Solving Systems of Equations

One process of **solving** a set of linear (i.e. no x^2, sin(x) etc.) equations involves *row operations*: (1) Multiply all the coefficients of an equation by a **non zero** constant; (2) **Interchange** the order of two equations in the list of equations; (3) **Add** one equation to another equation. Often (1) and (3) are combined into the rule *"Add a Multiple of one equation to another*".

When performing **row operations**, get the system of equations in triangular or reduced form. Then the system is easily solved by *backward substitution*. After getting the system in the desired form and two equations are the same, this means that these two lines are parallel. It is easier to solve equations by *concentrating on numbers* - forgetting the **variables** x_1, x_2, x_3,... Place the coefficients in an Array, called an augmented matrix. Operations on rows of matrices correspond to operations on *equations*. Equivalent matrices — **different** numbers, **different** equations; **same** solution.

$$x_1 - 2x_2 = 3,\ 2x_1 + x_2 = 4 \Leftrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

An array is just a way of **holding** and **referring** to a large number of values. An array is given a name, often chosen to indicate the *subject matter* we have in mind. The array A is said to be an $m \times n$ array with m rows and n columns. The value placed in the i^{th} row and the j^{th} column of the array is **referred** to as A(i,j) or A_{ij}.

Special Cases. (1) **Column vectors**: arrays with 1 row, n=1. Often denoted by lower case letters x,y,... (2) **Row vectors**, where m=1. Written x^t. (3) **Square arrays**, m=n. Much matrix algebra only works for *square arrays*. (4) Diagonal arrays. The main diagonal of a square array is the collection of entries for which i=j. A diagonal array is a square array with all of its off diagonal entries zero. Thus A(i,j)=0 unless i=j. (5) **Triangular arrays**. Upper Triangular arrays with all the entries above the *main diagonal are zero*. A(i,j) = 0 if i < j. (6) Unit array, I. Unit arrays are diagonal arrays with all diagonal entries = 1.

Addition of Arrays. (Only on matrices of the **same** size). (1) Equality. A and B are equal only if a(i,j) = b(i,j) *for all choices of i,j*. (2) Addition, C = A+B: C(i,j) = A(i,j) + B(i,j). (3) **Zero**, an array with all entries zero. (4) **Scalar Multiplication**. The product of an array A by a number k is an array C with entries C(i,j) = k×A(i,j). The product is *written as the matrix* kA.

Matrix arithmetic satisfies the following rules: (i) A + B = B + A (Commutative); (ii) A + 0 = A. (iii) There is an **array** called -A with A + (-A) = 0 so we can "subtract". (iv) (A+B)+C = A+(B+C) (Associative); (v) k(A+B) = kA+kB (Distributive); (vi) (f+k)(A) = fA+kA (Distributive); (vii) f(k(A)) = (fk)(A).

Multiplication of arrays. We require A to be an m×n matrix and B to be a n×p matrix, s.t. the product, C=AB, is an $m \times p$ array. Multiply as follows: The i^{th} row of the m×n array A is a **row vector** with n entries. The j^{th} column of the n×p array B is a **column vector** with n entries. Multiply corresponding entries of these 2 vectors. A *typical product* would be A(i,j)B(j,k). Add these n products, **summed** over the n values of k; Call the sum C(i,j).

Matrix **multiplication** satisfies the following rules: (i) AB and BA are *generally different*. BA may not be allowed *even if AB is*. (ii) A0=0; 0A=0; (iii) IA=A=AI; (iv) (AB)C=A(BC); (v) k(AB)=(kA)B=A(kB); (vi) A(B+C)=AB+AC; (vii) (A+B)C=AC+BC.

Solving sets of simultaneous equations

A **collection** of linear equations in some unknowns can be written as a matrix equation Ax = b. There are *several approaches* to solving this set of equations, or finding the entries in the unknown vector x. **Inverse**: Multiply the equation Ax=b by C so we have C(Ax)=Cb. Find a *matrix C* such that CA=I. Then, x = Ix = (CA)x = C(Ax) = C(b) = Cb. The particular matrix C is called the **Inverse** of the matrix A and is written A^{-1}. **Elementary row operations**. An alternative approach is to amend the set of equations and *produce another set*, but with the **same** solution.

Examples: **Matrix Multiplication** AB =

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} (2\times 1)+(5\times(-1)) \ +(3\times 2) & (2\times 0)+(5\times 1)+(3\times 1) \\ (1\times 1)+(2\times(-1)) \ +(-1\times 2) & (1\times 0)+(2\times 1)+((-1)\times 1) \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ -3 & 1 \end{pmatrix}$$

The **formula** for the entry of a product is $(AB)_{ik} = \sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha k}$. Hence, for example, $((AB)C)_{ij} = \sum_{\alpha=1}^{p} (AB)_{i\alpha} c_{\alpha j}$. We want to **substitute** for $(AB)_{i\alpha}$ using the above **formula**. As that formula has α as a count in its description, we change the *dummy symbol* from α to β. Therefore, $((AB)C)_{ij} = \sum_{\beta=1}^{p} [\sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha\beta}] c_{\beta j}$. This is a double **summation** over all values of α from 1 to n **inclusive** and for all values of β from 1 to p **inclusive**. $(= \sum_{\beta=1}^{p} \sum_{\alpha=1}^{n} (a_{i\alpha} b_{\alpha\beta}) c_{\beta j})$. Because $(a_{i\alpha}b_{\alpha\beta})c_{\beta j} = a_{i\alpha}(b_{\alpha\beta}c_{\beta j})$, we have **(AB)C = A(BC)**.

Write A^2 for AA, A^3 for AAA etc., and the rules of powers apply i.e. $(A^n)^m = A^{nm}$. **Definition**: B is the inverse of A if AB = I and BA = I.

Example: *Solve the following system* of equations using arrays: x+2y-6x=-3; x+4y+4z=9; 3x+10y+4z=15. These equations can be represented by the yellow array . By using row operations as shown in the **green** box, we can obtain solutions for the unknowns. From the last array (3rd row) we gather that z=0. By substituting in the 2nd row of the last array (2y+10z=12), we get y = 6. Similarly by substitution into **row 1**, (x+2y-6z=-3), we get the result x = -15. The process we use **here** is to try to get the matrix into echelon form, where we try to get zeros in a triangle on the bottom left. All entries **below a leading** term (the first non-zero entry in a row) are always zero. It is easier for a leading term to be 1 for easier calculation. Note: we do not need a *45° slope* for the leading terms.

$$\begin{pmatrix} 1 & 2 & -6 & -3 \\ 1 & 4 & 4 & 9 \\ 3 & 10 & 4 & 15 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & -6 & -3 \\ 0 & 2 & 10 & 12 \\ 0 & 4 & 22 & 24 \end{pmatrix} \begin{matrix} \\ [R_2 - R_1] \\ [R_3 - 3R_1] \end{matrix} \Leftrightarrow \begin{pmatrix} 1 & 2 & -6 & -3 \\ 0 & 2 & 10 & 12 \\ 0 & 0 & 2 & 0 \end{pmatrix} \begin{matrix} \\ \\ [R_3 - 2R_2] \end{matrix}$$

Examples involving constants. Q: Solve the **following** system of matrices: x-y-z = 1; 2x+3y+kz = 3; x+ky+3z = 2, for all *values of the constant k.* (see over for continuation...)

This system can be **represented** by the yellow array. After row operations to reduce to echelon form, which may involve taking away *multiples of k*, we get the green array as shown below. Consider this array. We have 3 cases of answers: (i) k not equal to 2 and not equal to 3; (ii) k=2; (iii) k=-3. In situation (i), by trying out values we see will always get a *unique solution*. However, situations (ii) and (iii) will give different situations. When k=-3, the **last** equation becomes 0x+0y+0z = 5. This is impossible, so when k=-3, there are no solutions. When k=2, the last equation becomes 0=0. This means that we only have *two equations*, y=1-4z and x=1-y+z. We can have <u>ANY</u> choice of z, which then fixes y and x like this: z = z, y = 1-4z, x = 5z. In **vector** notation, solution is all vectors **shown** in purple. (for all values of z)

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (3+k)(2-k) & 2-k \end{pmatrix}$$

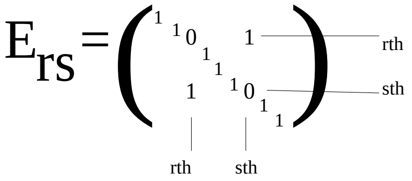
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5z \\ 1-4z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5z \\ -4z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$$

Elementary row operations and matrix multiplication

(1) To multiply row r of an array A by a **non zero scalar** α, this is the same as multiplying matrix A on the left by the *matrix* $E_r(\alpha)$, which is the same as the **matrix** I but with the value α in the position (r,r). Note the *inverse* of the matrix $E_r(\alpha)$ is the matrix $(E_r(\alpha))^{-1} = E_r(\alpha^{-1})$ [**multiplying** the rth row by $^1/_\alpha$].

(2) To add α copies of s to row r, this has a *corresponding elementary matrix* $E_{rs}(\alpha)$ which again is like I but with α in position (r,s) (Row r, column s).
The **inverse** of $E_{rs}(\alpha)$ is $E_{rs}(-\alpha)$.

(3) To *interchange rows r and s, this* has a matrix like I but with **alterations** as shown on the **right**.



Definition: Matrix B is row equivalent to matrix A if B *can be obtained* from A by elementary row operations. **<u>Theorem</u>**: If B is row <u>equivalent</u> to A then B=PA for some inevitable matrix P. **<u>Proof</u>**: B = $(E_k\ E_{k-1}\ ..\ E_1)A$, **where** $E_1...E_k$ are the sequence of elementary matrices corresponding to these *row operations*. So P = $E_kE_{k-1}...E_2E_1$, where $E_1...E_k$ are invertible. So $P^{-1} = (E_k..E_1)^{-1} = E_1^{-1}E_2^{-1}...E_k^{-1}$.

Theorem: If A is **row equivalent** to I, the identity matrix, then there exists an inevitable matrix P with PA = I in which case AP=I and $P=A^{-1}$. The method for reducing a matrix to echelon form can be used to *find the inverse of a matrix*. Example: to find the inverse of a matrix A, join to it on the <u>right hand side</u> a similar **<u>Identity</u>** matrix, and then use row operations to obtain the identity matrix on the *left hand side*. What you will be left with on the **right** hand side will be A^{-1}. **Note**: $(AB)^{-1} = B^{-1}A^{-1}$ (Change the order).

Additional rules: A(B-C)=AB-AC; a(B±C) = aB±aC ('a' is a *scalar*). $AA^{-1}=A^{-1}A=I$. $(A^{-1})^{-1} = A$. $(A^n)^{-1} = (A^{-1})^n$, *where* n=0,1,2,.... $(kA)^{-1} = {}^1/_kA^{-1}$ ('k' is a *scalar*), $(A^t)^t = A$. $(A+B)^t = A^t+B^t$; $(kA)^t = kA^t$ ('k' is a *scalar*); $(AB)^t = B^tA^t$.

Determinants

In a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** of this matrix is ad-bc. If ad-bc is not zero, then the matrix $\frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = A^{-1}$. If ad-bc=0, then *there is no inverse* to the matrix. The number ad-bc is called the determinant of A, written det(A) or |A|. This is **true** for n×n matrices A. The formula for e.g. 3×3 matrices, is take an **entry**, remove items from the same row and column, leave the numbers. You must *specify where you* are expanding from. This is important in terms of the sign of the answer - if you start from an **odd** row/column, then start with a +ve coefficient then alternate -ve, +ve, -ve etc., but if you start from an **even** numbered row/column e.g. 2nd column, then start with a -ve coefficient and *proceed* +ve, -ve, +ve, etc.

E.g. $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & 2 & 1 \end{vmatrix} = -1\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = 7 + 0 - 2 = 5$. (Expand by 2nd row).

Properties of determinants. (1) det(A) = det(A^t). (2) If a *row or column* of A consists entirely of zeros then det(A) = 0. (3) If **two rows** (columns) of A are interchanged giving B, then det(B) = -det(A). (4) If **two rows** (columns) of A are identical then det(A) = 0. (5) If a row (column) of A is *multiplied by k* giving B, then det(B) = k×det(A). (6) If a row (column) of A is a *multiple* of another row (column), det(A) = 0.

(7) If a row of A **consists** of a sum $(a'_{ij} + a''_{ij})$, where j = 1,2,..,n, then det(A) = det A' + det A'', where A' has i^{th} row a'_{ij} and A'' has i^{th} row a''_{ij} but is **otherwise** identical to A. Similarly for columns. (8) If a multiple of *one row* (column) of A is added to another row (column), then the determinant of A in **unchanged**. (9) If A and B are both n×n matrices then det(AB) = (det(A))(det(B)). (10) If A is *singular* then det(A) = 0. If A in *non-singular* then det(A) not equal to 0 and det(A^{-1}) = ${}^{1}/_{\det(A).}$

Applying the above. e.g. if A,B,C have all except the 1st row the same and $C_{ij} = a_{ij}+b_{ij}$, *then* det(C) = det(A) + det(B). Note: triangular matrices have determinant given by the product of the diagonal numbers.

Assignment 1

Note: after evaluating the **inverse** using the AI method, check AA^{-1} to see if it is the identity **matrix**. In a *question, the array shown in yellow* was obtained after row operations. Here there are four cases: μ=2, μ not equal to 2, λ=0, λ not equal to 0. (i) μ=2, λ=0: The **bottom row** becomes 0=0, so more than *one solution*. We can have any x, y and z (ii) μ = 2, λ ≠ 0: Again the **bottom** row is 0=0 with any z, $y = {}^{2-2x}/_{\lambda}$, $x = {}^{2-2z}/_{\lambda}$. (iii) μ ≠ 2, λ = 0. Here z=1 but (4-μ)z-2 = μz-2, which is **inconsistent**, so no solution. (iv) μ ≠ 2, λ ≠ 0: Here z=1 *again* but y and x also have definite values: $y = {}^{2-(4-\mu)}/_{\lambda}$; $x = {}^{2-\mu}/_{\lambda}$, so we have an **unique** solution.

$$\begin{pmatrix} \lambda & 0 & \mu & 2 \\ 0 & \lambda & 4-\mu & 2 \\ 0 & 0 & \mu-2 & \mu-2 \end{pmatrix}$$

Evaluating determinants.

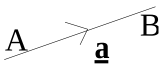
Q: $$\begin{vmatrix} 1 & 0 & -10 & 8 \\ 2 & 2 & 0 & 6 \\ 1 & 2 & 5 & 3 \\ 2 & 12 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -10 & 8 \\ 0 & 2 & 20 & -10 \\ 0 & 2 & 15 & -5 \\ 0 & 10 & 3 & -10 \end{vmatrix} = 10\begin{vmatrix} 1 & 20 & -2 \\ 1 & 15 & -1 \\ 5 & 3 & -2 \end{vmatrix} = 10\begin{vmatrix} 1 & 20 & -2 \\ 0 & -5 & 1 \\ 0 & -97 & 8 \end{vmatrix} = 10(-40+97) = 570$$

Q: $$\begin{pmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b3cd \end{pmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 2a & 3a+2b & 4a+3b+2c \\ 0 & 3a & 6a+3b & 10a+6b+3c \end{vmatrix} \begin{matrix} \\ R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix} =$$

$$a\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1 \end{matrix} = a^2\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} = a^4 \ .$$

Vectors

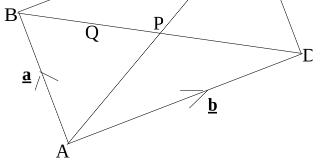
A vector has **two** properties, *magnitude and direction* e.g. velocity, acceleration, force, weight, displacement. A scalar has just *magnitude* e.g. speed, temperature, age, mass. Notation: **a**, $\underline{AB}$, $\overline{AB}$, $\overrightarrow{\mathrm{AB}}$. This is the **vector a** from A to B written **a** or $\overrightarrow{\mathrm{AB}}$, etc Geometrically, it is *as shown*. Direction is from A to B. The magnitude, written |AB| or ||AB||, is the "length of the line", always **greater** or **equal** than zero. Two vectors are equal if they have the *same magnitude and direction*.



Operation on vectors: scalar *multiplication* by λ. λ**a** is the vector with the same magnitude and *direction* as λ|a| if $\lambda \geq 0$. If $\lambda < 0$ then the direction is *reversed* and the magnitude is increased by |λ||**a**|. **Addition**. **a** + **b** is the sum of *two vectors* given by the parallelogram (visualise it). Rules: (i) **a** + **b** = **b** + **a** [*Commute*]; (ii) (**a**+**b**)+**c** = **a**+(**b**+**c**) [*Associative*]; (iii) (λμ)**a** = λ(μ**a**) [*Associative scalar multiplication*]. (iv) (λ+μ)**a** = λ**a**+μ**a**; λ(**a**+**b**) = λ**a**+λ**b** [*Distributive*].

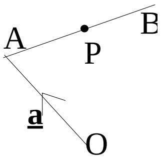
Axes: In **3 dimensions**, choose 3 unit vectors **i**, **j**, **k**, forming a right handed rectangular (orthogonal) *co-ordinate system*. **i**, **j**, **k** are unit vectors from the *origin along* the x, y, z axes. Components and co-ordinates of vector: **addition** rule can be used to decompose a vector: **a** = x**i**+y**j**+z**k**, where x**i**, y**j**, z**k** are the components; x, y, z are co-ordinates *using standard base* **i**, **j**, **k**. Vector **a** is *determined by the triple* (x, y, z). The **length** of the vector **a** is |**a**| = $\sqrt{(x^2+y^2+z^2)}$ (*pythagoras' rule*).

Example: diagonals of a parallelogram bisect each other. **Introduce** some vectors: $\overrightarrow{AB}$ = **a**; $\overrightarrow{AD}$ = **b**. By the *parallelogram rule*, $\overrightarrow{AC}$ = **a** + **c**. Let R be the *mid-point* of AC, so that AR = RC. Now it follows that $|\overrightarrow{AR}| = \frac{1}{2}|\overrightarrow{AC}|$; $\overrightarrow{AR}$ = ½(**a**+**b**). Let W be the *mid point* of BD. We have $|\overrightarrow{BQ}| = \frac{1}{2}|\overrightarrow{BD}|$. As $\overrightarrow{AB}$ = **a**, $\overrightarrow{BA}$ = -**a**. By the *parallelogram* rule, $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{BA} =$ **b**-**a**. Hence $\overrightarrow{BQ}$ = ½(**b**-**a**). *Starting* from A, $\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{BQ}$ = **a** + ½(**b**-**a**); $\overrightarrow{AQ}$ = ½**a** + ½**b**. We have shown that $\overrightarrow{AQ} = \overrightarrow{AR}$; so $Q=R$.



26th October 1998

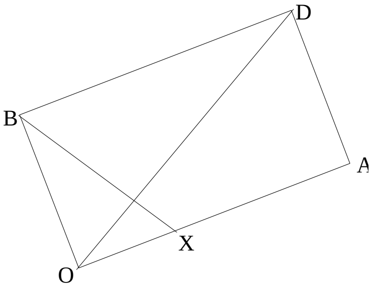
Geometric Vectors: Equation of a Line



A line is *given by 2 points, say A and B*. Let P be a point on the line from A to B. Pick an **origin**, O. A is determined by the vector $\overrightarrow{OA} = \underline{\mathbf{a}}$. Similarly $\overrightarrow{OB} = \underline{\mathbf{b}}$; and let $\overrightarrow{OP} = \underline{\mathbf{r}}$. *Question: find a formula for the vector* $\underline{\mathbf{r}}$ *in terms of* $\underline{\mathbf{a}}$ *and* $\underline{\mathbf{b}}$. *By the parallelogram rule,* $\overrightarrow{AB} = \underline{\mathbf{b}}-\underline{\mathbf{a}}$. $\overrightarrow{AP}$ is a multiple of AB if *P is on the line AB*. So $\overrightarrow{AP} = \lambda(\underline{\mathbf{b}}-\underline{\mathbf{a}})$. Again by the *parallelogram law*, $\underline{\mathbf{r}} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$. $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \overrightarrow{AP}$. $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda(\underline{\mathbf{b}}-\underline{\mathbf{a}})$.

Summary: *line through the points* A($\underline{\mathbf{a}}$), B($\underline{\mathbf{b}}$) is $\underline{\mathbf{r}} = \underline{\mathbf{a}}+\lambda(\underline{\mathbf{b}}-\underline{\mathbf{a}}) = (1-\lambda)\underline{\mathbf{a}} + \underline{\mathbf{b}}\lambda = \alpha\underline{\mathbf{a}}-\beta\underline{\mathbf{b}}$ $(\alpha+\beta=1)$. If $\lambda=0$, **then** $\underline{\mathbf{r}}=\underline{\mathbf{a}}$. If $\lambda=1$, $\underline{\mathbf{r}}=\underline{\mathbf{b}}$. If $\lambda>1$, or $\lambda<0$, then *outside the segment* AB. λ measures the proportion of the line segment AB *covered* by P. $\lambda = {}^{AP}/_{AB}$. Note: from **now on AP** will be used to *represent* $\overrightarrow{AP}$.

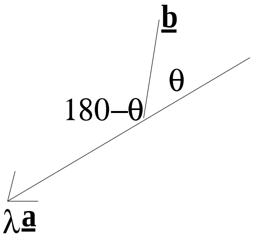
Example: Diagonals of a ||gram *bisect each other*. Choose O as one vertex, the others are $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{a}}+\underline{\mathbf{b}}$. So ***OD*** = $\underline{\mathbf{a}}+\underline{\mathbf{b}}$-O = $\underline{\mathbf{a}}+\underline{\mathbf{b}}$. So a point on ***OD*** is $\lambda(\underline{\mathbf{a}}+\underline{\mathbf{b}})$. λ is a fraction of the distance from O to D. ***AB*** = $\underline{\mathbf{b}}-\underline{\mathbf{a}}$. A point on **AB** is $\underline{\mathbf{a}}+\mu(\underline{\mathbf{b}}-\underline{\mathbf{a}})$. Lines *meet* when $\lambda(\underline{\mathbf{a}}+\underline{\mathbf{b}}) = \underline{\mathbf{a}}+\mu(\underline{\mathbf{b}}-\underline{\mathbf{a}})$; $\lambda\underline{\mathbf{a}}+\lambda\underline{\mathbf{b}} = (1-\mu)\underline{\mathbf{a}}+\mu\underline{\mathbf{b}}$; $(\lambda+\mu-1)\underline{\mathbf{a}} = (\mu-\lambda)\underline{\mathbf{b}}$. $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ are not || so $\lambda+\mu-1=0$ and $\mu-\lambda=0$ i.e. *equating coefficients* implies $\mu = \lambda = ½$ (meet ½ way along the *diagonals*).



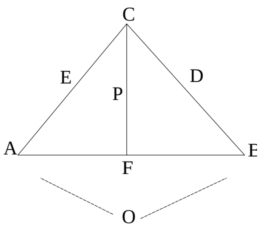
Example: Line joining the **vertex** of a ||gram to the midpoint of a side dividing the *diagonal in the ratio* 1:2. Chose *vectors*: $\underline{\mathbf{a}}$ = OA, $\underline{\mathbf{b}}$ = OB. Then X, the *midpoint* of OA is ½$\underline{\mathbf{a}}$. *Line* XB: $\underline{\mathbf{r}} = ½\underline{\mathbf{a}} + \lambda(\underline{\mathbf{b}}-½\underline{\mathbf{a}}) = (½-½\lambda)\underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$. $\underline{\mathrm{r}}$ is on the *diagonal* OD if $\underline{\mathbf{r}} = \mu(\underline{\mathbf{a}}+\underline{\mathbf{b}})$. Equating $½(1-\lambda)\underline{\mathbf{a}}+\lambda\underline{\mathbf{b}} = \mu\underline{\mathbf{a}}+\mu\underline{\mathbf{b}}$. So $½(1-\lambda)=\mu$ and $\lambda=\mu$. Hence $½(1-\lambda)=\lambda$ implies $\lambda={}^{1}/_{3}$. So $\mu=\lambda={}^{1}/_{3}$. Meet ${}^{1}/_{3}$ way **along** the lines.

Scalar or Dot Product

Given 2 vectors $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, form a number *scalar*, written $\underline{\mathbf{a}}.\underline{\mathbf{b}}$ defined by $\underline{\mathbf{a}}.\underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}|\cos\theta$ where $|\mathbf{a}|$ is the *size length* of vector $\underline{\mathbf{a}}$; θ is the angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$. *Special cases:* (1) $\underline{\mathbf{a}}.\underline{\mathbf{a}} = |\underline{\mathbf{a}}||\underline{\mathbf{a}}|\cos 0 = |\underline{\mathbf{a}}|^2$. Hence $|\underline{\mathbf{a}}| = \sqrt{(\underline{\mathbf{a}}.\underline{\mathbf{a}})}$ (2) $\underline{\mathbf{a}}.\underline{\mathbf{b}}=0$ which *occurs* when $\cos\theta=0$ i.e. $\theta=90°$. Such vectors are "orthogonal"/ "perpendicular".



Properties: (1) $\underline{\mathbf{a}}.\underline{\mathbf{b}} = \underline{\mathbf{b}}.\underline{\mathbf{a}}$ $(= |\underline{\mathbf{a}}||\underline{\mathbf{b}}|\cos\theta)$. (2) $(\lambda\underline{\mathbf{a}}).\underline{\mathbf{b}} = \lambda(\underline{\mathbf{a}}.\underline{\mathbf{b}})$ $(= |\lambda\underline{\mathbf{a}}||\underline{\mathbf{b}}|\cos\theta$ if $\lambda>0$ $= |\lambda||\underline{\mathbf{a}}||\underline{\mathbf{b}}|\cos\theta)$. *Note*: If $\lambda<0$ then angle θ changes by 180° So $\cos\theta$ *changes* by -1. (3) $\underline{\mathbf{a}}.(\underline{\mathbf{b}}+\underline{\mathbf{c}}) = \underline{\mathbf{a}}.\underline{\mathbf{b}} + \underline{\mathbf{a}}.\underline{\mathbf{c}}$.



Tutorial question: Prove the *medians of a triangle* meet at a point which divides each median in the ratio 1:2. Take $\underline{\mathbf{a}},\underline{\mathbf{b}},\underline{\mathbf{c}}$ to represent the vectors of the triangle. These *vectors are not independent* as they lie in the same plane, but the calculations will be more symmetrical.

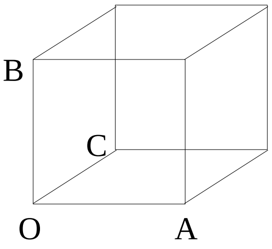
AB is the line $\underline{\mathbf{b}}-\underline{\mathbf{a}}$. F is half way *along the line* so F is $\underline{\mathbf{a}} + \frac{1}{2}(\underline{\mathbf{b}}-\underline{\mathbf{a}}) = \frac{1}{2}(\underline{\mathbf{a}}+\underline{\mathbf{b}})$. CF is the line $\frac{1}{2}(\underline{\mathbf{a}}+\underline{\mathbf{b}})-\underline{\mathbf{c}}$ so a point on CF is $\underline{\mathbf{c}} + \lambda(\frac{1}{2}(\underline{\mathbf{a}}+\underline{\mathbf{b}})-\underline{\mathbf{c}})$. So P is $\frac{1}{2}\lambda\underline{\mathbf{a}} + \frac{1}{2}\lambda\underline{\mathbf{b}} + (1-\lambda)\underline{\mathbf{c}}$. Similarly D is $\frac{1}{2}(\underline{\mathbf{b}}+\underline{\mathbf{c}})$ and a point on AD is $\underline{\mathbf{a}} + \mu(\frac{1}{2}(\underline{\mathbf{b}}+\underline{\mathbf{c}})-\underline{\mathbf{a}})$. So P is also $(1-\mu)\underline{\mathbf{a}} + \frac{1}{2}\mu\underline{\mathbf{b}} + \frac{1}{2}\mu\underline{\mathbf{c}}$. We cannot *equate coefficients* as the vectors are not <u>independent</u>, but if $\lambda=\mu={}^{2}/_{3}$, then these expressions are the **same**. Hence, with $\lambda=\mu={}^{2}/_{3}$, we **get** ${}^{1}/_{3}\underline{\mathbf{a}}+{}^{1}/_{3}\underline{\mathbf{b}}+{}^{1}/_{3}\underline{\mathbf{c}}$ is a vector on median AD and on median CF, by *symmetry it is also on BF*. Hence the point of intersection divides the **median** in the ratio 2:1.

Q: The *diagonals of a parallelopial* meet and bisect each other. A: Points on the diagonal are $\lambda(\underline{\mathbf{a}}+\underline{\mathbf{b}}+\underline{\mathbf{c}})$, $\underline{a}+\mu(\underline{\mathbf{a}}+\underline{\mathbf{b}}+\underline{c})$, $\underline{\mathbf{b}}+\alpha(\underline{\mathbf{a}}-\underline{\mathbf{b}}+\underline{\mathbf{c}})$ and $\underline{\mathbf{c}}+\theta(\underline{\mathbf{a}}+\underline{\mathbf{b}}-\underline{\mathbf{c}})$. Hence the **point** $\frac{1}{2}\underline{\mathbf{a}} + \frac{1}{2}\underline{\mathbf{b}} + \frac{1}{2}\underline{\mathbf{c}}$ is on all four diagonals *taking* $\lambda=\mu=\alpha=\theta=\frac{1}{2}$.

30th October 1998

Dot or Scalar Product

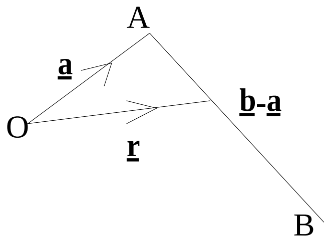
$a.b = |a||b|\cos\theta$. Calculation: use a set of the **normal vectors** i,j,k, where any vector $v = \alpha i+\beta j+\chi k$ for some *coefficients* α,β,χ. Also $i.i=1=j.j=k.k$ and $i.j=0=i.k=j.k$, etc. **General** dot product: $a = a_x i+a_y j+a_z k$; $b = b_x i+b_y j+b_z k$. $a.b = (a_x i+a_y j+a_z k).(b_x i+b_y j+b_z k) = a_x i.b_x i+a_x i.b_y j + \ldots = a_x b_x i.i + a_x b_y i.j + \ldots = a_x b_x+a_y b_y+a_z b_z$. Often refer to vector $\underline{\mathbf{a}}$ just by its *coordinates*, $\underline{\mathbf{a}} = (a_x\ a_y\ a_z)$. *Then* $(a_x\ a_y\ a_z).(b_x\ b_y\ b_z) = a_x b_x+a_y b_y+a_z b_z$.



Example: diagonals of a cube have the same length. Take one vertex as an origin, and then OA=a, OB=b and OC=c. Find the positions of the other corners a+b, a+c, a+b+c and b+c. *Diagonals are given by the vectors* a+b+c-O, b+c-a, a+b-c and a+c-b. a,b,c are orthogonal. $a.b = a.c = b.c = 0$. a,b,c have the same **length** k: $a.a = b.b = a.c. = k^2$. *Therefore*, $(b+c-a).(b+c-a) = b.b+c.c+(-1)a.(-1)a$ + [b.c etc. (<u>all zero</u>)] = $a.a + b.b. + c.c = 3k^2$. Hence **length** of the diagonal is $\sqrt{(3k^2)} = k\sqrt{3}$.

Q: Show that the **perpendicular bisectors** of sides of a triangle meet. Take the origin O as the point of intersection between the *perpendicular bisectors* of BC and AC. Take a = OA, b = OB and c = OC. Then $OD = \lambda b+(1-\lambda)c$ where $\lambda=\frac{1}{2}$ (Midpoint). $OD = \frac{1}{2}(b+c)$ and $OE = \frac{1}{2}(a+c)$. But O is chosen so that OD is the **perpendicular bisector** of BC. The line OD is perpendicular to the line BC. Vector BC = c-b. So we require $OD.BC = 0$ i.e. $\frac{1}{2}(b+c).(c-b) = 0$. Expand: $\frac{1}{2}b.c + \frac{1}{2}c.c - \frac{1}{2}b.b - \frac{1}{2}c.b = 0$; i.e. $c.c-b.b = 0$. Similarly, as O is on the *perpendicular bisector* of AC, $c.c-a.a = 0$. Hence $b.b-a.a = 0$; $\frac{1}{2}(a+b).(b-a) = 0$. So AB is perpendicular to OF i.e. O is on the perpendicular bisector of **all 3 edges**.

Lines & Planes



In $\mathbf{R}^2$, **the equation of a line** is $ax+by = c$ or $y = mx+\text{constant}$. The vector equation of a line: specify a point by a *vector* r from the origin. We need a point on the line i.e. a vector 'a'. We need the direction of the line: $r = a +\lambda d$, where $d = b-a$ or another point on the line. So $r = (1-\lambda)a + \lambda b$. **Equation** of a plane in $\mathbf{R}^3$. In vector form, it is specified by a point on the plane and a normal to the plane. *Suppose* P_0 is on the plane, with vector r_0. Suppose n is the normal to the plane. Let P be any point in $\mathbf{R}^3$ with **vector** r.

Then the **vector** P_0P lies in the plane and so is *perpendicular to n*. Therefore, $(P_0P).n = 0$ i.e. $(r-r_0).n = 0$, i.e. $r.n = r_0.n$ = constant. **Vector equation** of the plane is r.n = constant, where n is normal to the plane. *Cartesian Version*: r = xi+yj+zk; n = ai+bj+ck for some a,b,c. So r.n = ax+by+cz = d (constant is **equation** of plane). Given 3 points on the plane, how do we find the normal? Suppose *points* are p_1, p_2, p_3. Then (p_2-p_1) and (p_3-p_1) are vectors in the **plane**. So the normal n satisfies $(p_2-p_1).n = 0$; $(p_3-p_1).n = 0$. Q: How do we find a **vector** n perpendicular to 2 vectors a and b?

Vector Products (or Cross Products)

The **vector product** of two vectors u and v is a vector, written u×v or u^v. Two definitions: (1) u×v is the vector *perpendicular to u and v*, $|u\times v| = |u||v|\sin\theta$, where θ is angle between u and v. Direction of u×v is given by the RIGHT HAND RULE. (2) If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then $u\times v = (u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)$ or is **determinant** as shown in the yellow box. Example: the *plane* through p_1, p_2, p_3 has equation r.n = constant = p_1.n, where $n = (p_2-p_1)\times(p_3-p_1)$.

$$u\times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

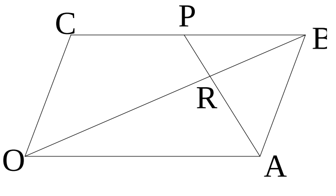
Rules of vector products: (1) u×v = -v×u. (2) u×u = 0. (3) Linearity: u×(v+w) = u×v + u×w; u×(λv) = λ(u×v). (4) $|u\times v|^2 = |u|^2|v|^2-|u.v|^2$. (5) *Formulae* for repeated products: u.(v×w) [scalar triple product], u×(v×w) [vector triple product].

Assignment 2

Q: The **vertices** A,B,C of a triangle have position vectors $\underline{\mathbf{a}} = 2\underline{\mathbf{i}}+4\underline{\mathbf{j}}-\underline{\mathbf{k}}$; $\underline{\mathbf{b}}=4\underline{\mathbf{i}}+5\underline{\mathbf{j}}+\underline{\mathbf{k}}$; $\underline{\mathbf{c}}=3\underline{\mathbf{i}}+6\underline{\mathbf{j}}-3\underline{\mathbf{k}}$. Find the lengths of the *vectors AB, BC and CA* and show that the triangle is right angled. Find the vector $\underline{\mathbf{p}}$ representing the point P distant 6 from A on the line AB.

A: AB = $\underline{\mathbf{b}}-\underline{\mathbf{a}} = (4\underline{\mathbf{i}}+5\underline{\mathbf{j}}+\underline{\mathbf{k}})-(2\underline{\mathbf{i}}+4\underline{\mathbf{j}}-\underline{\mathbf{k}}) = 2\underline{\mathbf{i}}+\underline{\mathbf{j}}+2\underline{\mathbf{k}}$ and $|AB|^2 = 2^2+1^2+2^2 = 9$. *Similarly*, BC = $\underline{\mathbf{c}}-\underline{\mathbf{b}} = -\underline{\mathbf{i}}+\underline{\mathbf{j}}-4\underline{\mathbf{k}}$ with $|BC|^2 = 18$. And CA = $\underline{\mathbf{a}}-\underline{\mathbf{c}} = -\underline{\mathbf{i}}-2\underline{\mathbf{j}}+2\underline{\mathbf{k}}$ with $|CA|^2 = 9$. Hence $|CA|^2+|AB|^2=|BC|^2$ so by *pythagoras we have a right angled* triangle. P is on the line AB, distant 6 from A. **Unit** vector in direction AB is ${}^{\mathbf{b-a}}/_{|AB|} = {}^{\mathbf{b-a}}/_3$. P is $6\times{}^{\mathbf{b-a}}/_3$ from A. $\underline{\mathbf{p}} = \underline{\mathbf{a}} + {}^6/_3(\underline{\mathbf{b}}-\underline{\mathbf{a}}) = 2\underline{\mathbf{b}}-\underline{\mathbf{a}}$. $\underline{\mathbf{p}} = 6\underline{\mathbf{i}}+6\underline{\mathbf{j}}+3\underline{\mathbf{k}}$.

Q: OABC is a ||gram with O *as origin and* $\underline{\mathbf{a}}$ and $\underline{\mathbf{c}}$ representing the vertices A and C. P is the midpoint BC. Find the equation of the line AP. Find the *position vector representing* R where AP meets OB and give the **ratios** OR:RB and AR:RP.



A OA=$\underline{\mathbf{a}}$, OC=$\underline{\mathbf{c}}$, OB=$\underline{\mathbf{a}}+\underline{\mathbf{c}}$ by ||gram law. OP = ½OC + ½OB = $\tfrac{1}{2}\underline{\mathbf{c}}+(1-\tfrac{1}{2})(\underline{\mathbf{a}}+\underline{\mathbf{c}})$ so OP = $\tfrac{1}{2}(\underline{\mathbf{a}}+2\underline{\mathbf{c}})$. Point R *is on* line AP if $\underline{\mathbf{r}} = \lambda\underline{\mathbf{a}}+(1-\lambda)\underline{\mathbf{p}}$. So $\underline{\mathbf{r}} = \lambda\underline{\mathbf{a}}+(1-\lambda)({}^{\mathbf{a}}/_2+\underline{\mathbf{c}}) = ({}^{1+\lambda}/_2)\underline{\mathbf{a}}+(1-\lambda)\underline{\mathbf{c}}$. Point **R** on OB is $\mu\underline{\mathbf{a}}+\mu\underline{\mathbf{c}}$. Hence ${}^{1+\lambda}/_2 = \mu = 1-\lambda$ so $\lambda = {}^1/_3$, $\mu = {}^2/_3$. Hence OR:RB = $\mu:1-\mu$ = 2:1; and AR:RP = $\lambda:1-\lambda$ = 1:2.

Q: Write the **planes** x+2y+3z=4 and 2x-3y+z=1 in *vector notation*. Find a vector parallel to the line L, of intersection of the planes. Find where the line L cuts the plane y=0. Hence give the vector **equation** of L.

A: r=(x,y,z) is on the plane x+2y+3z=4 if r.(1,2,3)=4. **Normal** to plane is $n_1 = (1,2,3)$. Similarly plane 2x-3y+z=1 is $r.n_2=1$ where normal $n_2=(2,-3,1)$. $n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix} =$ 11i+5j-7k = (11,5,-7) is **normal** to n_1 and n_2, and hence lies in **both** planes. Thus direction of L is (11,5,-7). Line L cuts y=0 where all *3 plane equations are satisfied* so x+2y+3z=4, 2x-3y+z=1, y=0. Hence x+3z=4 and 2x+z=1. So $x=-{}^{1}/_{5}$, y=0 and $z={}^{7}/_{5}$. Point on **a** is $(-{}^{1}/_{5},0,{}^{7}/_{5})$. Line L is $r = a+\lambda(n_1 \times n_2) = (-{}^{1}/_{5},0,{}^{7}/_{5})+\lambda(11,5,-7)$.

Q: **Let** **a**=**i**-8**j**-4**k**, **b**=**i**+2**j**+2**k**, **c**=3**i**+6**j**+2**k**. Find (i) a *unit vector perpendicular* to **a** and **b**; (ii) The *sine of the angle* between **b** and **c**; (iii) the *cosine of the angle* between **a** and **c**; (iv) the *vectors* (a×b)×c and a×(b×c).

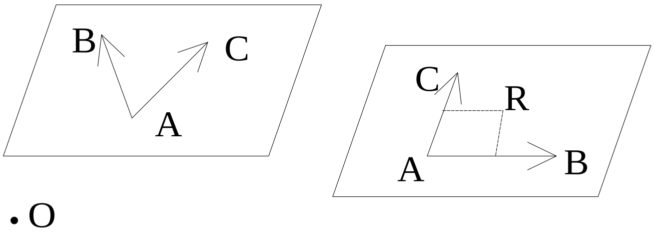
A: (i) **a**×**b** = $\begin{vmatrix} i & j & k \\ 1 & -8 & -4 \\ 1 & 2 & 2 \end{vmatrix}$ = -8**i**-6**j**+10**k**. *Unit* vector is $\frac{-8i-6j+10k}{\sqrt{200}}$.

(ii) **b**×**c** = $\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 6 & 2 \end{vmatrix}$ = -8**i**+4**j**. $\mathrm{Sin}\theta = \frac{|b\times c|}{|b||c|} = \frac{\sqrt{80}}{\sqrt{9}\ \sqrt{49}}$. (iii) $\mathrm{Cos}\theta = {}^{a.c}/_{|a||c|} = -{}^{53}/_{\sqrt{81}\sqrt{49}} = -{}^{53}/_{63}$.

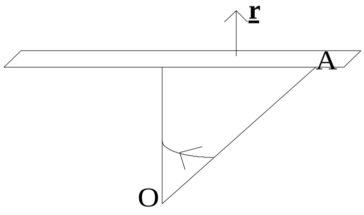
(iv) Using *previous calculations* in (i) and (ii), (**a**×**b**)×**c** = -72**i**+46**j**-30**k** and **a**×(**b**×**c**) = 16**i**+32**j**-60**k**. Note: be *careful with signs* when working out determinants.

Properties of Vectors

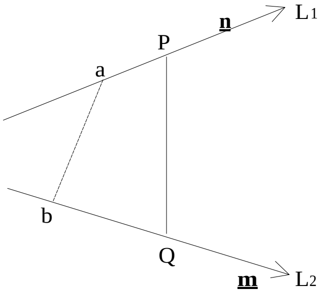
If you take a **triangle** and take the medians of this triangle, using these vectors we can produce a *new triangle*. Example (unrelated): Find the equation of the plane through an A, B, C. Pick an **origin** O. OA=**a**, OB=**b**, OC=**c**. R is a point on *the plane* where OR = **r** = ? We have a point A in the *plane and vectors* in the plane [**b**-**a**, **c**-**a**]. The vector **r**-**a** is in the plane so must be a *combination* of **b**-**a** and **c**-**a**. So AR = n(**b**-**a**) + p(**c**-**a**). Hence **r** = OA+AR; **r** = **a**+n(**b**-**a**)+p(**c**-**a**). **r** = (1-n-p)**a** + n**b** + p**c**. Or, **r** = $\frac{\alpha a+\beta b+\gamma c}{\alpha+\beta+\gamma}$.



Example: Distance from an **origin** to a **plane**. Given an equation of a plane, **r**.**n**=d. Let A be a point on the plane. OA is a vector from the origin to a *point on the plane*. Distance = projection of OA in direction n = OA.**n̂**, where **n̂** is a unit vector in this *direction*. Now OA = **a** and **a**.**n** = d. So OA.**n̂** = $a.{}^{n}/_{|n|} = {}^{d}/_{|n|}$. If **plane** is $\alpha z+\beta x+\chi y=d$ with **n**=(α,β,χ), distance = ${}^{d}/_{\sqrt{(\alpha^2+\beta^2+\chi^2)}}$.



Example: Distance *between 2 lines*. L_1: $r = \mathbf{a}+\lambda\mathbf{n}$; L_2: $r = \mathbf{b}+\mu\mathbf{m}$. Minimal distance *is along a line PQ*. Must have PQ perpendicular to L_1 and to L_2. Direction of PQ is *perpendicular* to **n** and **m**. Therefore PQ in the *direction* **m**×**n**. The distance is the *projection* of AB onto PQ i.e. the **projection** of (**b**-**a**) onto **m**×**n**. *Distance* $= (b-a).\frac{(m\times n)}{|m\times n|}$.



Vector Spaces

Start with a field k. For our purposes, k = **R** (*Real numbers*) or k = **C** (*Complex numbers*). A real (k=**R**) Vector Space is a SET, V, of elements x,y,z,u,v... i.e. V = {x,y,z,u,v,...}. We have **two** operations on V. (1) Addition, +. If $u \in V$ and $v \in V$ then there is an element $w+v \in V$. (2) Scalar Multiplication. If $u \in V$ and $k \in K$ then there's an *element* $ku \in V$.

To be a vector space, this must **satisfy 10 axioms**. (1) $u+v \in V$ (*Closed under addition*). (2) $(u+v)+w = u+(v+w)$. This is for all u,v,w, in V (*Associativity*). (3) $u+v = v+u$. This is for **all** u,v in V, written $\forall\, u, v \in V$. (*Commutative*). (4) There *exists a vector* 0 in V, for which $u+0 = 0+u = u$ for **all** u. (5) $u+(-u) = 0$ **for** all u in V. (6) $ku \in V$ for **all** k in K, u in V. (7) $k(u+v) = ku+kv$ for **all** k in K, u,v in V. (8) $(k+L)u = ku+Lu$ for **all** k,L in K and **all** u in V. (9) $(kL)u = k(Lu)$ for **all** k,L in K and **all** u in V. (10) $1u = u$, *where 1 is the identity* in K.

You must check the **above axioms** to make sure something is a vector space. *Examples*. (1) $k\times k = k^2$ (In $\mathbf{R}^2$) = $\{(x,y) \mid x,y \in k\}$. If $u = (x_1,y_1)$; $v = (x_2,y_2)$, define $u+v=(x_1+x_2,\ y_1+y_2)$ and $ku = (kx_1,\ ky_1)$. (2) $V = k^n$. Let $u = (x_1,x_2,...,x_n)$ and $v = (y_1,y_2,...,y_n)$ then $u+v = (x_1+y_1,\ x_2+y_2,\ ...,\ x_n+y_n)$ and $ku = (kx_1,\ kx_2,\ ....\ ,\ kx_n)$. (3) V = k[x] — all polynomials in x *over the field k*: can add polynomials or scalar multiply poly's. (4) V = set of all **polynomials** of degree ≤ 3 (*Subset* of (3), $u = a_0+a_1x+a_2x^2+a_3x^3$ $[a\in k]$; $ku = ka_0+ka_1x+ka_2x^2+ka_3x^3$).

(5) V = set of **all functions** f: **R** → **R,** $k\in\mathbf{R}$. Add and Scalar Multiply functions. f+g is a function *whose value at x* is f(x)+g(x). $(f+g)(x) = f(x)+g(x)$; $(\lambda f)(x) = \lambda f(x)$. (6) V = set of *continuous* functions. Defined on the **closed** interval f: [a,b] → **R**. Notation ([a,b]=V). (7) V = set of *solution vectors* $(x_1,...x_m)$ of a given set of *linear equations*. (8) V = set of all m×n arrays of real **numbers**. With matrix addition A+B and scalar multiplication λA ($\lambda\in\mathbf{R}$; $A,B\in V$). (9) V = the set of all *twice differentiable functions* f(x) such that $D^2f+a(x)Df+b(x)f = 0$.

Deciding if it is a vector space. If you can find a *counter example,* then the description is invalid. For example, consider V = {all polynomials of **degree equal** to 5}. We require $ku\in V$ for all k in K and u in V. *When k*=0, u is not an element of V. **Consider** this: V = {all ordered pairs for *which y = 2x+1*}. Consider (1,3). When k=2, ku = (2,6). But for x=2, (2,5) is the correct ordered pair, **not** (2,6). Consider V = {all functions f which are *continuous* on the interval [0,1] and for which f(0) = 1}. Consider $f_a(x)+f_b(x)=g(x)$ [This is $u+v\in V$], g(0) = 2 (≠1).

Linear Dependence / Independence

Definition: **Given a vector space** V, a set of vectors $S = \{V_1, V_2, ...\}$ is linearly dependent if there *exists* $a_1, a_2, ... \in \mathbf{R}$, not all **zero**, with $a_1v_1+a_2v_2+a_3v_3+...+a_mv_m = 0$. **Conversely**, if $a_1v_1+a_2v_2+...+a_mv_m = 0$, this *implies* $a_1=a_2=...=a_m=0$, then $\{v_1,...,v_m)$ in linearly Independent. Definiton: **Span**: Given a set $S = \{v_1,v_2,...,v_m\}$ in V, the **span** of S is the set L(S) of vectors $x = a_1v_1+a_2v_2+...+a_mv_m$ ∀ *choices* of $a_1...a_m$. Theorem: A **subset** W of a vector space V is a subspace if it *satisfies all 10 axioms* of a vector space. But we only need 2 (technically 3): W is a **subspace** if (1) $w_1,w_2\in W$ implies $w_1+w_2\in W$ (*closed under addition*); (2) $w_1\in W$, $\lambda\in\mathbf{R}$ **implies** $\lambda w_1\in W$; (*closed under scalar multiplication*) (3) W is not **the empty set** or $0 \in W$.

Examples of **subspaces** W of vector spaces V. (1) Take *W=V*. (2) Take W = {0}. (3) V = R[x] — polynomials in x, S = $\{1,x,x^2\}$. W = L(S) = *polynomials of degree* ≤ 2, $a+bx+cx^2$. (4) V = R[x], S = $\{1, 1+x, 1+x+x^2, x-x^2, -x^2\}$. W = L(S) = all *polynomials* $a + b(1+x) + x(1+x+x^2) + d(x-x^2) + e(1-x^2)$. (5) V = R[x], S = $\{1, 1+x, 1+x+x^2\}$. W = L(S). (6) V = R[x], S = $\{1+x, x-x^2, 1-x^2\}$. W = L(S) = all *polynomials of degree* ≤ 2.

Basis

Definition: A set S of vectors is a Basis *for a subspace W* if (1) Vectors of S are **Linearly Independent**; (2) S **spans** W i.e. W = L(S). *Definition*: The number of vectors in S is the **dimension** of W. Example: show that $e_1 = (0, 2, -4)$; $e_2 = (1, -2, -1)$; $e_3 = (1, 0, 3)$ form a **basis** for $\mathbf{R}^3$. Show two things: *Linear Independence* and *Span*.

A: Show **LI**: Assume LD, *so assume* $ae_1+be_2+ce_3=0$ i.e. a(0 2 -4) + b(1 -2 -1) + c(1 0 3) = (0 0 0) i.e. 0a+1b+1c=0; 2a-2b=0; -4a-b+3c=0. Solve for a,b,c. **Matrix** equation is shown in *yellow*. The columns are the vectors e_1,e_2,e_3. By row operations we reduce the array to the *green configuration*. We have an unique solution: 8c=0 which **means** c=0; b+c=0 **implies** b=0, and 2a-2b=0 **implies** a=0. So only possible when a=b=c=0 so *linearly independent*.

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & -2 & 0 \\ -4 & -1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Span: Show any vector (x y z) is a **combination** of e_1 e_2 e_3. Solve $ae_1+be_2+ce_3$ = (x y z) for a,b,c. i.e. b+c=x, 2a-2b = y; -4a-b+3c = z. Or in an **augmented** matrix and manipulated as shown in **purple**. At the end we see 8c = 5x+2y+z; c = $^1/_8$(5x+2y+z); b+c=x, **so** b = x-$^1/_8$(5x+2y+z) = $^1/_8$(3x-2y-2); *and* a = $^1/_8$(3x+2y-z). So any (x y z) is a known **combination** of e_1 e_2 e_3 — they span.

$$\begin{pmatrix} 0 & 1 & 1 & x \\ 2 & -2 & 0 & y \\ -4 & -1 & 3 & z \end{pmatrix}\sim\begin{pmatrix} 2 & -2 & 0 & y \\ 0 & 1 & 1 & x \\ 0 & -5 & 3 & 2y+z \end{pmatrix}\sim\begin{pmatrix} 2 & -2 & 0 & y \\ 0 & 1 & 1 & x \\ 0 & 0 & 8 & 5x+2y+z \end{pmatrix}$$

Theorem: Let S = $\{x_1\ x_2, ..., x_d)$, $x_i \neq 0$. **Then** S is LD iff some x_r is a linear combination of its predecessors. Proof: If $x_r=a_1x_1+a_2x_2+...+a_{r-1}x_{r-1}$, then $a_1x_1+a_2x_2+...a_{r-1}x_{r-1}+(-1)x_r+ax_{r-1}+ax_d = 0$. Not all **coefficients** are zero so here S is LD. Converse: *assume S is LD*. Assume $a_1x_1+...+a_dx_d=0$, not all the a's are zero. Pick as a_r the last non *zero entry* in this list. Hence $a_1x_1+a_2x_2+...+a_rx_r=0$, $a_r \neq 0$. **Divide** by $a_r \neq 0$; $x_r = \frac{-a_1}{a_r}x_1 - \frac{a_2}{a_r}x_2 - ... - \frac{a_{r-1}}{a_r}x_{r-1}$.

Theorem: Let $V_1...V_n$ be a basis for a **vector space** V. Let W = $\{w_1...w_m\}$ be a subset of V. *Assume* W is LI. Then m≤n. If m<n then W can be *extended to a basis*. Proof: Let S = $\{V_1..V_n\}$. S is a *basis* for V. **S spans V.** $w_1 \in V$, w_1 is dependent on the *vectors* in S. w_1 is a linear combination of **vectors** of S. Hence S ∪ $\{w_1\}$ is LD. $\{V_1V_2V_3...V_n,\ w_1\}$ is *dependent*. $\{w_1V_1V_2..V_n\}$ is *Dependent*. So some V_i is **dependent on the previous** vectors. Hence S_i = $\{w_1V_1V_2...V_i...V_n\}$ [We delete V_i] spans the *same vector space* as $\{V_1...V_n\}$ i.e. **V**.

Repeat process. $S_1 \cup \{w_2\}$ is dependent. $\{w_1w_2V_1V_2...V_i...V_n\}$ is Dependent. $\{w_1w_2\}$ is LI. *Remove vector*, first dependent on its predecessors, NOT w_1 or w_2. S_2 = $\{w_1w_1V_1\ V_2... V_j..V_i...V_n\}$. Spans same vector space & is independent. Repeat, eventually get $S_m=\{w_1w_2..w_m\ V_aV_bV_c\}$, a set of n vectors which are LI and *spans same space* as S i.e. **vector space** V.

18th November 1998

Q: Show **whether** u=(1,2,3,4) and v=(4,3,2,1) are LD. A: Assume LD, so au+bv=0 i.e. a(1,2,3,4)+b(4,3,2,1) = 0; a+4b=0; 2a+3b=0; 3a+2b=0; 4a+b=0. **Solve these** for a and b: only can when *a=b=0*, so LI.

Assignment 3

(1) Show that the **subset** S of $\mathbf{R}^2$ of vectors $\begin{pmatrix}x\\y\end{pmatrix}$, where y = 2x, is a *subspace*. A: S is *non-empty*: $\begin{pmatrix}0\\0\end{pmatrix}$ belongs to S. S is *closed under scalar multiplication*: Suppose $\lambda \in \mathbf{R}$ and $\begin{pmatrix}x\\y\end{pmatrix}\in$ S. **Then** y=2x. **Hence** λy=2λx. *Hence* $\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}$ *also belongs to* S. S is *closed under addition*: **Suppose** $\begin{pmatrix}x\\y\end{pmatrix}$ and $\begin{pmatrix}u\\v\end{pmatrix}$ *both belong* to S. Then y=2x and v=2u. Hence y+v = 2(x+u). Hence $\begin{pmatrix}x+u\\y+v\end{pmatrix}\in$ S. Thus $\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}x+u\\y+v\end{pmatrix}\in$ S.

(2) Are the vectors v_1=(1 0 -1 1), v_2=(0 1 -3 2), v_3=(-1 2 -1 0), v_4=(0 4 0 -1) in $\mathbf{R}^4$ LD? If they are, express v_4 as a *linear combination* of v_1 v_2 v_3. A: **Suppose** $Av_1+Bv_2+Cv_3+Dv_4=0$. Then A-C=0; B+2C+4D=0; -A-3B-C=0; A+2B-D=0. **As** $Av_1+Bv_2+Cv_3+Dv_4$ = (A-C, B+2C, -A-3B-C, A+2B-D), thus A = C, 3B = -A-C = -2C, so B = $-{}^2/_3$C. 0 = A+2B = C$-{}^4/_3$C = $-{}^1/_3$C. And B+2C+4D = 0 which is *compatible with the above*. Thus v_1,v_2,v_3,v_4 are dependent and we *<u>have</u>* $Cv_1-{}^2/_3Cv_2+Cv_3-{}^1/_3Cv_4 = 0$ or $3v_1-2v_2+3v_3 = v_4$.

(3) Let T be the **linear transformation** T: $\mathbf{R}^3\rightarrow\mathbf{R}^3$ *given by* T(x,y,z) = (x+y, y-z, 2x+y+z). Write down the matrix A <u>such that</u> Tv=Av for vectors v of $\mathbf{R}^3$. Find **bases** for the kernel and image of T. A: The **transformation** can be expressed as the yellow section. The *kernel* of A: $A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$. See *below:*

$$A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+y\\y-z\\2x+y+z\end{pmatrix}\text{ where } A=\begin{pmatrix}1&1&0\\0&1&-1\\2&1&1\end{pmatrix}$$

$$\begin{matrix}x+y=0\\y-z=0\\2x+y+z=0\end{matrix}\Leftrightarrow\begin{matrix}y=z\\x=-z\\\text{any } z\end{matrix}\Leftrightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}\Leftrightarrow\begin{pmatrix}-z\\z\\z\end{pmatrix}.\text{ Kernel = set of all vectors }\begin{pmatrix}-z\\z\\z\end{pmatrix}.\text{ Basis is }\begin{pmatrix}-1\\1\\1\end{pmatrix}\text{ and dimension = 1.}$$

Image of A: *Dimension (Kernel)+ Dimension (Image*) = 3. Hence Dimension (Image) = 2. Image is spanned by **columns** of A, by vectors (1 0 2), (1 1 1), (0 -1 1). Two of these will form the *bases for the 2-dimensional* image space. A possible basis is (1 0 2), (1 1 1). These are independent. The **3rd column** is dependent of these: (0 -1 1) = -(1 1 1) + (1 0 2).

(4) e_1 e_2 e_3 is the *standard basis of* $\mathbf{R}^3$. Vectors f_1 f_2 f_3 are *defined* by $f_1=e_1$, $f_2=e_1+e_2$; $f_3=e_1+e_2+e_3$. Prove that they are **linearly independent** and span $\mathbf{R}^3$. Express the *vector* v=(1 1 1) as a linear <u>combination</u> of f_1 f_2 f_3. A: e_1 e_2 e_3 are *linearly independent and span* as *<u>standard</u>* basis for $\mathbf{R}^3$. Suppose $Af_1+Bf_2+Cf_3=0$. **Then** $Ae_1+B(e_1+e_2)+C(e_1+e_2+e_3) = 0$. **Hence** $(A+B+C)e_1 + (B+C)e_2 + Ce_3 = 0$. **But** as e_1 e_2 e_3 are independent there is no *relationship between them* so A+B+C=0, B+C=0, C=0. **Hence** A=B=C=0, therefore f_1 f_2 f_3 are *independent*. (ii) The vector v is (1 1 1) or (1 0 0)+(0 1 0)+(0 0 1). <u>Hence</u> v = $e_1+e_2+e_3$ as these are *standard basis*. Suppose $Af_1+Bf_2+Cf_3=v$. Then $Af_1+Bf_2+Cf_3 = e_1+e_2+e_3$, **hence** $(A+B+C)e_1+(B+C)e_2+Ce_3 = e_1+e_2+e_3$. *Therefore* A+B+C=1, B+C=1 and C=1; hence A=B=0 and C=1, **so** $v=f_3$.

Linear Transformation, Kernel, Image

Take 2 *vector spaces:* V, W. A linear transformation T assigns *to each vector* $x \in V$ a vector $Tx \in W$, $T: V \to W$; and **satisfies** $T(x+y)=Tx+Ty$ and $T(\lambda x)=\lambda Tx$. T *sends lines to lines*. T is defined and **defines a matrix A** once we have a basis for V, W. Example: V=$\mathbf{R}^3$, W=$\mathbf{R}^3$. Standard basis of $\mathbf{R}^3$ in **both** cases. Vector in V is given by column $\begin{pmatrix}x\\y\\z\end{pmatrix}$. Take $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+2y+3x\\2x+4y+8z\\3x+6z+15z\end{pmatrix}$. $T: V \to W$. T is the yellow matrix, A.

$$\begin{pmatrix}1&2&3\\2&4&8\\3&6&15\end{pmatrix}$$

<u>Kernel</u> (Null Space) = $\{\underline{\mathbf{x}} \in v$ such that $T(\underline{\mathbf{x}})=0\}$. This is a *subspace* of V (Theorem). Example: find *kernel of T above*. $T(\underline{\mathbf{x}}) = 0$ implies $A\underline{\mathbf{x}} = 0$.

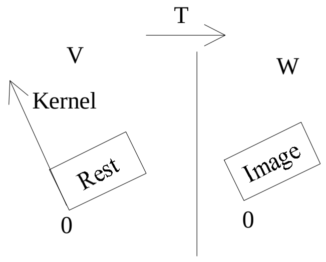
$$\begin{matrix}x+2y+3z=0\\2x+4y+8z=0\\3x+6y+15z=0\end{matrix}\ .\quad \begin{matrix}\text{Row ops}\\\text{on A:}\end{matrix} \sim\begin{pmatrix}1&2&3\\0&0&2\\0&0&6\end{pmatrix}\begin{matrix}\\R_2-2R_1\\R_3-3R_1\end{matrix},\sim\begin{pmatrix}1&2&3\\0&0&2\\0&0&0\end{pmatrix}\begin{matrix}\\\\R_3-3R_2\end{matrix}\quad\Rightarrow\quad\begin{matrix}x+2y+3z=0\\2z=0\end{matrix}\quad\Rightarrow z=0;\ x=-2y\ .$$

The kernel is all **multiples** of $\begin{pmatrix}-2\\1\\0\end{pmatrix}$: a vector *subspace of dimension* 1 and basis $\begin{pmatrix}-2\\1\\0\end{pmatrix}$.

Image Space. *Theorem*: The set of all vectors T(x) is a subspace of W, the image of T. Take the **standard basis** of V: $e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$. A typical *vector* $\begin{pmatrix}x\\y\\z\end{pmatrix}$ is $xe_1+ye_2+ze_3$. Typical *vector in image* is $T\begin{pmatrix}x\\y\\z\end{pmatrix}$ or $A\begin{pmatrix}x\\y\\z\end{pmatrix} = Txe_1+Tye_2+Tze_3$ and so is a *linear combination* of Te_1, Te_2, Te_3. Te_1, Te_2, Te_3 **span** the image, where:

$$Te_1=A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1&2&3\\2&4&8\\3&6&15\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}.\ \begin{matrix}\text{Te}_1\text{ is the first}\\\textit{column}\text{ of A.}\end{matrix}\quad Te_2=A\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix}.\ Te_3=A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}3\\8\\15\end{pmatrix}\ .$$

Theorem: **The columns of A span** the image of T. Image of T is spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\4\\6\end{pmatrix}\begin{pmatrix}3\\8\\15\end{pmatrix}$. But $\begin{pmatrix}2\\4\\6\end{pmatrix} = 2\times\begin{pmatrix}1\\2\\3\end{pmatrix}$. The **image** is spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}3\\8\\15\end{pmatrix}$. These two are *independent*. The image space is of dimension 2 with <u>basis</u> W, $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}3\\8\\15\end{pmatrix}$. The kernel is of dimension 1 with *basis* V, $\begin{pmatrix}-2\\1\\0\end{pmatrix}$. Theorem: *Dimension of Kernel + Dimension of Image = Dimension of V.*



Example: Consider B as shown on the right. It **features** in Tx = Bx, where T = $\mathbf{R}^3$ = V → W = $\mathbf{R}^3$. To get the Kernel, *perform row operations* on B and you get the 2nd array. From it, we see that x+4y+5z=0 and y+z=0; the kernel is $\{z\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ for any z$\}$. The *kernel has basis* $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ and dimension 1. The image is <u>spanned</u> by the columns of B, f_1, f_2 and f_3. Because $f_3=f_1+f_2$, the image is *spanned* by f_1, f_2, forms a basis. **Dimension** of image = 2.

$$B=\begin{pmatrix}1&4&5\\2&5&7\\3&6&9\end{pmatrix}\sim\begin{pmatrix}1&4&5\\0&1&1\\0&0&0\end{pmatrix}$$

The Kernel of a line as a transformation in a subspace

Proof: Given $T: V \to W$, Kernel = {x=V s.t. Tx=0} = S. Check *conditions for a subspace*: (1) S is non empty. T(0) = A0 = 0 so $0 \in S$. (2) S is *closed under multiplication*. Suppose $x \in S$ and $\lambda \in \mathbf{R}$. Then T(x) = Ax = 0 by **supposition**. Hence $T(\lambda x) = A(\lambda x) = \lambda Ax = \lambda.0 = 0$. **Thus** $T(\lambda x) = 0$ so $\lambda x \in S$ by the definition of S. We have *shown that* $x \in S$ implies $\lambda x \in S$. (3) S is closed under **addition**. Suppose $x,y \in S$. Then T(x)=0 and T(y)=0. **Hence** T(x+y) = A(x+y) = Ax+Ay = T(x)+T(y)=0. *Thus* $x+y \in S$ by the **definition** of S [$x,y \in S$ implies $x+y \in S$]. Hence S **is** a subspace.

Theorem: The range of T is a *subspace of W*. Range = Image = **R** = {y∈W s.t. y=T(x) for some x∈V}. (1) **R** *is non-empty*: 0 = T(0) = A0 so 0∈**R**. (2) *Closed under multiplication*. Suppose y∈**R** and λ is a **real** number. Then y = T(x) for some x. **Hence** λT(x) = T(λx) so λy ∈ Range.

25th November 1998

Past Paper, June 1997

Section I: *True or False*. Q: If matrix B is **obtained** from matrix A by elementary row operations, can A be obtained from B by elementary row operations? A: True: just inverse each operation *in the reverse order*. Q: Can a system of **m** linear equations in **n** variables, with n>m, have a unique solution, m,n∈N. A: False. When *m=2 and n=3*, x+y+z+1=0 and x+y+z+2=0 has **no** solution. Q: If none of the vectors ($v_1v_2v_3$), which are subsets of $\mathbf{R}^3$, are multiples of one of the others, is this set **linearly independent**? A: False. Consider V_1=(10,9,8) V_2=(6,6,6) V_3=(16,15,14). None are *multiples* but because $v_3=v_1+v_2$ then the set is LD, not LI.

Q: If **matrices** A & B are s.t. AB=BA, and A is *invertible*, is $A^{-1}B=BA^{-1}$? A: True: AB=BA; $A^{-1}(AB) = A^{-1}(BA)$ [*pre-multiply by A^{-1}*]; IB = $A^{-1}BA$ [*using associativity*]; $IBA^{-1} = A^{-1}BA^{-1}$ [*post-multiply by A^{-1}*]; $IBA^{-1} = A^{-1}BI$ [*I=1*]; $\mathbf{BA^{-1} = A^{-1}B}$. QED. Q: If A is an m×n matrix with rank(A) = m, is the *linear transformation* x → Ax one to one? A: (Rank = Dimension of Image) **False**. Take A=(1,2). Rank of A = 1. But $A({}^{2}{}_{0}) = A({}^{0}{}_{1})$: Not 1-to-1.

Section II: Q: Solve the matrix equation A**x**=B, where the arrays are as shown in *yellow*, either by inverting A directly or by simplifying the *augmented matrix* (A:B) as in the usual case of linear equations. If the columns of B are added together to give b, state, without **detailed** calculation, the solution of Ax=b. A: To invert A directly we must *use new theory...*

$$A=\begin{bmatrix}1&3&8\\2&4&11\\1&2&5\end{bmatrix},\ B=\begin{bmatrix}-3&5\\1&5\\3&4\end{bmatrix},\ x=\begin{matrix}x_1\\x_2\\x_3\end{matrix}$$

<u>MINORS,</u> M_{ij}. M_{ij} = Determinant of an n-1×n-1 matrix formed by **removing** row i and column j. For the matrix A shown, we have *minors* as shown in *green*. Calculate the minors for all values of i and j, and place the minors in a *matrix M*. (purple). **<u>COFACTORS</u>**. From the matrix M, create a matrix C, where the *entries* $C_{ij} = (-1)^{i+j}M_{ji}$ i.e. **Transpose** and *change signs*. Now by **calculation,** you can verify that AC is the *identity matrix* (Not in all cases). If you calculate the <u>Determinant</u> of A, you will get 1. Now in **general**, AC = Det(A)I, so $A^{-1} = {}^{1}/_{\det(A)}\times C$. Summary: Calculate the **minors**, then the **cofactors**, and *divide by the determinant* to get the inverse matrix.

$$M_{11}=\begin{vmatrix}4&11\\2&5\end{vmatrix}=-2.\ M_{12}=\begin{vmatrix}2&11\\1&5\end{vmatrix}=-1.\ M_{21}=\begin{vmatrix}3&8\\2&5\end{vmatrix}=-1.$$

$$M=\begin{pmatrix}-2&-1&0\\-1&-3&-1\\1&-5&-2\end{pmatrix}.\ C=\begin{pmatrix}-2&1&1\\1&-3&5\\0&1&-2\end{pmatrix}$$

Back to the question: You can **solve** Ax=B by calculating x = BA^{-1}. Now suppose we form a *new* vector <u>**b**</u>, which is the sum of the *columns* of B, so <u>**b**</u> = $({}^{2}{}_{67})$. Now solve Ax=<u>**b**</u>. We already have $A\begin{pmatrix}10&-1\\9&10\\-5&-3\end{pmatrix}=\begin{pmatrix}b_1&b_2\end{pmatrix}=\begin{pmatrix}-3&5\\1&5\\3&4\end{pmatrix}$, i.e. $A\begin{pmatrix}10\\9\\-5\end{pmatrix}=b_1;\ A\begin{pmatrix}-1\\10\\-3\end{pmatrix}=b_2$.

So $A\begin{pmatrix}10&+&-1\\9&+&10\\-5&+&-3\end{pmatrix}=b_1+b_2=$ <u>**b**</u>, i.e. $A({}^{9}{}_{19\text{-}8})$ = <u>**b**</u>.

$$P=\tfrac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.\ P^2=\tfrac{1}{4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\tfrac{1}{4}\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}=P$$

Question 3: A **square matrix** P is said to be idempotent if $P^2=P$. Show that $P = \frac{1}{2}[{}^{1}_{-1}\,{}^{-1}_{1}]$ is idempotent. We see that it is from the yellow box. Q: Show that if P is *idempotent and invertible*, then P=I. A: Assuming that P is invertible implies P^{-1} exists, $PP^{-1} = P^{-1}P = I$. We also **know** $P^2=P$ in this case. So $PP=P$; $P^{-1}(PP) = P^{-1}(P)$; $(P^{-1}P)P = P^{-1}P$; $IP = I$; $P = I$. QED. (iii) Show that the only **eigenvalues** of an idempotent matrix are *0 and 1*. A: $P-\lambda I=\frac{1}{2}\begin{pmatrix}1-\lambda & -1\\ -1 & 1-\lambda\end{pmatrix}$. $\mathrm{Det}(P-\lambda I) = \frac{1}{2}((1-\lambda)(1-\lambda)-(-1\times-1)) = \frac{1}{2}((1-\lambda)^2-1) = \frac{1}{2}(1-2\lambda+\lambda^2-1) = \frac{1}{2}(\lambda^2-2\lambda) = \frac{1}{2}(\lambda(\lambda-2))$. So either $\frac{1}{2}\lambda = 0$; implying $\lambda=0$, or $\lambda-2=0$; meaning $\lambda=2$. **Other method**: Given $P^2=P$ and $Px-\lambda x$, $P^2=Px$; $P(Px)=Px$; $P(\lambda x)=\lambda x$; $\lambda\lambda x = \lambda x$; $\lambda^2 x = \lambda x$; $\lambda^2=\lambda$. So $\lambda = 0$ or 1 — these are the only numbers when *squared give themselves*.

$$\begin{pmatrix} u \end{pmatrix}\begin{pmatrix} \quad u^t \quad \end{pmatrix}=\begin{pmatrix} \quad P \quad \end{pmatrix}$$

(iv) If **<u>u</u>** is a non-zero vector, *such that* $u^tu = 1$, show **that** uu^t is idempotent. This question uses *transpose rules*: $A^{tt}=A$; $(AB)^t \neq A^tB^t$; $(AB)^t = B^tA^t$. A: We're given $u^tu = 1$. **Form** $P = uu^t$ as shown in *green*. Now show that $P^2=P$. What is P^2? It is $(uu^t)(uu^t) = (u)(u^tu)(u^t)$; $P^2 = uu^t$. But $P = uu^t$, so $P=P^2$. QED.

<u>Properties of Vector Spaces</u>

Given a vector space V over **F**, then we have a *set V of vectors*, also an addition of vectors +, a *scalar multiplication* ., subject to the axioms. We say that V is closed with respect to + and ., so the sum **v+u** of any two vectors **v**,**u** ∈ V is a vector in V and *similarly the scalar multiple*, λv, of $v\in V$ by a scalar $\lambda\in F$, is a *vector in V*. Subspace: A subspace of V is a subset of V that itself forms a **vector** space, but with the SAME *addition and scalar multiplication* as V and with the same **F**.

The *span of a set S* is the set of all possible linear combinations of S. We say that a space W is <u>spanned</u> by $v_1...v_n$ if ANY vector in W can be expressed as a *linear combination* of these $v_1...v_n$. $v_1...v_n$ are said to be *linearly dependent* iff there exists scalars $k_1...k_n$ not all zero s.t. $k_1v_1+...k_nv_n=0$. If this is **not the case**, the set is linearly *independent* i.e. $k_1=k_2=...=k_n=0$ is the only **possible** solution.

Basis: A basis for a space V is a *spanning set which* is also linearly independent. Dimension: Dimension of a <u>space V</u> is the number of **vectors** in any basis for V.

Linear Transformation: given vector spaces V,W both over **F**, a mapping T: V→W is called a *linear transformation* if (1) $T(v+w)=Tv+Tw$; (2) $T(kv)=kT(v)$ for all $v,w\in W$, $k\in$ **F**. Kernel of T is ker(T). **Rank**: The *dimension of the image space*. The dimension of the kernel space is the nullity of T.

Row Space: This is the subspace of $\mathbf{F}^n$ spanned by the rows of A. Then dim(row space) = row rank of A. **Similarly dim**(column space) = column rank of A. But these are equal, so *Rank A = row rank of A = column rank of A.*

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue of a square matrix* if there exists a non-zero vector x with $Ax=\lambda x$. Definition: x is an *eigenvector corresponding* to an eigenvalue λ, if $Ax=\lambda x$ (any such vector x, *including* x=0, will do). Definition: The collection of all **Eigenvectors** with **Eigenvalues** λ is called an Eigenspace. It is a subspace. **Result**: λ is an eigenvalue $\Leftrightarrow$ $\mathrm{Det}(A-\lambda I) = 0$. If *A is an n×n matrix*, $\mathrm{Det}(A-\lambda I)$ is a *polynomial* in λ of degree n. So at most n roots; at **most** n eigenvalues.

4th December 1998

Linear Transformations

T: $\mathbf{R}^n \to \mathbf{R}^n$ is a function *satisfying* $T(x+y)=Tx+Ty$ **and** $T(\lambda x)=\lambda Tx$. Q: Given $T(A) = A+A^t$, is it a *linear transformation*? A: Because $T(A+B) = (A+B)+(A+B)^t = (A+B)+(A^t+B^t) = (A+A^t)+(B+B^t) = T(A)+T(B)$, **and** $T(\lambda A) = (\lambda A)+(\lambda A)^t = \lambda A+\lambda A^t = \lambda(A+A^t) = \lambda T(A)$, then it is a *linear transformation*. The kernel of T is where $T(A) = 0$.

Assignment 4

(1) Find values of the **real number f** such that the following system of equations has (i) *no* solution; (ii) a *unique* solution; (iii) *infinitely many* solutions. $x_1+x_2-x_3=1$, $2x_1+3x_2+fx_3=3$, $x_1+fx_2+3x_3=2$. A: Put the *equations in matrix* form: A**x**=**b**, where **b**, A and the *Augmented matrix* are as shown below:

$$\boldsymbol{b}=\begin{pmatrix}1\\3\\2\end{pmatrix}, A=\begin{pmatrix}1&1&-1\\2&3&f\\1&f&3\end{pmatrix}. \begin{bmatrix}1&1&-1&1\\2&3&f&3\\1&f&3&2\end{bmatrix} \sim \begin{bmatrix}1&1&-1&1\\0&1&f+2&1\\0&f-1&4&1\end{bmatrix} \begin{matrix}\\R_2-2R_1\\R_3-R_1\end{matrix} \sim \begin{bmatrix}1&1&-1&1\\0&1&f+2&1\\0&0&(3+f)(2-f)&2-f\end{bmatrix} \begin{matrix}\\\\R_3-(f-1)R_2\end{matrix}$$

Applying **row operations** to the Augmented matrix as shown above, we see that there are

$$f=2:\begin{pmatrix}1&1&-1&1\\0&1&4&1\\0&0&0&0\end{pmatrix}; f=-3:\begin{bmatrix}1&1&-1&1\\0&1&-1&1\\0&0&0&5\end{bmatrix}; f\neq 2, f\neq -3:\begin{bmatrix}1&1&-1&1\\0&1&f+2&1\\0&0&3+f&1\end{bmatrix}$$

3 cases. (i) f=2. this **produces** 0=0 in the bottom row, which means that x_3 can take **any** value of k. x_2 is *expressed* as 1-4k and x_1 as 5k. In this situation we have *infinitely many solutions*, with $\underline{\mathbf{x}} = ({}^{0}{}_{10})+k({}^{5}{}_{-41})$. In the 2nd case, f=-3, we have **no solution**, because the bottom row is inconsistent: $0x_3\neq 5$. In the **3rd situation**, $f\neq 2$, $f\neq -3$, we have *unique solutions*. x_3 is ${}^{1}/_{3+f}$, $x_2 = 1-(f+2)x_3 = {}^{1}/_{3+f} = x_3$. **And** $x_1 = 1-x_2+x_3=1$.

(2) Find **eigenvalues** and **eigenvectors** for the matrix A *given* by $A = \begin{pmatrix}-14&12\\-20&17\end{pmatrix}$. A: λ is an eigenvalue of the **matrix** A if $A(xy) = \lambda(xy)$ i.e. if $\begin{pmatrix}-14&12\\-17&17\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}$. So we **require** $-14x+12y=\lambda x$ and $-20x+17y=\lambda y$. So we **solve** $12y=(14+\lambda)x$ and $20x=(17-\lambda)y$. Hence ${}^{12}/_{17-\lambda} = {}^{14+\lambda}/_{20}$ or $240 = 238+3\lambda-\lambda^2$. **Thus** the *eigenvalues* λ satisfy $\lambda^2-3\lambda+2=0$, so $\lambda=1$ **or** $\lambda=2$. *Case* $\lambda=1$: *Eigenvectors* require $-14x+12y=x$, so $12y=15x$. **Eigenvectors** are $k({}^{4}{}_{5})$ for *any* $k\in\mathbf{R}$. **Case** $\lambda=2$: *Eigenvectors* require $-14x+12y=2x$, $12y=16x$, *Eigenvectors* are $k({}^{3}{}_{4})$ for *any* $k\in\mathbf{R}$.

(3) Explain why the *function* T: $\mathbf{R}^2 \to \mathbf{R}^2$ given by T(x,y) = (xy,2y) is **not** a linear transformation but S: $\mathbf{R}^2 \to \mathbf{R}^2$ given by S(x,y) = (x+y,2y) **is** one. A: A linear transformation satisfies T(u+v)=T(u)+T(v) and T(λu)=λT(u). We need an **example** to prove that T(x,y)=(xy,2y) is not a linear transformation. Now T(1,1)=(1,2) and T(3,3)=(9,6) ≠ 3(1,2). So **take** λ=3 with u=(1,1). Then T(λu)≠λT(u) in *this case*. S is a linear transformation. **Take** u(x,y) and v(a,b). *Then* λS(u) = λS(x,y) = λ(x+y,2y) = (λx+λy,2λy) = S(λx,λy) = S(λu). **And** S(u)+S(v) = (x+y,2y)+(a+b,2b) = (x+a+y+b,2(y+b)) = S(x+a, y+b) = S(u+v).

(4) Let the *linear transformation* T be given by T(x,y,z) = (x+3y-4z, 3x+4y-7z, -2x+2y). Write down the **matrix A which** describes this transformation. Consider the three columns of this matrix as three <u>vectors</u> e_1 e_2 e_3. Show that these vectors are *linearly dependent*. Give a basis for the **range** or **image space** of T.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+3y-4z \\ 3x+4y-7z \\ -2x+2y \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 3 & 4 & -7 \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \text{ So } A = \begin{pmatrix} 1 & 3 & -4 \\ 3 & 4 & -7 \\ -2 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ -7 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha+3\beta \\ 3\alpha+4\beta \\ -2\alpha+2\beta \end{pmatrix}$$

A: The transformation T is given by the matrix A as shown on the right. The **columns** of A are e_1=(1,3,-2); e_2=(3,4,2), e_3=(-4,-7,0). If e_3 is dependent on e_1 & e_2 then e_3 = $\alpha e_1+\beta e_2$. So we have what is shown in green. **Hence** -4=α+3β; -7=3α+4β; 0=-2α+2β. **Therefore** α=β and -7=7α <u>so</u> α=β=-1. Check to confirm that $e_3=-e_1-e_2$. Now the **image** of T is spanned by the columns of A, or by e_1 e_2 e_3. As e_3 is **dependent** on e_1 & e_2, the image of T is *spanned* by e_1 & e_2. e_1 & e_2 are *independent*, hence e_1 & e_2 form a basis for the **image** of T.

<u>University of Leeds Past Paper, 1994</u>

(1) (a) Find the **solution set** of the system of equations x-2y+3z+w=3, 2x-4y+7z+2w=5, x-2y+2x+w=1, 2x-6y+9z+3w=6. (b) *Explain which* of the following are <u>subspaces</u> of $\mathbf{R}^3$: (i) S_1={(x,y,z)∈$\mathbf{R}^3$: x-2y+5z=0 and 3x+6y-z=0}; (ii) S_2={(x,y,z)∈$\mathbf{R}^3$: x+y+z=1}; (iii) S_3={(x,y,z)∈$\mathbf{R}^3$: x=0 or y=0}.

$$\begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 2 & -4 & 7 & 2 & 5 \\ 1 & -2 & 2 & 1 & 1 \\ 3 & -6 & 9 & 3 & 6 \end{pmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A: (a) Produce the augmented matrix *shown in yellow* and execute row operations to produce the matrix shown on the **right**. From that, we see that z=1 and x=2y-3z-w+2 i.e. x=2y-w-1. The **solution set** is as follows (Stuff in *brackets* supposed to be in columns not rows): {(x, y, z, w) s.t. z=1, x=2y-w-1} = {(2y-w-1, y, 1, w) **for any** y and w} = {y(2 1 0 0) + w(-1 0 0 1) + (-1 0 1 0) such that y,w are *any real numbers*, y,w∈**R**}. What we have is a *plane through* (-1 0 1 0) and 2 vectors through it.

(b) S_2 is not a *vector space*: (0,0,0) does not **belong** to S_2. Also (1,0,0)+(1,0,0) is not an element of S_2. (iii) (0,1,1)∈S_3. (1,0,1)∈S_3. But (0,1,1)+(1,0,1) is **not** an element of S_3.

(2) For **which values** of k is the set of vectors {(1,1,-1,2), (2,-1,3,4), (-3,3,-7,k)} a linearly independent *subset* of $\mathbf{R}^4$? (b) For which **values** of k is the set of vectors {(1,1,1), (1,2,3), (1,-1,k), (1,-3,k)} a *spanning set* for $\mathbf{R}^3$?

A: Suppose A(1,1,-1,2) + B(2,-1,3,4) + C(-3,3,-7,k) = (0,0,0,0). Then the **following** is true: A+2B-3C=0, A-B+3C=0, -B+3B-7C=0; 2B+4B+kC=0. Solving the *first three equations*, we get what is shown on the left. <u>Solving</u>, B=2C (can be *anything*); A=-2B+3C; A=-C.

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 1 & -1 & 3 & 0 \\ -1 & 3 & -7 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 5 & -10 & 0 \end{pmatrix}$$

If also the **4th** equation is true, then -2C+8C+kC=0 so (6+k)C=0. When k=-6, then any C, A=0C, B=2C *satisfies all 4 equations* i.e. the 3 vectors are **dependent**. When k≠-6 then (6+k)C=0, ***implying*** **C=0. So the only solution is A=B=C=0;** the vectors **are** linearly independent.

(b) *Do the vectors contain a basis of 3 vectors*? We want 3 linearly independent vectors, with v_1=(1,1,1), v_2=(1,2,3), v_3=(1,-1,k) and v_4=(1,-3,k). If $v_3=Av_1+Bv_2$ then A+B=1, A+2B=-1, A+3B=k. **Solving** these, B=-2 and 2B=k-1, *hence* -4=k-1; k=-3. Unless k=-3, v_3 is not a linear **combination** of v_1,v_2 i.e. v_1,v_2,v_3 form a *basis* for $\mathbf{R}^3$ (spans). If k=-3 then the 4 vectors **are** v_1(1,1,1), v_2(1,2,3), v_3(1,-1,-3) [*depends* on v_1,v_2], v_4(1,-3,-4). **Now** are v_1, v_2, v_4 <u>dependent</u>? If $v_4=Av_1+Bv_2$ then A+B=1, A+2B=-3, A+3B=-3. These equations imply that B=-4 and 2B=-4. These are <u>impossible</u> — you cannot have A=1 <u>**and**</u> A=-3. So again in this case v_4 is *independent* of v_1,v_2, so $v_1v_2v_3v_4$ <u>always spanning</u> the set.

Q: Explain **which** of the following sets of vectors are *bases* for $\mathbf{R}^3$. (i) {(1,1,0), (0,1,1), (1,2,1)}. (ii) {(1,1,0), (0,1,1), (1,2,3)}. (iii) {(1,1,0,0), (0,1,1,0), (1,2,3,0)}. (iv) {(1,1,0), (0,1,1), (1,2,3), (3,2,1)}. (v) {(1,1,0), (0,1,1)}.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{pmatrix} \sim\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & x \\ 1 & 1 & 2 & y \\ 0 & 1 & 3 & z \end{bmatrix} \sim\sim \begin{bmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y-x \\ 0 & 0 & 2 & z-(y-x) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 3 & 0 \end{bmatrix} \sim\sim \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

A: (i) (*Yellow*). Here b+c = 0 so b=-c; and a+(-b)=0 so a=b. There is a **nontrivial** solution so it is not a basis. [Also **we can see** straight away from $V_1+V_2=V_3$ that they are **not** a basis] (ii) (*Green*). Here *c=0, b=0, a=0,* so it is Linearly Independent. **Does** it span? i.e. show $ae_1+be_2+ce_3$=(x,y,z). **Expanding** out, a+c=x, a+b+2c=y, b+3c=z. Looking at the *blue* arrays, we see that in the end we get 2x=z-y+x; c=½(x-y+x). And a=x-½(z-y+x), b=y-x-½(z-y+x). Therefore they **are** LI **and** they span and so *form a basis* for $\mathbf{R}^3$. (iii) they are <u>not</u> vectors for $\mathbf{R}^3$ so do not **form** a basis. (iv) (*Purple*). A nontrivial solution exists so not a basis for $\mathbf{R}^3$. (v) *Not a basis* — only 2 vectors.

(3) Using the **matrices** shown, determine whether the following *matrix products* exist, and calculate which <u>do</u> exist. (i) AB. (ii) BA. (iii) AC. (iv) CA. (v) AB^t. (vi) B^tA^t. (vii) B^tC.

$$A=\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 4 & 0 \\ 1 & 2 \end{bmatrix}; B=\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}; C=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

A: (i) Yes (ii) No (iii) No (iv) No (v) Yes (vi) Yes (vii) No. Calculate in the **usual** manner.

(3b) For **each of the following** definitions of a mapping t: $\mathbf{R}^2 \rightarrow \mathbf{R}^3$, determine whether t is a linear *transformation*. (i) t(x,y) = (x,y,x+y+1). (ii) t(x,y) = (x,y,x+y). (iii) t(x,y) = (x,y,xy). (iv) t(x,y) = (0,0,0).

A: (i) Let u=(0,0) and v=(1,1). **Because** T(u)+T(v)≠T(u+v), (0,0,1)+(1,1,3)≠(1,1,3), then t(x,y) is *not a linear transformation*. (ii) Take u(x,y) and v(a,b). λT(u) = λ(x,y,x+y) = (λx, λy, λ(x+y)) = (λx, λy, λx+λy) = T(λx, λy) = T(λu). **Now** T(x) = (a,b,a+b). T(u)+T(v) = (x,y,x+y)+(x,b,a+b) = (x+a,y+b,x+a+y+b) = T(x+a, y+b) = T(u+v). So *here it is a L.T.* (iii) **Take** u(1,1) and v(2,2). Because T(u+v) ≠ T(u)+T(v), (3,3,9) ≠ (1,1,1)+(2,2,4), then it is **not** a L.T. (iv) This **is** a L.T.

(4) Calculate the **inverse** of the matrix A. There are **two** methods for this: (i) By performing *elementary row operations* on a pair of matrices AI to get IA^{-1}; or (ii) To use the *Minors, Cofactors and Determinant* method.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 5 & 11 & 15 \\ 3 & 6 & 10 \end{bmatrix}$$

A: (i)

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 6 & 1 & 0 & 0 \\ 5 & 11 & 15 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right) \sim \begin{bmatrix} 1 & 2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ R_2 - 5R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{20}{2} & -2 & -3 \\ 0 & 1 & 0 & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & 1 \end{pmatrix} R_1 - 2R_2 - 3R_3 .$$

$$\text{So } A^{-1} = \tfrac{1}{2}\begin{bmatrix} 20 & -4 & -6 \\ -5 & 2 & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

(ii) First Calculate the **Minors** and place *them in a table* e.g. $M_{11} = \begin{vmatrix} 11 & 15 \\ 6 & 10 \end{vmatrix} = 20$; $M_{12} = \begin{vmatrix} 5 & 15 \\ 3 & 10 \end{vmatrix} = 5$ etc., and then calculate the Determinant, Det(A) = (2×20)-(4×5)+(6×-3) = 2. Manipulate the **Minors** table (Cofactors) and divide *through* by $^1/_{\det(A)}$, thus producing

$$A^{-1} = \tfrac{1}{2}\begin{pmatrix} 20 & -4 & -6 \\ -5 & 2 & -0 \\ -3 & -0 & 2 \end{pmatrix}. \text{ Check } AA^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 4 & 6 \\ 5 & 11 & 15 \\ 3 & 6 & 10 \end{pmatrix}\begin{bmatrix} 20 & -4 & -6 \\ -5 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exam Paper: January 1999

SECTION 1 (Compulsory)

(1) Given square matrices A and B where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 & 2 \\ 4 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \text{ calculate}$$

(i) A + B
(ii) $A^t + B^t$
(iii) AB
(iv) det(A)
(v) adj(A)
(vi) A^{-1}
(vii) AA^{-1}

Use Gaussian elimination to

(viii) solve Ax = d where d = $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$. Explain why the solution is unique.

[20 marks]

SECTION 2 (Answer 2 out of 4 questions)

(2) Find the general solution of the following equations for all real values of λ and μ, indicating those values for which the equations have (i) a unique solution, (ii) more than one solution, (iii) no solution.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= \mu \\ \lambda x_1 + x_2 + x_3 &= 0 \\ (2\lambda - 1)x_1 + (3-\lambda)x_2 + 3x_3 &= \mu \end{aligned}$$

[15 marks]

(3) (a) If **a** = (-4, 1, -3) and **b** = (2, 5, -1) find a third vector **c** perpendicular to both **a** and **b**, so that {**abc**} form a basis set of right handed axes. Express the vector **d** given by **d** = (4, -7, -12) in terms of this basis. Show that **d**, **c**, and 2**a**-2**b** are coplanar.

[8 marks]

(b) Let the points A, B, C, D have position vectors
(1, -1, 0) (-1, 2, 1) (5, -1, 2) (11, -7, 1) respectively.
Find the equations of the line joining A to C and of the line joining B to D. Do these lines meet? **[7 marks]**

(4) (a) State the conditions under which a non empty subset of a vector space is a subspace. **[2 marks]**

(b) Prove that the subspace U of $\mathbb{R}^4$ of vectors (a, b, c, d) satisfying a = b = c+d is a subspace. State the dimension of U and find a basis. **[6 marks]**

(c) The linear transformation f: $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ is defined by f(x) = Ax where A is the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Find the dimension of the kernel and image of f. Find bases for these subspaces.

[7 marks]

(5) Explain the meaning of the terms, linearly dependent, spanning set and basis. Show that the vectors $\mathbf{x}_1$ = (1, -1, 3, 2); $\mathbf{x}_2$ = (-1, 1, 1, -2) and $\mathbf{x}_3$ = (2, -1, 1, 3) are linearly independent. Show that one of the vectors $\mathbf{x}_4$ = (1, 2, 0, 2) and $\mathbf{x}_5$ = (-1, 0, 6, -1) belongs to the subspace spanned by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and one does not. Express the one which does as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. **[15 marks]**

(Questions done: 1, 3, 5)