresponding to the repeated root α of the **auxillary** equation.

Examples of Homogenous Equations

Q: $(D^2+3D)y = 0$. **Let** $y = Ae^{mx}$ $(A\neq 0)$ so $(D^2+3D)Ae^{mx} = 0$; $A(m^2+3m)e^{mx} = 0$. As Ae^{mx} is non zero, **then** $m^2+3m = 0$, $m(m+3)=0$, m=0 or m=-3. Therefore $y = Ae^{0x} + Be^{-3x}$; $y = A+Be^{-3x}$ (A,B *arbitrary* constants). Q: $(D^2-4D+13)y = 0$. **Let** $y=Ae^{mx}$ so $A(m^2-4m+13)e^{mx} = 0$. Ae^{mx} is not zero, so $(m^2-4m+13) = 0$, $(m-2)^2+3^2 = 0$, $(m-2+3i)(m-2-3i) = 0$; m=2-3i or m=2+3i. So $y = Ae^{(2-3i)x} + B^{e(2+3i)x} = e^{2x}(Ae^{-i3x}+Be^{+i3x})$. Now using **Euler's Relation, where $e^{i\theta} = \cos\theta+i\sin\theta$ and $e^{-i\theta} = \cos\theta + i(-\sin\theta) = \cos\theta - i\sin\theta$**. So $y = e^{2x}(A(\cos(3x)-i\sin(3x)) + B(\cos(3x)+i\sin(3x))) = e^{2x}[E\cos(3x)+F\sin(3x)]$, where E=A+B, F=i(-A+B) and E & F are **arbitrary** constants.

Q: $(D^2-2D+1)y = 0$, *solve* (a) given y=0 when x=0. (b) y=1, $^{dy}/_{dx}=0$ when x=0. A: $y=Ae^{mx}$ *implies* $A(m^2-2m+1)e^{mx} = 0$. With Ae^{mx} not zero, $(m^2-2m+1) = 0$; $(m-1)(m-1) = 0$. $(m-1)^2 = 0$. So m=1 **twice**. $e^{mx} = e^x$ is *one independent solution;* the 2nd independent solution is where the root occurs **twice**: xe^x. So $y = Ae^x+Bxe^x$ (A,B arb. constants). (a) y=0 when x=0, so $0 = Ae^0+B.0.e^0$; 0=A. So $y = Bxe^x$, b arb. *constant*. (b) y=1, $^{dy}/_{dx}=0$ when x=0. So $1 = Ae^0+B.0.e^0$; 1 = A. **Therefore** $y = Ae^x+Bxe^x$. When A = 1, $y = e^x+Bxe^x$. 2nd: $^{dy}/_{dx} = 0$ when x = 0, so $0 = {}^{dy}/_{dx} = {}^{d}/_{dx}(e^x+Bxe^x) = e^x+B(e^x+xe^x) = 1+B$; so B = -1. **Therefore** $y = e^x-xe^x$; $y = e^x(1-x)$.

Non Homogenous Equations

We now have $b_nD^ny+b_{n-1}D^{n-1}y+\ldots+b_1Dy+b_0y = f(x)$. Suppose y_p (particular) is a solution of the above and y_c (complementary) is a solution of $b_nD^ny+b_{n-1}D^{n-1}y+\ldots b_1Dy+b_0y = 0$. **Then** $y = y_c+y_p$ is a solution of the *first* equation also. The **first** equation implies $F(D)y = f(x)$. The 2nd implies $F(D)y = 0$. So $F(D)y_p = f(x)$ and $F(D)y_c = 0$. $F(D)(y_c+y_p) = F(D)y_c+F(D)y_p$. Because F(D) is **linear**, $F(D)(y_c+y_p) = f(x)$. So y_c+y_p is also a solution of the non-homogenous equation above.

From $y = c_1y_1+c_2y_2+\ldots+c_ny_n$, we know that the **general** solution of $F(D)y = 0$ is $y_c = c_1y_1+c_2y_2+\ldots+c_ny_n$, where $y_1,y_2,\ldots,y_n$ are linearly **independent** solutions. y_p is called the particular integral of $F(D)y_p = f(x)$. y_c is called the complementary function. y_c is the general solution to the corresponding *homogenous* equation (= 0). Theorem: The **general** solution of the red equation is given by $y = y_c+y_p$, where y_p is a particular solution of the non-homogenous equation, and y_c is the general solution of the **homogenous** equation. So any solution of the red equation can be found in *this* way.

Example: $2D^2y+5Dy+2y = 5+2x$. First **Solve** $2D^2y_c+5Dy_c+2y_c = 0$. Let $y_c=Ae^{mx}$, so the Auxiliary **equation** is $2m^2+5m+2 = 0$; $(2m+1)(m+2) = 0$; $m = -½$ or $m = 2$. *Therefore* $y_c = c_1e^{-1/2x} + c_2e^{-2x}$ (c_1,c_2 arb. const). By **inspection**, $y_p=x$ is a *particular* solution of the equation. So the general solution of the Blue equation is $y = y_c+y_p$; $y = c_1e^{-x/2}+c_2e^{-2x}+x$. Now **suppose** that at $x=0$, $y=0$ and $D_y=2$. Show that $c_1={}^2/_3$ and $c_2=-{}^2/_3$ to **give** $y=({}^2/_3)e^{-x/2}-{}^2/_3e^{-2x}+x$; $y = ({}^2/_3)(e^{-x/2}-e^{-2x})+x$.

Examples 7

Section 1. (i) $(D-2)(D+3)y = 0$. A: Let $y = Ae^{mx}$, so $Ae^{mx}(m-2)(m+3) = 0$. As $e^{mx} \neq 0$, $(m-2)(m+3) = 0$. So $m = 2$ or $m = -3$. Therefore $y = Ae^{2x}+Be^{-3x}$ (A,B arbitrary constants). (ii) $(D^2-4D-5)y = 0$. A: Let $y = Ae^{mx}$ so $A(m^2-4m-5)e^{mx} = 0$; $m^2-4m-5 = 0$ because $Ae^{mx} \neq 0$. $(m+1)(m-5) = 0$; $m = -1$ or $m = 5$, so $y = Ae^{-x}+Be^{5x}$.

(iii) $(D^4-16)y = 0$. A: A.E. is $(m^2-2^2)(m^2+2^2) = 0$, $(m-2)(m+2)(m-2i)(m+2i) = 0$. So $m = 2$ or -2 or $2i$ or $-2i$. So $y = Ae^{2x}+Be^{-2x} +Ee^{i2x}+Fe^{-i2x}$. Now $e^{i\theta} = \cos\theta+i\sin\theta$ (Euler's relation) and $e^{-i\theta} = \cos\theta-i\sin\theta$, so $Ee^{i2x}+Fe^{-i2x} = E(\cos(2x)+i\sin(2x))+F(\cos(2x)-i\sin(2x)) = G\cos(2x)+H\sin(2x)$, where $G = E+F$ and $H = i(E-F)$. So $y = Ae^{2x}+Be^{-2x}+G\cos(2x)+H\sin(2x)$. (iv) $(D^3-4D^2+D)y = 0$. A.E. is $m(m-2)^2-(\sqrt{3})^2) = 0$; $m(m-2-\sqrt{3})(m-2+\sqrt{3}) = 0$. So $m = 0$ or $2+\sqrt{3}$ or $2-\sqrt{3}$. **So** $y = Ae^{0x} + Be^{(2+\sqrt{3})x} + Ce^{(2-\sqrt{3})x}$. *Therefore* $y = A+e^{2x}(Be^{\sqrt{(3)}x}+Ce^{-\sqrt{(3)}x})$.

(v) $(D^2+4D+20)y = 0$. A.E. is $(m+2)^2+4^2 = 0$; $(m+2-4i)(m+2+4i) = 0$. So $m = -2+4i$ or $m = -2-4i$. So $y = Ae^{(-2+4i)x}+Be^{(-2-4i)x} = e^{-2x}(Ae^{-4x}+Be^{-i4x}) = e^{-2x}(A(\cos(4x)+i\sin(4x)) + B(\cos(4x)-i\sin(4x))) = e^{-2x}(E\cos(4x)+i\sin(4x)) + B(\cos(4x)-i\sin(4x)))$, where $E = A+B$, $F = i(A-B)$ are **arbitrary** constants.

(vi) $(D^4-6D^2+8D-3)y = 0$. A.E. (after long division) is $(m-1)^3(m+3) = 0$. So $m=1$ (3 times) or $m=-3$. **Corresponding** to the 3 times repeated root $m=1$, we have the *solution* $y = Ae^x+Bxe^x+Cx^2e^x$. Also corresponding to $m=-3$ we have $y_2 = Ee^{-3x}$. So $y = y_1+y_2 = e^x(A+Bx+Cx^2)+Ee^{-3x}$ is the general solution to the *equation*. (vii) $(D^4-33D^2+100D-84)y = 0$. After long division, the A.E. is $(m-2)^2(m-3)(m+7) = 0$, so $y = Ae^{-7x} + (Bx+C)e^{2x} + Ee^{3x}$.

Section (2): (i) $(D^2+5D+6)y = e^x$. $(D^2+5D+6)y_c = 0$. Let $y_c = Ae^{mx}$ so $A(m^2+5m+6)e^{mx} = 0$; $(m^2+5m+6) = 0$. $m=-2$ or $m=3$, so $y_c = Ae^{-2x}+Be^{-3x}$. For y_p, try $y_p = Ce^x$ so $(D^2+5D+6)y_p = (D^2+5D+6)(Ce^x) = C(1^2+5.1+6)e^x = e^x$; $C.12 = 1$; $C = {}^1/_{12}$. Therefore $y_p = {}^1/_{12}e^x$. Now $y = y_c+y_p = Ae^{-2x}+Be^{-3x}+{}^1/_{12}e^x$.

(ii) $(D^2+D-2)y = xe^{-2x}$. A: Use $y_c = Ae^{mx}$ to get $m=1$ or $m=-2$. So $y_c = Ae^x+Be^{-2x}$. For y_p, try $y_p = (C_1x+C_2x^2)e^{-2x}$. (No need to *include* C_0e^{-2x} since it **is** in y_c). So $(D^2+D-2)\{(C_1x+C_2x^2)e^{-2x}\} = xe^{-2x}$. $Dy_p = (C_1+2C_2x)e^{-2x}+(C_1x+C_2x^2)(-2)e^{-2x} = (C_1+(2C_2-2C_1)x-2C_2x^2)e^{-2x}$. $D^2y_p = ((2C_2-2C_1)-4C_2x)e^{-2x} + (C_1+(2C_2-2C_1)x-2C_2x^2)(-2)e^{-2x} = ((2C_2-2C_1-2C_1)+(-4C_2-2(2C_2-2C_1))x + 4C_2x^2)e^{-2x}$. Therefore $D^2y_p+Dy_p-2y_p = \{(2C_2-4C_1) + (-8C_2+4C_1)x + 4C_2x^2 + (C_1) + (2C_2-2C_1)x - 2C_2x^2 - 2C_1x-2C_2x^2\}e^{-2x} = xe^{-2x}$. Therefore *because* e^{-2x} is non zero, $(2C_2-3C_1) + (-8C_2+4C_1+2C_2-2C_1-2C_1)x + (4C_2-2C_2-2C_2)x^2 = x$. Now **equating** coefficients of powers of x, $2C_2-3C_1 = 0$; $-6C_2 = 1$; therefore $C_2 = -{}^1/_6$ and $C_2 = -{}^1/_9$, giving $y_p = (-{}^x/_9-{}^{x^2}/_6)e^{-2x}$. So $y = y_c+y_p = Ae^x+Be^{-2x}-{}^x/_{18}(2+3x)e^{-2x}$.

(iii) (D-3)(D+1)y = x². A: $y_c = Ae^{3x}+Be^{-x}$. Let $y_p = C_0+C_1x+C_2x^2$. $Dy_p = C_1+2C_2x$ and $D^2y_p = 2C_2$. Now $D^2y_p-2Dy_p-3y_p = 2C_2-2(C_1+2C_2x)-3(C_0+C_1x+C_2x^2) = x^2$. $(2C_2-2C_1-3C_0) + (4C_2-3C_1)x - 3C_2x^2 = x^2$. Comparing **powers**, $2C_2-2C_1-3C_0=0$; $-4C_2-3C_1=0$, $-3C_2=1$. Therefore $C_2 = -{}^1/_3$, $C_1 = {}^4/_9$ and $C_0 = -{}^{14}/_{27}$. So $y = y_c+y_p = Ae^{3x}+Be^{-x}-{}^1/_3x^2+{}^4/_9x-{}^{14}/_{27}$.

(iv) (D-3)(D+1)y = e^x. A: $y_c = Ae^{3x}+Be^{-x}$. Let $y_p = Ce^x$. Now $(D^2-2D-3)y_p = e^x$; $C(1^2-2.1-3)e^x = e^x$. $-4Ce^x = e^x$; $-4C=1$; $C=-¼$. So $y = y_c+y_p = Ae^{3x}+Be^{-x}-(e^x/4)$. (v) (D-3)(D+1)y = x^2+e^x. A: $y_c = Ae^{3x}+Be^{-x}$. Now, if $F(D)y_{p_1} = f_1(x)$ and $F(D)y_{p_2} = f_2(x)$ then $F(D)(y_{p_1}+y_{p_2}) = F(D)y_{p_1}+F(D)y_{p_2} = f_1(x)+f_2(x)$ [F(D) **linear**]. So we *have* $yp_1=x^2$ from previous work and $yp_2=e^x$ from *previous* work. So $y_p = y_{p_1}+y_{p_2} = -{}^1/_3x^2+{}^4/_9x-{}^{14}/_{27}-(e^x/4)$. So $y = y_c+y_p = Ae^{3x}+Be^{-x}-{}^1/_3x^2+{}^4/_9x-{}^{14}/_{27}-(e^x/4)$.

(vi) (D-3)(D+1)y = x^2e^x. A: $y_c = Ae^{3x}+Be^{-x}$. Let $y_p = (C_0+C_1x+C_2x^2)e^x$. So $D^2y_p-2Dy_p-3y_p = x^2e^x$, where $Dy_p = (C_1+2C_2x)e^x+(C_0+C_1x+C_2x^2)e^x$; $D^2y_p = (2C_2)e^x + (C_1+2C_2x)e^x + (C_1+2C_2x)e^x + (C_0+C_1x+C_2x^2)e^x = [(2C_2+2C_1+C_0) + (2C_2+2C_2+C_1)x + (C_2)x^2]e^x$. **Therefore** $x^2e^x = [(2C_2+2C_1+C_0) + (4C_2+C_1)x + C_2x^2]e^x - 2[(C_0+C_1) + (2C_2+C_1)x + C_2x^2]e^x - 3(C_0+C_1x+C_2x^2)e^x$. This *implies* $(2C_2+2C_1+C_0-2(C_0+C_1)-3C_0) + ((4C_2+C_1) - 2(2C_2+C_1) - 3C_1)x + (C_2-2C_2-3C_2)x^2 = x^2$. Now comparing *coefficients* we get $2C_2-4C_0=0$, $-4C_1=0$, $-4C_2=1$. From this we *get* $C_2=-¼$, $C_1=0$, $C_0=-{}^1/_8$. **Therefore** $y_p = (-{}^1/_8-{}^1/_4x^2)e^x$. *Therefore* $y = y_c+y_p = Ae^{3x}+Be^{-x}-{}^1/_8(1+2x^2)e^x$.

Note: There is **another** example sheet in the notes (8) involving *physics applications*.

The Particular Integral

Method of Undetermined Coefficients

How do we find a **particular** integral of the *non-homogenous* d.e. $b_nD^ny+b_{n-1}D^{n-1}y+ ... + b_1Dy+b_0y = f(x)$? Example: n=2, $b_2D^2y+b_1Dy+b_0y = e^{2x}$. Try $y = e^{ax}$ then $A(b_2a^2+b_1a+b_0)e^{ax} = e^{ax}$. As e^{ax} is not zero, $A(b_2a^2+b_1a+b_0) = 1$; $A = {}^1/_{(b_2a^2+b_1a+b_0)}$ *provided* $(b_2a^2+b_1a+b_0)$ is not zero. Now **consider** $D^2+4Dy+4y = 4x^2+6e^x$. Try $Ax^2+Bx+C+Ee^x$, so $4Ax^2+(8A+4B)x+(2A+4B+4C)+9Ee^x = 4x^2+6e^x$. Compare **coefficients** of x^2, x^1, x^0, e^x. So 4A=4, 8A+4B=0; 2A+4B+4C=0 and 9E=6. From these we *deduce that* A=1, B=-2, $C={}^3/_2$ and $E={}^2/_3$. Therefore $y = y_p = x^2-2x+{}^3/_2+{}^2/_3e^x$. The **complementary** function, y_c, is $y_c = (C_1+C_2(x))e^{-2x}$. The **general** solution of $D^2y+4Dy+4y = 4x^2+6e^x$ is $y = y_g = y_c+y_p = (c_1+c_2x)e^{-2x}+x^2-2x+{}^3/_2+{}^2/_3e^x = y_c+y_p$.

In **general**, this method works provided the f(x) in $b_nD^ny+b_{n-1}D^{n-1}y + ... + b_1Dy+b_0y = f(x)$ is only a **sum** of terms, each of which *possesses* only a finite number of independent derivatives. In practice, terms like a, x^k, e^{ax}, sin(ax), cos(ax), etc. have only a **finite** number of independent derivatives, but a term like ${}^1/_x$ has an infinite number of *linearly independent derivatives*. So our method cannot be used if f(x) contains a term like ${}^1/_x$.

Useful Rules

(1) If $f(x) = a_0+a_1x+...+a_rx^r$, then **try** $y = y_p = a_0+a_1x+...+a_rx^r$ (a **polynomial** of the same degree as f(x): degree r). If $b_0 = 0$, the trial solution y_p must be of **degree** r+1. If $b_0=0$ and $b_1=0$, y_p must be of *degree* r+2, etc.

(2) If $f(x) = ce^{ax}$, then **try** $y_p = De^{ax}$. This fails if e^{ax} is a term in the *complementary* function y_c. In this case, try $y_p = \alpha xe^{ax}$, or if this fails (if a is a **double** root of the A.E.) try $y_p = \alpha x^2e^{ax}$, etc.

(3) If $f(x) = c_1\sin(ax)+c_2\cos(ax)$ then try $y_p = \alpha_1\sin(ax) + \alpha_2\cos(ax)$. If sin(ax) or cos(ax) lies in y_c, then the **trial** solution fails, so try $y_p = x(\alpha_1\sin(ax)+\alpha_2\cos(ax))$, etc.

(4) If f(x) contains a **sum** or **product** of the above types, then try the appropriate sum or product as a *trial solution*.

Example: $(D^2+D-2)y = xe^{-2x}$. A: The **complementary** solution y_c is the general solution of the homogenous equation $(D^2+D-2)y = 0$. A.E. $m^2+m-2 = 0$; $(m+2)(m-1) = 0$, **meaning** m=-2 or m=1. So $y_c = Ae^x+Be^{-2x}$. For y_p, try $y_p = (c_1x+c_2x^2)e^{-2x}$. (Note: no **need** to include the term $e^{-2x}c_0$ since this lies in the *complementary* function, y_c). So ... $c_1=-{}^1/_9$, $c_2 = -{}^1/_6$, meaning $y_p = (-{}^x/_9-{}^{x^2}/_6)e^{-2x}$. So $y_g = y_c+y_p$.

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SECTION 1 (Compulsory)

(1) (a) If $y = \tan^{-1}(x)$ show that $(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$. **[6 marks]**

(b) If $x = \tan(t)$, $y = \tan(pt)$, where p is a constant, show that $(1+x^2)\frac{d^2y}{dx^2} = 2(py - x)\frac{dy}{dx}$. **[7 marks]**

(c) Find the particular solution of the differential equation $\frac{dy}{dx} + \frac{2}{x}y = 1$, $x > 0$ which satisfies $y = 0$ when $x = 1$. **[7 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) Consider the function $f(x) = x^4e^{-x}$, $-\infty < x < \infty$.

(a) Determine the stationary points and stationary values of f. **[3 marks]**

(b) Find which stationary values are turning values and determine their nature. **[4 marks]**

(c) Examine f for inflexion points and determine their position. **[4 marks]**

(d) Sketch f showing the positions of the stationary points, turning points and inflexion points. **[4 marks]**

(3) By writing $y = \text{cosech}^{-1}(x)$ in the form $\text{cosech}(y) = x$ and using the definition $\sinh(y) = (e^y - e^{-y})/2$, show that $\text{cosech}^{-1}(x) = \log_e\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$, $x \neq 0$. **[8 marks]**

Hence, or otherwise, show that $\frac{d}{dx}(\text{cosech}^{-1}(x)) = \frac{-1}{\sqrt{x^4+x^2}}$, $x \neq 0$. **[7 marks]**

(4) Show that if $I_n = \int_0^{\pi/2} \cos^n(x)dx$ then $I_n = (\frac{n-1}{n})I_{n-2} - \frac{1}{n^2}$, $n \geq 2$. **[10 marks]**

Deduce that $I_4 = {}^1/_{64}(3\pi^2 - 16)$.
(Hint: write $\cos^n(x) = \cos(x)\cos^{n-1}(x)$ and use integration by parts.) **[5 marks]**

(5) (a) Find the solution to the differential equation $\frac{dy}{dx} + y\tan(x) = \sin(2x)$ given that at x=0, y=0. **[8 marks]**

(b) Find the general solution to the differential equation $2x^3\frac{dy}{dx} = y^2 + 3xy^2$, $x > 0$. Hence show that if $y = 4/3$ when $x=1$, then $y \to 1$ as $x \to \infty$. **[7 marks]**

(Questions done: 1, 2, 4)