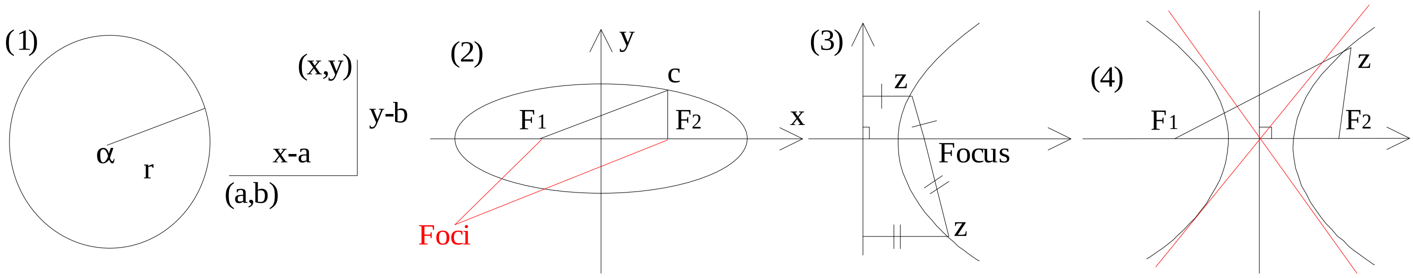
Analysis is a more careful account of calculus. Calculus implies *calculate*, which implies doing sums. Analysis implies more understanding — needed because ideas need to be expressed in other situations (in coming years). Example: **Continuity & Limits** are basic ideas in science & maths. Limits are concerned with *approximation*. Like to say an output of a radio etc., is a CTS. function of the position of its volume knob. But the answer is a signal. What does continuity **mean**?

Famous Curves: Circle, Ellipse, Parabola, Hyperbola



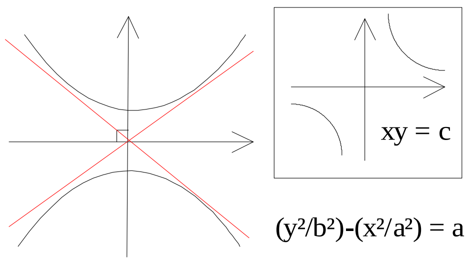
(1) **Circle**. $\alpha = a+ib = a+(\sqrt{-1})b = (a,b) \in$ ‘^{2}. *If* $z = x+iy$ $[(\sqrt{-1}) = (0,1)]$, $z = (x,y) = x(1,0) + y(0,1)$. The **circle** is the set of points z whose distance from a fixed point $\alpha = (a,b)$ is a fixed number r. Looking at the *2nd part of the diagram*, we see = $\{(x,y): (x-a)^2+(y-b)^2 = r^2\}$, or in **complex** number notation, $|z-\alpha| = r$. It is a famous curve, with **circumference** = $2\pi r$; **area** = πr^2 (assuming you understand *arc length and area!).*

(2) **Ellipse**. It is the *set of points* $|z-F_1| + |z-F_2|$ = constant. **Cartesian** Equation: $\{(x,y): {}^{x^2}/_{a^2}+{}^{y^2}/_{b^2} = 1\}$. (3) **Parabola**. The y-axis is the *directrix*. Distance from z to the directrix = distance from z to the focus. Cartesian Equation: $y^2 = 4cx$. (4) **Hyperbola**. Distance from z to F_1 minus the distance from z to F_2 is **constant** (the *modulus* of) i.e. independent of z.

The curves are **famous** for several reasons: (1) *conic sections:* we can create different curves by cutting a conic section in different ways. (2) They occur in **celestial** mechanics i.e. in the orbits of planets. Kepler: planets describe *ellipses* with the sun as the focus. (3) Influential in Newton's law of **gravitation** (the formulation of). This notion is due to the *inverse square law* of planets.

Hyperbola

If you can draw an **ellipse** with a loop of string, how do you draw a hyperbola? Good question! Cartesian equation: ${}^{x^2}/_{a^2}-{}^{y^2}/_{b^2} = 1$. When x is large, then y is large; the **equation** is near to ${}^{x^2}/_{a^2}-{}^{y^2}/_{b^2} = 0$. **Assuming** $a,b>0$, $({}^x/_a-{}^y/_b)({}^x/_a+{}^y/_b) = 0$ i.e. $y = {}^b/_a x$, $y = -{}^b/_a x$. Fact (not proved): take any 2nd degree equation: $by^2+ax^2+2fx+2gy+2hxy+c = 0$ (a,b,... not all 0): this **determines** a conic section (translated).



Warning: we have to include a *pair of lines* as a conic sections (crossing lines). Proof: part of algebra, summarised as completing the square. Question: What do we mean by a **curve**? 1st answer: a set of points satisfying an equation. **Problems**: are $x^2+y^2 = 0$, $0x+0y = 0$ and $0x+0y = 1$ okay?

Parametric Representation

$x = x(t)$, $y = y(t)$, t **parameter**. Think of $(x(t), y(t))$ as giving a *function* from the real line into $\mathbf{R}^2$, c : ‘ → ‘ ^{2}, $t \to (x(t), y(t))$. Usually, **domain** of c = an interval in **R**. c is differentiable or continuous. <u>Continuous</u>: "no sudden gaps". <u>Differentiable</u>: "no corners". It is important to be able to **switch** from one representation of information to another or others.

Examples. (1) *Circle of radius r* centred (0,0). $x = r\cos(t)$, $y = r\sin(t)$ (because $\cos^2 t+\sin^2 t = 1$ for all t). Centred (p,q): $x = p+r\cos(t)$, $y = q+r\sin(t)$. (2) **Ellipse**. ${}^{x^2}/_{a^2}+{}^{y^2}/_{b^2} = 1$ can be described *parametrically* as $x = a\cos(t)$, $y = b\sin(t)$ (implies ${}^{x^2}/_{a^2}+{}^{y^2}/_{b^2} = 1$). (3) **Parabola**: $y^2 = 4ax$. Here, $x = 2at$, $y = at^2$ or $({}^{y^2}/_{4a}, y)$. (4) **Hyperbola**: $x = a\cosh(t)$, $y = b\sinh(t)$. *Formula*: $\cosh^2(t)-\sinh^2(t) = 1$ *implies* ${}^{x^2}/_{a^2}-{}^{y^2}/_{b^2} = 1$. **Maple** can be used to plot these.

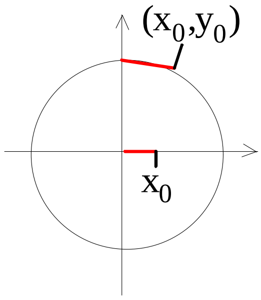
Merits of Parametric against Cartesian

Parametric is more *general* that a function definition $y = f(x)$. We're interested in functions because there is a <u>calculus</u> for functions i.e. differentiate and integrate. The formula $x^2+y^2 = 1$ defined a set of **points** in $\mathbf{R}^2$, namely the set of points (x,y) *such that* $x^2+y^2 = 1$. But this set of points is not the graph of a function, because a set $A \subseteq$ ‘ ^{2} is a graph of a function iff $\forall\, x \in$ ‘ , $\exists$ at most one $y \in$ ‘ s.t. $(x,y) \in A$.

Note: we allow for a given x, one $(x,y) \in A$. Given that, we get a **function** F_A: ‘ → ‘ (it sends $x \in \mathbf{R}$ to a unique $y \in \mathbf{R}$ such that $(x,y) \in A$). There exists a **Domain** A_F in this set of *points* x s.t. there exists y with $(x,y) \in A$. The notation f : ‘ → ‘ is sometimes called a **partial** function — e.g. the domain of f for f: ‘ → ‘, $f(x) = \sqrt{x}$, is $[0,\infty)$. Analysis is about **partial** functions. Not fully recognised in *advanced* literature.

We are able to **differentiate** $x^2+y^2 = 1$ implicitly. What does it mean? (a) If I have a differentiable function y(x) such that $x^2+y(x)^2 = 1$ for all x, then $2x+2y{}^{dy}/_{dx} = 0$. Examples: $y(x) = \sqrt{(1-x^2)}$, $y(x) = -\sqrt{(1-x^2)}$ ($\sqrt{x} \geq 0$). **Note**: $y'(\pm 1)$ does not exist (Dividing by zero). (b) It goes the other way: the *implicit function theorem.*

Example: If (x_0,y_0) satisfies $x_0^2+y_0^2 = 1$, and the **tangent** to the curve is not vertical, then there exists a *differentiable* function $y = y(x)$ defined by the *function theorem* (Finney & Thomas). Very important in <u>higher dimensions</u> Thus, on the whole, <u>parametrised</u> curves are easier to deal with.



Curve Sketching

Maple has an **implicit** plot and a **parameter** plot. *Maple* and *Mathematica* call algorithms conveniently. Need to switch between <u>Cartesian</u> & <u>Parametric</u>. Examples: $x = 1+t^{2/3}$; $x-1 = t^{2/3}$; $(x-1)^3 = t^2$. **And** $y = 2-t^{1/5}$, $y-2 = -t^{1/5}$; $(y-2)^5 = -t$. So **now** $(y-2)^{10} = (x-1)^3$. x(t) and y(t) **satisfies** this equation. But we have <u>lost</u> information by squaring. The equation we got *looks nice*, but the original form is more *helpful*.

Tangents to curves

Basic facts: Suppose x = x(t), y = y(t) are *differentiable*. Then the slope of the tangent at t is $\frac{\dot{y}(t)}{\dot{x}(t)} = \frac{dy/dt}{dx/dt}$. The equation of a line with slope m through the point (x_0,y_0) is $y-y_0 = m(x-x_0)$. **Proof**: It is a line: only x's, y's, constants. It is *satisfied* by (x_0,y_0). It has slope m. So the tangent at t has equation $y - y(t_0) = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}(x - x(t_0))$ or $\dot{x}(t_0)(y - y(t_0)) = \dot{y}(t_0)(x - x(t_0))$. **Example**: $x(t) = t^2+1$; $y(t) = t^3-t^2$. The Tangent at t is $2t(y-t^3-t^2) = 3t^2-2t(x-(t^2+1))$. Then *simplify*!

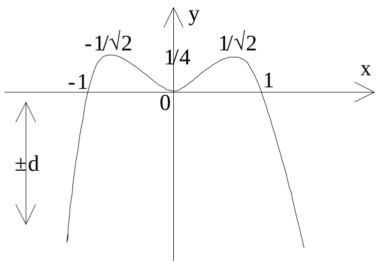
Workshop 1

Strategies for dealing with **new** situations. (a) Start: write down the *question*, the *initial data* and what is *required*. (b) Look in **other** problems and lecture notes for analogous situations. Work **forwards** and **backwards**. Look at the form of the formulae. Try to **transform the problem**. Problem: how to cope with failure!

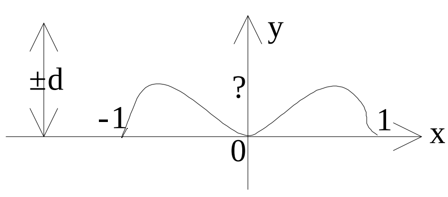
Q: Express the following curves given *parametrically* in Cartesian form. (i) $x = 2\cos(t)-3\sin(t)$ and $y = 2\sin(t)+3\cos(t)$. A: $x^2 = 4\cos^2(t)+9\sin^2(t)-12\cos(t)\sin(t)$ and $y^2 = 4\sin^2 t + 9\cos^2(t) + 12\cos(t)\sin(t)$. **So** $x^2+y^2 = 4(\cos^2(t)+\sin^2(t)) + 9(\sin^2(t)+\cos^2(t)) + 12\cos(t)\sin(t) - 12\cos(t)\sin(t)$; $x^2+y^2 = 4+9$; $x^2+y^2 = 13$. (ii) $x = 4\cos(2t)$ **so** $x^2 = 16\cos^2(2t)$. And $y = 2\sin(2t)$ so $y^2 = 4\sin^2(2t)$. Now $x^2/4+y^2 = 4\cos^2(2t)+4\sin^2(2t)$; $x^2/4+y^2 = 4(\cos^2(2t)+\sin^2(2t))$; $x^2+4y^2 = 16$.

(iii) $x = 1+t^3$ so $(x-1)^5 = t^{15}$. **And** $y = -2-t^5$ so $t^{15} = (-2-y)^3$. Equating, $(x-1)^5 = (-2-y)^3$. (iv) $x = \sec(t)-1$ and $y = \tan(t)$. **As** $1+\tan^2(t) = \sec^2(t)$, *then* $\tan^2(t) = \sec^2(t)-1$. We can say that $(x+1)^2 = \sec^2 t$. $x^2+2x+1 = \sec^2(t)$; $x^2+2x = \sec^2(t)-1$; $x^2+2x = \tan^2(t)$. And because $y^2 = \tan^2(t)$, then $x^2+2x = y^2$; $x(x+2) = y^2$.

Q: Sketch the **family** of functions $z = x^2-x^4+c$. Hence sketch the **family** of curves $y^2 = x^2-x^4+d$ for varying d. Find dy/dx at the point (x,y) on one of the **last** set of curves. A: Define $y = x^2-x^4$ so that $y = x^2(1-x^2)$; $y = x^2(1-x)(1+x)$, **therefore** x = 0, 1 or -1 Find stationary points using $y' = 2x-4x^3 = x(2-4x^2) = 2x(1-2x^2) = 2x(1-(\sqrt{2})x)(1+(\sqrt{2})x)$. **Therefore** when $y' = 0$, $x = 0$ or $1/\sqrt{2}$ or $-1/\sqrt{2}$. Using $y'' = 2-12x$, we can **check** the nature of the **points** and say that there is a MIN at (0,0), a MAX at $(1/\sqrt{2}, ¼)$, and another MAX at $(-1/\sqrt{2}, ¼)$. Therefore we can **complete** the sketch as shown. Note: ±d **shifts** the graph *up or down*.



Now **define** $y^2 = x^2-x^4$; $y = \sqrt{[x^2(1-x^2)]} = x\sqrt{[(1+x)(1-x)]}$. Cannot have *-ve square root*, so we must have $|x| < 1$. Sketch: like the above, but no **graph** for $|x| > 1$ (undefined), graph is "taller" i.e. taking the **square** root of numbers less than 1 increases their magnitude. Find dy/dx at the point (x,y) for $y^2 = x^2-x^4+d$. **Differentiating** $y^2 = x^2-x^4+d$, we get $2y.dy/dx = 2x-4x^3$. $dy/dx = 2x(1-2x^2)/2y$; $dy/dx = x(1-2x^2)/y$.



9th February 1999

Lissajous Figures

Remember, $\sin(t+2\pi) = \sin(t)$. The period is 2π. $y = \sin(2t)$ has a periodic of period π because $\sin(t+2\pi) = \sin(t)$. What is the **curve** $x = \sin(t)$; $y = \sin(t)$? A: $x = y$. Actually a *straight line*. It moves backwards & forwards along $x = y$, between (1,1) and (-1,-1). It is the *"Harmonic Oscillator"*. Next step: $x = \sin(t)$, $y = \sin(2t)$. With $t = 0$, we get (0,0). With $t = {}^{\pi}/_{2}$, we get (1, 0). With $t = \pi$, we get (0,0). With $t = {}^{3\pi}/_{2}$, we get (-1,0). When $t = 2\pi$, we get (0,0). Then **periodic**.

When is the *tangent horizontal*, and what is the **angle** between the tangents at the origin? Tangent is <u>horizontal</u> when $\dot{y} = 0$, where $\dot{y} = {}^{dy}/_{dt}$. But $\dot{y} = 2\cos(2t)$, so $\dot{y} = 0$ when $\cos(2t) = 0$ i.e. $2t = {}^{\pi}/_{2}$, $t = {}^{\pi}/_{4}$; or ${}^{3\pi}/_{2}$, or... If $t = {}^{\pi}/_{4}$, $x = {}^{1}/_{\sqrt{2}}$ and $y=1$. **Tangent** at (0,0): $\dot{x} = \cos(t)$; $\dot{y} = 2\cos(2t)$: $t = 0$, $\dot{x} = 1$, $\dot{y} = 2$. Comes **back** to the origin at $t = \pi$, $\dot{x} = -1$, $\dot{y} = 2$. **Slopes** at the origin are 2 and -2. *Angle* between tangents is $2({}^{\pi}/_{2}-\tan^{-1}(2))$.

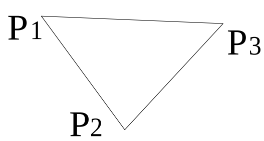
Suppose that $x = \sin(2t)$ (**period** π) and that $y = \sin(3t)$ (**period** ${}^{2\pi}/_{3}$). *Questions*: When does it meet the <u>axes</u>? A: $y(t) = \sin(3t)$. This is equal to zero when $3t = n\pi$, with $n\in$ ‹ i.e. $t = {}^{n\pi}/_{3}$. Meets the x-axis at $t = 0, {}^{\pi}/_{3}, {}^{2\pi}/_{3}, \pi, {}^{4\pi}/_{3}, \ldots$ **Meets** the y-axis at $t = 0, {}^{\pi}/_{2}, \pi, {}^{3\pi}/_{2}, 2\pi, \ldots$ Now we can **calculate** the tangents, where $\dot{y} = 0$. $\dot{y} = 3\cos 3t = 0$ when $3t$ is an odd multiplier of ${}^{\pi}/_{2}$ $[{}^{(2n+1)\pi}/_{2}]$. $t = {}^{(2n+1)\pi}/_{6}$. First **case**: when $t = {}^{\pi}/_{6}$, $x = \sin(2t) = \sin({}^{\pi}/_{3}) = \sqrt{({}^{3}/_{2})}$. Do a lot now by **symmetry**.

Done $\sin(2t)$, $\sin(3t)$. *What about* $\sin(4t)$, $\sin(15t)$, etc.? Done tangent. Next: **normals** to curves. Suppose $x = x(t)$, $y = y(t)$. **Slope** of tangent at t is $\frac{\dot{y}(t)}{\dot{x}(t)}$. So the slope of the *normal* is $-\frac{\dot{x}(t)}{\dot{y}(t)}$. **Equation** of the normal at t is $y-y(t) = -\frac{\dot{x}(t)}{\dot{y}(t)}(x-x(t))$.

Example: $x = \cos(t)$, $y = \sin(t)$. So *the normal* at $\cos(t)$, $\sin(t)$ is $(\cos(t))(y-\sin(t)) = \sin(t)(x-\cos(t))$ i.e. $\cos(t)y = \sin(t)x$ (**cancellation**). *Normal to a circle* passes through the centre. Later — a curve defines its <u>family</u> of tangents. Can you **recover** a curve from its family of tangents? Interesting: *starting with a curve*, we can get a family of normals. Are they <u>tangents</u> to something? (Envelopes). Reminder: A curve may have **no** tangents at a point, e.g. $y = |x|$ (see saw). There exists curves with *no tangents anywhere*, e.g. the **Koch** snowflake — an example of a <u>fractal</u> curve.

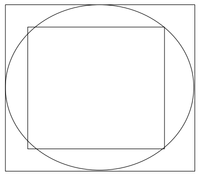
10th February 1999

Arc Length

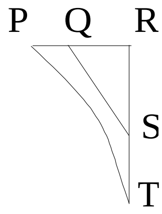


Original problem: length of a line is defined by *Pythagoras' theorem*: distance $(p_1, p_2) = \sqrt{[(x_2-x_1)^2+(y_2-y_1)^2]}$ = length $(p_1p_2) \geq 0$. **Properties**: (1) $\text{dist}(p_1,p_2) = 0$ iff $p_1 = p_2$. (2) $\text{dist}(p_1, p_2) = \text{dist}(p_2, p_1)$ (Symmetry). (3) **Triangle** Inequality: $\text{dist}(p_1, p_3) \leq \text{dist}(p_1, p_2) + \text{dist}(p_2, p_3)$. The **length** of a polygonal line = sum of the lengths of its pieces. Problem of the meaning of length for a **curved** line is of a different order of magnitude, and involves *approximation*.

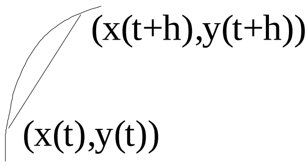
Old idea: perimeter of a circle = 2πr. Archimedes had <u>rigorous</u> arguments of approximation. **Looking** at the diagram, perimeter of inside < perimeter of circle < outside perimeter. **Concepts** in maths: *arc length, rate of change, area, volume, continuity, probability, symmetry,...*



Idea: choose **more** points when approximating polygons. Prove that as you do so, the outside perimeter gets *smaller* and the inside perimeter gets bigger. Start with PR+RT. With another point we have PQ+QS+ST < PQ+QR+RS+ST = PR+RT. Similarly for **inside**, inside perim. (getting bigger with more points) < ? < outside perimeter (getting smaller). Hopefully the two things will get to the same thing — the <u>arc length.</u> **Archimedes** did this brilliantly (Killed in 212BC).



Now, we use **calculus** (if possible). We have a shown curve (x(t), y(t)). **Length** of P(t)P(t+h) = $\sqrt{(x(t+h)-x(t))^2+(y(t+h)-y(t))^2}$. Now explain rather than *prove* (proof = <u>complete</u> explanation). x(t+h)-x(t) = h$\dot{x}$(t) + h(something small). And y(t+h)-y(t) = h$\dot{y}$(t) + h(something small). So the **length** of P(t)P(t+h) should be *approximately* $h\sqrt{\dot{x}(t)^2+\dot{y}(t)^2} + h(\textit{something small})$. When you take h to be **small** i.e. polygons with many sides, expect to get the <u>length</u> $\int_a^b \sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\,dt$.

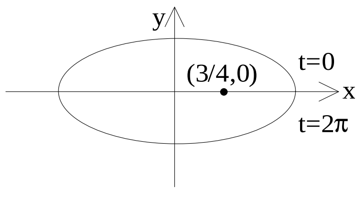


Example: x = cos(t), y = sin(t). So $\dot{x}$ = -sin(t), $\dot{y}$ = cos(t). $\dot{x}^2+\dot{y}^2$ = sin²(t)+cos²(t) = 1, so the arc length is $\int_0^\theta dt = \theta$. **Example**: x = acos(t), y = bsin(t). But we *can't* do $\int_0^\theta \sqrt{a^2\sin^2(t)+b^2\cos^2(t)}\,dt$. This is a starting point for a theory of "<u>elliptic functions</u>". Need **new** types of functions for an "analytic solution" i.e. a formula for e.g. $\int \frac{dt}{\sqrt{1+t^4}}$ *and* $\int \frac{dt}{\sqrt{1-t^3}}$. May have to resort to *numerical* integrals.

Example: x = cos³(t), y = sin³(t), an asteroid. $\dot{x}$ = 3cos²(t)(-sin(t)); $\dot{y}$ = 3sin²(t)(cos(t)). So we have $\dot{x}^2+\dot{y}^2$ = 9(cos4(t)(sin²(t))+sin4(t)(cos²(t)) = 9(cos²(t)sin²(t))(cos²t+sin²t) = 9cos²(t)sin²(t). So $\sqrt{\dot{x}^2+\dot{y}^2}$ = |3cos(t)sin(t)|. Suppose that we consider the range $0 \le t \le \pi/2$, then |3cos(t)sin(t)| = 3cos(t)sin(t) in this interval. **Arc** length in the 1st quadrant is $\int_0^{\pi/2} 3\cos(t)\sin(t)dt = [\frac{3}{2}\sin^2(t)]_0^{\pi/2} = \frac{3}{2}$. So the **total** arc length = 4×$\frac{3}{2}$ = 6. [<u>**Sign**</u>: If you forgot the sign, arc length = $[\frac{3}{2}\sin^2(t)]_0^{2\pi}$ = 0]. The arc **length** is always ≥ 0. *Integration* for area: $\int_a^b f$ gives the <u>signed</u> area while $\int_a^b |f|$ gives the <u>unsigned</u> area.

<u>Workshop 2</u>

Q: Find the **parametric** equations and a parameter interval for the motion of a particle that starts at (a,0) and *traces the circle* x²+y² = a² (a) once clockwise, (b) twice counterclockwise. (a) -t[0,-2π]. (b) t[0,4π]. Q: Find the **point** on the ellipse x = 2cos(t), y = sin(t), 0≤t≤2π, closest to the **points** ($\frac{3}{4}$,0).



A: **Distance** from ($^3/_4$, 0) to a point on the **ellipse** is $\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2]}$ = $\sqrt{[(2\cos(t)-^3/_4)^2+(\sin(t)-0)^2]}$. Distance **squared** = $(2\cos(t)-^3/_4)^2+\sin^2(t) = 4\cos^2(t) - ^{12}/_4\cos(t) + ^9/_{16} + \sin^2(t)$ = ... = $3\cos^2(t)-3\cos(t)+^9/_{16}+1$. Now **differentiating**, we have $-6\cos(t)\sin(t)+3\sin(t)$. *Minimum* points occur when the differential is 0, so set $0 = (\sin(t)(-6\cos(t)+3)$. So **sin**(t) = 0; t = 0, or $-6\cos(t) = -3$; $\cos(t) = ½$, $t = ^\pi/_3$. So a **point** on the curve when $t = ^\pi/_3$ is $(1,^{\sqrt{3}}/_2)$. **Distance** is $\sqrt{[(1-0.75)^2+(^{\sqrt{3}}/_2-0)^2]} = \sqrt{[(^1/_4)^2+(^{\sqrt{3}}/_2)^2]} = \sqrt{[^1/_{16}+^3/_4]} = \sqrt{[^{13}/_{16}]} = ^{\sqrt{13}}/_4$.

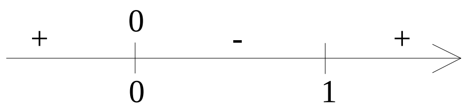
16th February 1999

The **arc length** can sometimes be worked out with known formulae or functions. At the other extreme, there exists curves *with no arc length*. Example: The Koch snowflake. At each stage, the new length is equal to $^4/_3\times$(old length) = $(1+^1/_3)$(old length). At **stage** n, we get a length of $(1+^1/_3)^n$(first length). But $(1+^1/_3)^n > 1+(n\times^1/_3)$, so the length tends to ∞. You can check that the area tends to a finite limit. It is not quite so clear how this **tends** to a continuous curve (x(t), y(t)). *Intuitively*, assuming this is OK, the resulting curve has no **tangent** anywhere — so *differential calculus* does not apply. This is an example of a **fractal** (Mandelbrot).

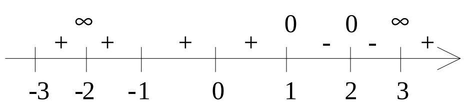
Reminder: $^{dy}/_{dx} = f'(x)$ gives the **slope** of the tangent at (x, f(x)). Such a tangent need not *exist* e.g. y = |x| has no tangent at x = 0. Meaning: No **line** through (0,0) "approximating" the curve. **More** later! If the tangent exists at x = a, then we say that f(x) is *differentiable* at a.

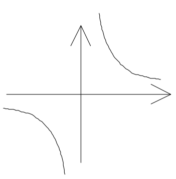
Basic Facts. (1) If f'(x) exists for all x in [a,b], with f'(x) = 0 in this interval, then f is **constant** on [a,b]. (2) Same *assumptions*, but $f'(x) \geq 0$ on the interval, I. Then f is *increasing* on I i.e. $x \leq x'$ and $x,x' \in I \Rightarrow f(x) \leq f(x')$. (3) $f'(x) > 0$ on I. Then f is **strictly** increasing on I i.e. $x < x'$ and $x,x' \in I \Rightarrow f(x) < f(x')$. (4) $f'(x) \leq 0$. **Decreasing**: $f(x) \geq f(x')$. (5) $f'(x) < 0$. Strictly **decreasing**: $f(x) > f(x')$.

Example. $f(x) = x^6-2x^3+½$. Determine the *intervals* on which f is increasing or decreasing. Solution: $f'(x) = 6x^5-6x^2 = 6x^2(x^3-1)$. [Notes: $x^3-a^3 = (x-a)(x^2+ax+a^2)$. $x^3-1 = (x-1)(x^2+x+1)$. $x^n-1 = (x-1)(x^{n-1}+x^{n-2}+...+x+1)$. Proof: RHS: $x^n+x^{n-1}+...+x^2+x$ {from (x)(...)} - $x^{n-1}-...-x^2-x-1$ {from (-1)(...)}. Formula: $1+x+...+x^{n-1} = (x^n-1)/(x-1)$.] Using the **notes**, $6x^2(x^3-1) = 6x^2(x-1)(x^2+x+1)$. Now $(x^2+x+1) = (x+½)^2+¾$ = always +ve. The diagram *shows* the sign analysis of f'(x). Thus f is strictly increasing for $x \geq 1$ and strictly decreasing for $x < 1$, $x > 0$. f(0) = ½, f(1) = -½.



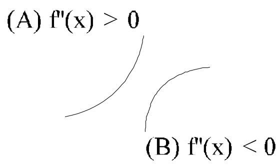
Note: Factorise **before** analysing f'(x) = 0: e.g. for f'(x) = $6x^5-6x^2$, we have a tendency to write f'(x) = 0 when $6x^5-6x^2$ = 0. This is **correct**, but not as **useful**. Example (2). Suppose that $f'(x) = ((x-1)(x-2)^2)/((x-3)^{17}(x+2)^{22})$. What *can* you say? We are interested in the sign of f'. Important: $(x-3)^{17} < 0$ for $x \geq 3$, < 0 for $x < 3$.





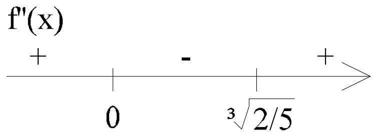
Example (3). $y = ^1/_x$; $^{dy}/_{dx} = -^1/_{x^2}$, which is < 0. Looks like you should **conclude that** $y = ^1/_x$ is strictly decreasing. But looking at the graph, we see that this is *false*. Careful: we write "on an interval I". Problem: $y = ^1/_x$ is not defined for x = 0. We can say that $y = ^1/_x$ is **decreasing** for x > 0, and/or for x < 0.

What does $d^2y/dx^2 = f''(x)$ tell us?



$^{d^2y}/_{dx^2} = {}^{d}/_{dx}({}^{dy}/_{dx})$. (A) On an **interval**, $\frac{d^2y}{dx^2} > 0$ implies that $^{dy}/_{dx}$ is strictly *increasing* on that interval. Described as *"concave up"*. Similarly, for (B), $^{d^2y}/_{dx^2} < 0$ on an **interval** implies that $^{dy}/_{dx}$ is strictly *decreasing* on I. Described as *"concave down"*.

Example: $f'(x) = 6x^5-6x^2$. $f''(x) = 30x^4-12x = 6x(5x^3-2) = 30x(x^3-{}^2/_5) = 30x(x-\sqrt[3]{{}^2/_5})(x^2+ax+a^2)$ [$a = \sqrt[3]{{}^2/_5}$]. See the **diagram** for the sign analysis.



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Parametrically, $^{d^2y}/_{dx^2} = \frac{d}{dt}(\frac{\dot y}{\dot x}) \times \frac{dt}{dx} = \frac{\ddot y\dot x-\dot y\ddot x}{(\dot x)^2} \times \frac{1}{\dot x} = \frac{\dot x\ddot y-\ddot x\dot y}{(\dot x)^3}$. So if $\dot x > 0$, then the **sign** of ($^{d^2y}/_{dx^2}$) is the *sign* of $(\dot x\ddot y - \ddot x\dot y)$.

Curvature of a plane curve

Idea: **curvature** is the rate at which the unit tangent *changes* direction with distance. To see how to calculate the curvature, first **reparameterise** with respect to distance. We are given a curve $x = x(t)$, $y = y(t)$. *Assume that* $\dot y$ and $\dot x$ exist, and that **arc length** exists i.e. $s(t) = \int_{t_0}^{t} \sqrt{\dot x^2 + \dot y^2}\, dt$. Assume that s(t) is strictly *increasing* i.e. there is no interval on which s(t) is constant i.e. no **interval** on which x(t), y(t) is fixed.

Then you can **tell** where you are on a curve by the distance from t_0 i.e. write $t = t(s)$ (distance from t_0). Consider the functions $x = x(t(s))$, $y = y(t(s))$. These are *new* functions "describing" the same curve. We tend to **write** $x(s)$, $y = y(s)$. To calculate curvature, the slope of tangent is $\theta = \tan^{-1}(\dot y/\dot x)$. We **want** $^{d\theta}/_{ds}$. *Now* $^{d\theta}/_{ds} = {}^{d\theta}/_{dt}\times{}^{dt}/_{ds} = \frac{1}{1+(\dot y/\dot x)^2} \times \frac{d}{dt}(\frac{\dot y}{\dot x}) \times \frac{dt}{ds} = \frac{\dot x^2}{\dot x^2+\dot y^2} \times \frac{\ddot y\dot x-\dot y\ddot x}{\dot x^2} \times \frac{1}{(\dot x^2+\dot y^2)^{1/2}} = \frac{\dot x\ddot y-\ddot x\dot y}{(\dot x^2+\dot y^2)^{3/2}}$. But curvature is just the **modulus** of this, so k = *curvature* = $\left|\frac{\dot x\ddot y-\ddot x\dot y}{(\dot x^2+\dot y^2)^{3/2}}\right|$.

Examples. (1) A **line:** $x = at+b$, $y = ct+d$. Here, $\ddot x = \ddot y = 0$ so $k = 0$ (surprise!). (2) Now *consider* a circle: $x = a\cos\theta$, $y = a\sin\theta$. Now $\dot x = -a\sin\theta$, $\dot y = a\cos\theta$; $\ddot x = -a\cos\theta$, $\ddot y = -a\sin\theta$. So $\dot x^2 + \dot y^2 = a^2$; and it follows that $\dot x\ddot y - \ddot x\dot y = +a\sin\theta a\sin\theta-(-a\cos\theta)(a\cos\theta) = a^2\sin^2\theta+a^2\cos^2\theta = a^2$. So the **curvature** is $|a^2/(a^2)^{3/2}| = |{}^1/_a| = {}^1/_{\text{radius}}$.

This suggests the **general definition** for a curve: radius of *curvature* at t = $^1/_{\text{curvature at t}}$. Another *explanation* (no detail!): Circle of curvature at t is the **circle** which touches the curve at t, and has the **radius** of curvature in the right direction. At the point t, work out k(t), or $r = {}^1/_{k(t)}$. Where is the centre? Direction of normal is $-(\dot x/\dot y)$, or vector $(\dot y, -\dot x)$.

So the **centre** is $(x(t), y(t)) + r(t)\frac{(\dot y,-\dot x)}{(\dot y^2+\dot x^2)^{1/2}}$. *But* $r(t) = {}^1/_{k(t)} = \frac{(\dot x^2+\dot y^2)^{3/2}}{|\dot x\ddot y-\ddot x\dot y|}$. So we can *simplify*! One way of **describing** the circle of curvature is with *second* order contact with the curve i.e. second order derivatives agree at that point! No problem in **calculating** k(t) or r(t) as long as you can calculate $\dot x, \ddot x, \dot y, \ddot y$.

Example: $x = 2\cos t$, $y = 3\sin t$. Calculate the max and min of the curvature.

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Workshop

Q: For the **curve** $x = t^3$, $y = {}^{3t^2}/_3$, what is the *arc length* from 0 to $\sqrt{3}$? A: $\dot{x} = 3t^2$, $\dot{y} = 2t$. So we have $\dot{y}^2+\dot{x}^2 = 9t^4+4t^2 = t^2(9t^2+4)$. So the **arc length** is $\int_0^{\sqrt{3}} t(9t^2+4)^{1/2}dt = [k[9t^2+4]^{3/2}]_0^{\sqrt{3}}$, where $k = {}^1/_{27}$. So the arc **length** is ${}^1/_{27}(31-4)^{3/2}$.

Q: Find the **radius** of curvature. A: Radius of curvature = ${}^1/_{\text{curvature at } t} = \left|\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}\right|$. Q: Determine the **intervals** on which a function f is (i) *increasing* or *decreasing*; (ii) concave *up or concave down*. A: For (i), analyse the **sign** change of ${}^{dy}/_{dx}$. For (ii), analyse the **sign** change of ${}^{d^2y}/_{dx^2}$. Q: Find the **radius** of curvature for a curve given in *Cartesian* form. A: In this kind of question, <u>differentiate</u> to find y', y'' (perhaps *implicitly*), and then use the Cartesian form of the **equation**: radius of curvature $=\left|\frac{(1+(dy/dx)^2)^{3/2}}{d^2y/dx^2}\right|$, where ${}^{d^2y}/_{dx^2} = \dot{x}\ddot{y} - \ddot{x}\dot{y}$; $(\dot{y}^2+\dot{x}^2) = (1+({}^{dy}/_{dx})^2)$.

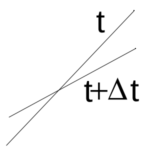
23rd February 1999

Envelopes

Definition: Given a polynomial F in **R[**x,y,t], fix a *real* number t in **R**. Then the variety in $\mathbf{R}^2$, defined by $F(x,y,t) = 0$, is *denoted* by $V(F_t)$, and the family of <u>curves</u> determined by F consists of the **varieties** $V(F_t)$ as t varies over **R**. Definition: Given a *family* $V(F_t)$ of curves in $\mathbf{R}^2$, its *envelope* consists of all points (x,y) in $\mathbf{R}^2$ with the *property* that $F(x,y,t) = 0$, ${}^{\delta}/_{\delta t}F(x,y,t) = 0$ for **some** t in **R**.

Easy example: a family of lines e.g. start with a curve, then the *tangents* form a family of lines. Question: <u>given</u> a family of lines, are they all tangents to something? A: No, only **sometimes**. A curve may be given *implicitly* by $F(x,y) = 0$ or *parametrically* by $x = x(t)$, $y = y(t)$. A **family** of curves means we need another *parameter* e.g. $F(x,y,\lambda)$ for various λ. Or, $x = x(t,\lambda)$; $y = y(t,\lambda)$.

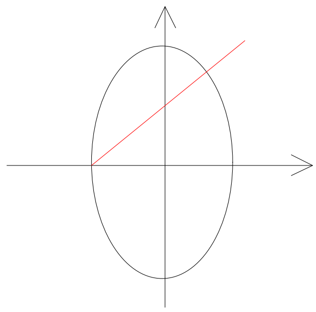
See the **class** handout: a family of curves of *constant radius* whose centres lie on the <u>parabola</u> $y^2 = 4x$. Get an inside curve, a tangent to all circles. Aim: find these curves from the **formula** $(x-t)^2+(y-t^2)^2 = 4$. We give an intuitive idea. Calculations need **Groebner** basis — hard. Idea (diagram): We are given a *family* of curves $F(x,y,t) = 0$. Looking for "**intersections**" of $F(x,y,t) = 0$ and $F(x,y,t+\Delta t)$.



We expect $F(x,y,t+\Delta t) = F(x,y,t) + \Delta t.{}^{\delta f}/_{\delta t}(x,y + ...((\Delta t)^2))$. Up to **approximation**, we expect to have to eliminate t from $F(x,y,t) = 0$ and ${}^{\delta f}/_{\delta t}(x,y,t) = 0$. Define an **envelope** as the curve (in several pieces?) given by $F(x,y,t) = 0$ and ${}^{\delta f}/_{\delta t}(x,y,t) = 0$. Note: ${}^{\delta f}/_{\delta t}$ means differentiate w.r.t. t, **treating** x and y as constants.

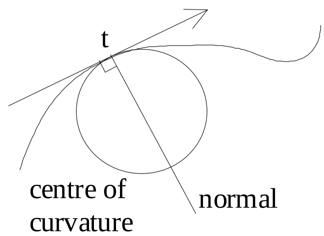
Example. Find the *envelope of the normals* to the ellipse ${}^{x^2}/_4 + {}^{y^2}/_9 = 1$. Method. **Parametric** form is $x = 2\cos t$, $y = 3\sin t$; *implying* $\dot{x} = -2\sin t$, $\dot{y} = 3\cos t$. **Normal** at t is $y - 3\sin t = (-\dot{x}/\dot{y})(x-2\cos t)$; $3\cos t(y-3\sin t) = 2\sin t(x-2\cos t)$. Now let $F(x,y,t) = 2x\sin t-3y\cos t + 5\cos t\sin t = 0$ (1); and **let** ${}^{\delta f}/_{\delta t} = 2x\cos t+3y\sin t+5(\cos^2 t-\sin^2 t) = 0$. The **aim** is to find x and y in terms of t. Do ${}^{(1)}/_{\cos t}$ giving $3y = 2x\tan t+5\sin t$. Now **substitute** in (2): $2x\cos t+(2x\tan t+5\sin t)\sin t + 5(\cos^2 t-\sin^2 t) = 0$, ..., $x = -{}^5/_2\cos^3 t$ and hence $y = {}^5/_3\sin^3 t$.

For normals, find $y - y(t) = -\frac{\dot{x}(t)}{\dot{y}(t)}(x-x(t))$; set it to zero; differentiate; solve **both** equations in terms of a parameter.



So we have **obtained** the curve $x = -\frac{5}{2}\cos^3 t$, $y = \frac{5}{3}\sin^3 t$ $(0 \le t \le 2\pi)$. Or, in **implicit** form, $\cos^2 t + \sin^2 t = (-\frac{2x}{5})^{2/3} + (\frac{3y}{5})^{2/3} = 1$. Maybe, this is less *satisfactory* than the parametric form. Now, what does this **mean**? A formula is not enough — what's the picture? Could plot $x = 2\cos t$, $y = 3\sin t$ (ellipse) and our **obtained** curve on the same picture (as shown). On the diagram, if $0 < t < \frac{\pi}{2}$, p(t) on the ellipse is in the *1st* quadrant, but p(t) on the *envelope* is in the 2nd quadrant.

Notice what we have is the elimination problem — given 2 equations in 3 variables, which are $F(x,y,t) = 0$ and $\frac{\delta f}{\delta t}(x,y,t) = 0$; find (x,y) in **terms** of t, (or, y, t in terms of x, ...). Usually done only **locally**. Compare $x^2+y^2 = 1$ to $y = \sqrt{(1-x^2)}$ or $y = -\sqrt{(1-x^2)}$. Also, in our *example*, we have shown that from $F(x,y,t) = 0$ and $\frac{\delta f}{\delta t}(x,y,t) = 0$, we can **deduce that** $x = -\frac{5}{2}\cos^3 y$ and $y = \frac{5}{3}\sin^3 t$. **Quick** note: 2+3 = 5. In general, what happens with $x = a\cos t$, $y = b\sin t$: do we *deduce* that $c = -\frac{a+b}{a}\cos^3 t$ and $y = \frac{a+b}{b}\sin^3 t$?



Note: on the **envelope** of the normals, the general definition is that the evolute of a curve is the envelope of the *normals* (where that exists). FACT: it is also equal to the curve traced by the centres of curvature. This can be proved. The epicycloid is the path traced by a **point** on a circle rolling outside another circle. It is famous *historically* because it describes the motions of planets — described first as **epicycles**, finally replaced by **Kepler's** laws.

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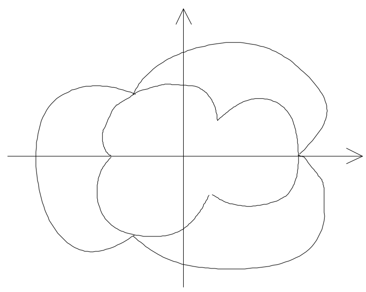


An **evolute** is the locus of the *centres of curvature* (the envelope) of a plane curve's normals. The original curve is then said to be the **involute** of its evolute. The evolute of an epicycloid is as shown. This evolute is actually **another** epicycloid.

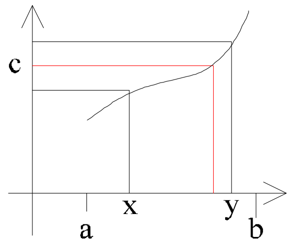
We now need to explain **continuity** & its implications. Abstraction is an important property in maths, and a reason for its *success*. We are able to have one theory describing many situations. Advantages of abstractions: (1) we are able to describe **many** known examples, (2) we can often **simplify** proofs — concentrate on what is needed; (3) the theory is **available** for new examples.

Examples of *applications* of continuity: (1) Signal processing — an output could be a signal; (2) Fractals — approximations e.g. Koch snowflake. Back to simple things: **functions** acting on sets. Let f be a function; A a set. If A is in the *domain* of f, we know what is meant by f(A). Let $f(A) = \{f(x): x \in A\}$. Examples: (First a **reminder** on the intervals of **R**: I, a subset of **R**, is an interval, if for all $x,y \in I$ *such that* $x < y$, if $z \in$ **R** and $x < z < y$, **then** $z \in I$.

A **list** of all intervals of **R:** [a,b] (CLOSED); (a,b], [a,b) (½ OPEN); (a,b) (OPEN); [a, ∞) (CLOSED); (a, ∞) (OPEN); (-∞, a] (CLOSED); (-∞, a) (OPEN); (-∞, ∞) = **R**; {a}; f. These are the **only** intervals on **R**. Not intervals: $[1,2]\cup[3,4]$ or $[1,3]\setminus\{2\}$.

Example: in $y = x^2$, if $a = [1,2]$, then $f(x) = [1,4]$. But if $b = [-1,2]$, then $f(b) = [0,4]$ i.e. do not make the *standard error* that the range is $[-1^2, 2^2]$. Now take $f(x) = {}^{1}/_{x}$. $f([1,2]) = [{}^{1}/_{2}, 1]$. **Note**: in $[a,b]$, with $a < b$, **take** $f([-1,2]) = (-\infty,-1)\cup({}^{1}/_{2}, \infty)$. This is *not* an interval. In determining $f([1,2])$ when $f(x) = x^2$, we have made an *implicit* assumption.

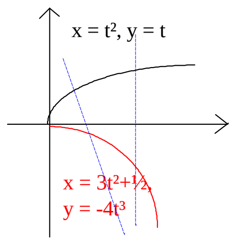
Intermediate Value Theorem



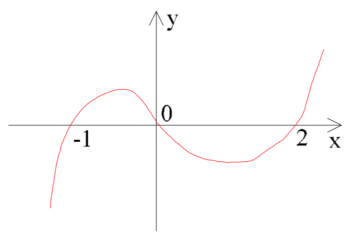
A **continuous** real function defined on an interval takes *any value between* any two given ones. In terms of symbols (more explicit). In the picture, $I = [a,b]$. **Domain** of f is the interval I. If $x,y \in I$, with $f(x) < f(y)$ and $f(x) < c < f(y)$, then *there exists a* $z \in [x,y]$ (assuming $x < y$) such that $f(z) = c$. In other words, assuming $x < y$, then $f([x,y])$ is **also** an interval. Methodological point: the *definition* has gained a **quantifier**.

The assumption that the domain is an **interval** is essential e.g. domain$({}^{1}/_{x})$ = **R**/{0} — **not** an interval. *Simple* example: Let f(x) be 1 for $x > 0$ and -1 for $x < 0$. This a **step** function, $f(x) = {}^{x}/_{|x|}$. $f([-1,2])$ or **any** interval = {-1,1}. An *immediate* application of the IVT is that $\sqrt{2}$ exists, and lies *between 1 and 2*. **Proof**: Look at $f(x) = x^2$: at $y = 4$, we have $x = 2$, **therefore** at y=2, this comes from $x = \sqrt{2}$. $\sqrt{2}$ is defined by an **infinite** process or an **approximation** process.

25th February 1999



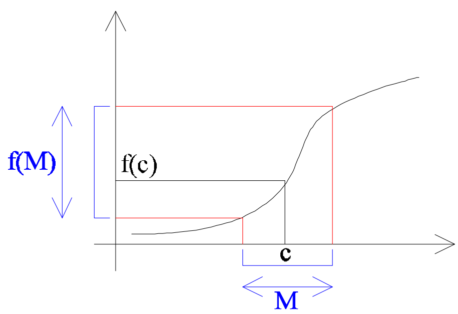
Q: Find the **envelope** of the normals to the parabola (t^2,t). Sketch the *parabola*, the above envelope and some *normals* on the same diagram. A: $x = t^2$ so $\dot{x} = 2x$; and $y = t$ so $\dot{y} = 1$. Now **normal** at t is $y-y_0 = -(\dot{x}/\dot{y})(x-x_0)$; $y-t = -{}^{2t}/_{1}(x-t^2)$; $y-t = -2t(x-t^2)$; $y-t = -2tx+2t^3$; $y-t+2tx-2t^3 = 0 = F(x,y,t)$. **Differentiating**, ${}^{\delta f}/_{\delta t} = -1+2-6t^2 = 0$. From ${}^{\delta f}/_{\delta t}$, we say that $x = 3t^2+{}^{1}/_{2}$. We then *substitute* this into the F(x,y,t) equation to get $y = -4t^3$. **Sketch** is as shown.



Q: Sketch the **function** f given by $f(x) = x(x+1)(x-2)$. Determine the *following* sets: $f([0,2])$, $f((-2,0])$, $f([{}^{1}/_{2}, \infty))$. A: For f(x), find *roots min/max* points as usual; i.e. roots at 0, -1 and 2; min at (1,-2) and max at $(-{}^{2}/_{3}, {}^{16}/_{27})$. Now looking at the **sketch**, we see that $f([0,2]) = [-2,0]$, $f((-2,0]) = (-8, {}^{16}/_{27}]$ and $f([{}^{1}/_{2}, \infty)) = [-2, \infty)$.

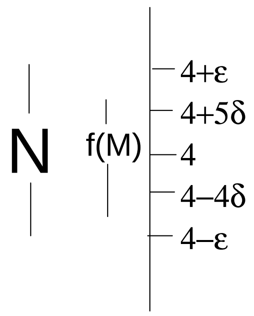
2nd March 1999

Continuity of a real function F. The domain and range of F is a subset of **R**. (Warning! We deal with domains which are *subsets* of **R)**. The idea of continuity is based on limits. F is continuous at C if C is a *subset* of the domain of F, $\lim_{x\to c} f(x) = f(c)$. F is continuous at C if F(C) is approximated by the values of f(x) for x near to c. To make this more **precise**, we'll use F(M).

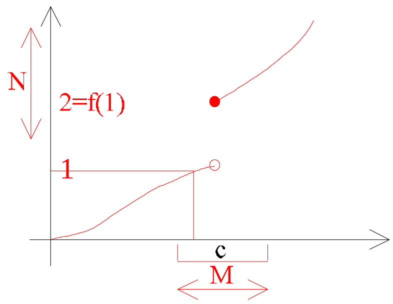


Look at **open** intervals M containing c: a neighbourhood of c. Consider f(M), then f(c) is in f(M). Definition: F is continuous at c if, given any neighbourhood N of f(c), there exists an M in c such that f(M) is a **subset** of N.

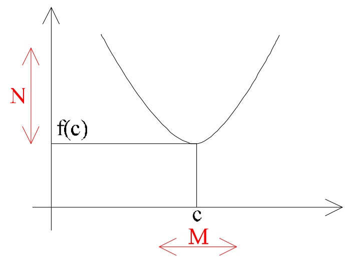
Example: $f(x) = x^2$, $c = 2$. In this case f, is *increasing* near 2 — in some <u>neighbourhood</u> of 2. Locally increasing at 2. Consider $f(2+\delta) = (2+\delta)^2 = 4+4\delta+\delta^2$. If $0<\delta<1$, **this** is $< 4+4\delta+\delta = 4+5\delta$. $f(2-\delta) = 4-4\delta+\delta^2$. This is *greater* than $4-4\delta$ if $\delta>0$. **Thus** for $0<\delta<1$, $4-4\delta < f(2-\delta) < f(2) < f(2+\delta) < 4+5\delta$. Thus if $M = (2-\delta, 2+\delta)$, then f(M) is a subset of $(4-4\delta, 4+5\delta)$. So f(M) is a **subset** of N if $5\delta < \varepsilon$. What we get from this *exercise* is some idea of how to approximate f(2). <u>Requests</u>: Get within n of f(2): then make $\varepsilon = n$ e.g get within $^1/_{10^2}$: make $\varepsilon = {}^1/_{10^2}$.



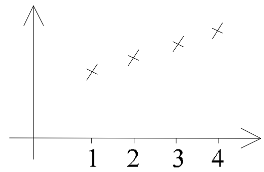
Estimation and *approximation* are crucial. Let us compare f(x) $= x^2+1$ for $x \geq 1$ and $f(x) = x^2$ for $x < 1$. f is not **continuous** at 1. How does this fit in with the definition? Intuitively, f(1) is not *approximated* by f(x) for $x < 1$. More formally, Let $N = (2-{}^1/_2, 2+{}^1/_2)$. Then for all *neighbourhoods* M of 1, f(M) contains <u>points</u> < 1; f(M) is not contained in N. Note: We are **emphasising** the geometry by looking at the *behaviour* of f on intervals — considering f(M). Better to write in terms of **inequalities**.



Emphasis here: suppose M is $(c-\delta, c+\delta)$, $\delta \geq 0$. Then x in M is *equivalent* to $c-\delta < x < c+\delta$. This is *equivalent* to $|x-c| < \delta$, the distance of x from c ($< \delta$). **Similarly**, if $N = (f(c)-\varepsilon, f(c)+\varepsilon)$, then f(x) in N is *equivalent* to $f(c)-\varepsilon < f(x) < f(c)+\varepsilon$, which is **equivalent** to $|f(x)-f(x)| < \varepsilon$. **Logical** point: the definition is of the form $\forall$ N $\in$ f(c), $\exists$ M $\in$ C such that f(M) is a *subset* of N. The negation: $\exists$ N $\in$ f(c) **such that** $\forall$ M $\in$ C, f(M) is <u>not</u> a subset of N. Notes: $\exists$, "*there exists*", and $\forall$, "*for all*", are **quantifiers**. So where a function is <u>not</u> continuous, we need only produce **one** N, then **no** M for C will have f(M) as a **subset** of N.

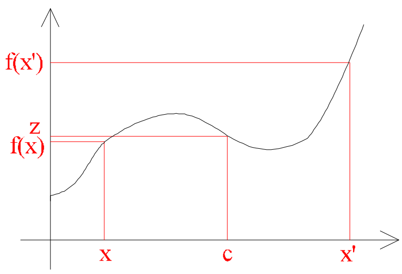


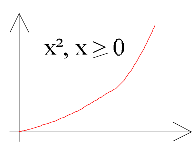
Further points: (1) Say f is continuous *for all c of its domain*. Example: f(x) = log(log(sin(x))). This is <u>defined</u> iff log(sin(x)) > 0 which is iff sin(x) > 1. Graph **of** the function: there is no graph, it is ϕ. But the function is *continuous*, because domain f = ϕ. (2) $f(x) = x+1$, $x \in \{1, 2, 3,4\}$. This is **also** continuous.



Want to say $\sqrt{x}$ is continuous; *domain* is $[0, \infty)$. $F(M) = \{f(x), x \in M$ in domain of f$\}$. **Reminder** of IVT: If f is continuous on an interval I, then f takes all values *between* any 2 given values. $f(x) < z < f(x')$, $x,x' \in I$. This **implies that** there exists a "c" such that $f(c) = z$. We've assumed the above in *saying that the* domain of $\sqrt{x}$ is $[0, \infty)$.

Let $z \in [0, \infty)$. Then $\exists$ a **+ve integer M** such that a $< n^2$. We know $0 < z < n^2$ by IVT. There *exists* a "c" in $[0, \infty)$ such that $f(c) = z$ i.e. $\sqrt{z}$ exists. This *illustrates* that the IVT is subtle — depends a lot on <u>things</u> from **R**. IVT implies that $\sqrt{x}$ *exists* for x>0.





Assignment 1

Q: Find a **parametric** representation in terms of *polynomials* of t for $(x+1)^7 = (y-2)^{11}$. A: Let $x+1 = t^{11}$. Then we can *take* $y-2 = t^7$. So the **parametric** form is $x = -1+t^{11}$, $y = 2+t^7$. Note: an expression using *fractional* powers is **NOT** a polynomial.

Q: **Consider** $x = (2t+5)^{3/2}$, $y = at+{}^{t^2}/_2$. Find a value for a such that $\dot{x}^2+\dot{y}^2$ is a **perfect square** in t. A: Work out that $\dot{x}^2+\dot{y}^2$ is $t^2 + 2(9+a)t + (45+a^2)$. This is a perfect square in t if and only if $({}^{(18+2a)}/_2)^2 = (45+a^2)$ [think of *completing the square*]. So $(9+a)^2 = (45+a^2)$; ...; $a = -2$. From this, we can calculate formulas for **arc length** and **curvature** when a = -2.

```
MyGraph := proc( a, b )
   local c, d, interval;
   c := a; d := b;
   interval := t=0..2*Pi;
   plot([sin(c*t), cos(d*t), interval]);
end;
```

Then e.g.

```
MyGraph(3,1)
```

Q: Write a **Maple** procedure to plot $x = \sin(at)$, $y = \cos(bt)$, $t = 0..2\pi$ for various a and b. How does the *case* a = 3, b = 1 differ from a = 9, b = 3? How does a = 5 differ for **odd** and **even** b? Find equations of the *tangent* and *normal* for one of your examples. A: The *procedure* is as shown in the box. For a = 9, b = 3, we have **exactly** the same motion as for a = 3, b = 1, but we *"go around the loop"* 3 times during the interval for a = 9, b = 3, instead of 1 time for a = 3, b = 1. So we are <u>travelling</u> along the path 3 times as fast as before. The **difference** in the a = 5, b = odd or even cases is the following: the case b = odd gives a *closed curve*, but the case b = even seems to give an <u>open</u> curve. (see the **assignment** for more). To get the *equation* of the tangent we use $y-y_0 = (\dot{x}/\dot{y})(x-x_0)$, and for the **normal** we use $y = y_0 = -(\dot{y}/\dot{x})(x-x_0)$. [for **tangent** use m, for **normal** use $-{}^1/_m$].

Q: Let $f(x) = 2x^7-7x^2+1$. Find the **intervals** in which f is *increasing*, *decreasing*, *concave up*, *concave down*. A: We **analyse** the sign of f'(x) for intervals *decreasing & increasing*, and analyse the **sign** of f''(x) for concavity. See definitions. Note: We need to *prove* why $x^4+x^3+x^2+x+1 = 0$ has no roots i.e. why it is always +ve. **Think** about this.

<u>*3rd March 1999*</u>

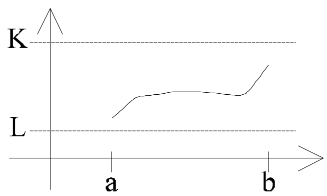
Explanation of the truth of the IVT

We are given a **continuous** function defined on [a,b]. *Suppose that f(a) < f(b) and f(a) < c < f(b).* How do we get a c such that f(c) = z? Where does <u>continuity</u> come in? Ideas to approximate c. **Unsophisticated** method: construct a sequence of points, u_n, v_n, such that $u_n < v_n$, and $f(u_n) \le z$, $f(v_n) \geq z$ as follows:

$u_0 = a$, $v_0 = b$. Consider $f({}^{a+b}/_2)$ — does it lie *above* or *below* z? Given $f(u_n) \le z$, $f(v_n) \geq z$; don't know *about* $(u_n+v_n)/2$. Let $u_{n+1} = u_n$ if $f((u_n+v_n)/2) \geq z$ and $(u_n+v_n)/2$ if $f((u_n+v_n)/2) \le z$. And let $v_{n+1} = v_n$ if $f((u_n+v_n)/2) \le z$; and $(u_n+v_n)/2$ if $f((u_n+v_n)/2) \geq z$. So $f(u_{n+1}) \le z$; $f(v_{n+1}) \geq z$. Also, the dist. **between** u_{n+1} & v_{n+1} is **half** the dist. between u_n & v_n. So u_n & v_n tend to the *same* point, say c. By <u>continuity</u>, $f(u_n) > f(c)$; $f(u_n) \le z \Rightarrow f(c) \le z$. And $f(u_n) \geq z \Rightarrow f(c) \geq z$. **So** f(c) = z.

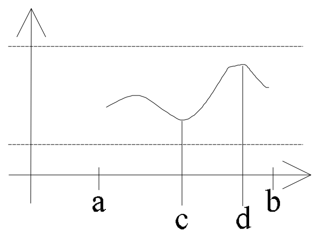
Example: $f(x) = x^5-2x-1$; f(1) = -2, f(2) = 27. f is a cts. polynomial (no *fractional* powers). ${}^{P(t)}/_{Q(t)}$ is a **rational** f^n — also $t^2+{}^1/_t+{}^1/_{t+1}$. Expect a 0 in f **between** 1 and 2 by the IVT. Calculate f(1.5) = +ve, f(1.25) = -ve, *etc*.

Fact #2 (General properties of cts. f^{ns}). A cts. f^n on a **closed** interval $[a,b]$ is bounded on $[a,b]$, i.e. $\exists$ K,L $\in$ **R** s.t. $\forall$ x $\in$ [a,b], L $\leq$ f(x $\leq$ K. Explanation: **suppose** f is unbounded (not bounded above, say) Then given any k $\in$ **R**, $\exists$ x $\in$ [a,b] s.t. f(x) $\geq$ k.

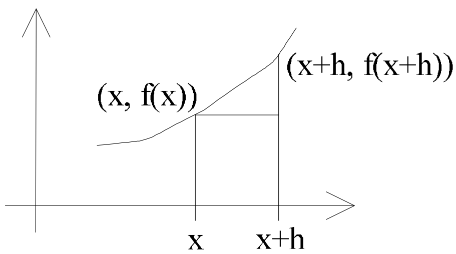


$f(x) = {}^1/_x$ is unbounded on (0,1]. Cannot <u>define</u> f(0) to get a cts f^n at 0. Midpoint $u_1 = {}^{a+b}/_2$. f is unbounded above on at least **one** of $[a, {}^{a+b}/_2]$ and $[{}^{a+b}/_2, b]$. Choose the 1st one, repeat, and get *smaller* intervals, I_n, with I_{n+1} = ½ the <u>length</u> of I_n. I_{n+1} is **contained** in I_n; f is *unbounded* on I_n. These approximate to a <u>point</u> c $\in$ [a,b] so that f(c) is defined. But f cannot be **cts** at c because c is in I_n, and on each of these intervals, f is *unbounded*. This explains Fact 2.

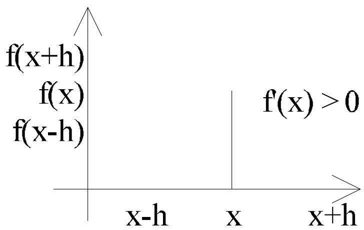
Fact #3. Let f be a cts f^n on a **closed** interval [a,b]. Then f has a max and a min on [a,b] i.e. there *exists* c,d $\in$ [a,b] s.t. $\forall$ x $\in$ [a,b], f(c) $\leq$ f(x) $\leq$ f(d). In summary, "f **attains** its bounds". The proof/explanation uses sup & inf: discussed **later**. This is very useful in explaining facts on differentiation e.g. why f' > 0 on an interval *implies* f is increasing on I.



Need to go back to the **meaning** of f'(c) — "the *slope of the tangent* at x" (if it exists). Consider f(x) = |x|. **There is no** tangent at x = 0. You can't always apply differentiation — this does not tell us that |x| has a **minimum** at x = 0. The basic picture is as shown. We can define a line joining **two** points, but not 1 point. "Zeno": *non-existence of motion*.



The problem of **drawing** a tangent was solved by the notion of a <u>limit</u>, $f'(x) = \lim_{h\to0} {}^{f(x+h)-f(x)}/_h$. (19th century, **Cauchy**). Intuitively, ${}^{f(x+h)-f(x)}/_h$ gets **near** to f'(x) as h gets **near** to 0. Consequently, if f'(x) > 0, then ${}^{f(x+h)-f(x)}/_h > 0$ for *small enough* h. When h > 0 and <u>small</u> enough, f'(x) > 0 **implies** f(x+h) > f(x). When h < 0 and **small** enough, f'(x) > 0 implies f(x+h) < f(x). *Hence* if x gives a <u>minimum</u> at f, then we could have f'(x) > 0. **Similarly we can get** f'(x) < 0, so that f'(x) = 0.



4th March 1999

<u>Worksheet 5</u>

Q: Let f be the **real** function $f(x) = 2-2x+x^2+2x^3$. Find a,b > 0 such that for $0<\delta<1$, $2-a\delta < f(1-\delta) < f(1) < f(1+\delta) < 2+b\delta$. Hence find an r > 0 such that $|x-1| < r \Rightarrow |f(x)-2| < 10^{-6}$. A: f(1) = 2-2+1+2 = 3. First we **need** $f(1+\delta) < 2+b\delta$. Now $f(1+\delta) = 2-2(1+\delta)+(1+\delta)^2+2(1+\delta)^3 = \ldots = 3+6\delta+7\delta^2+2\delta^3$. Now *because* $0<\delta<1$, $\delta^3<\delta^2<\delta$. And so $3+6\delta+7\delta^2+2\delta^3 < 3+6\delta+7\delta+2\delta = 3+15\delta$. Let us now look at $f(1-\delta)$. $f(1-\delta) = \ldots = 3-6\delta+7\delta^2-2\delta^3 > 3-8\delta$ if $0<\delta<1$, since *then* $\delta^2>0$, $-\delta^3>-\delta$. We require r > 0 such that $|x-1| < r \Rightarrow |f(x)-3| < 10^{-6}$. From the above, it's *sufficient* if $15r < 10^{-6}$ i.e. $r < {}^2/_3 10^{-7}$.

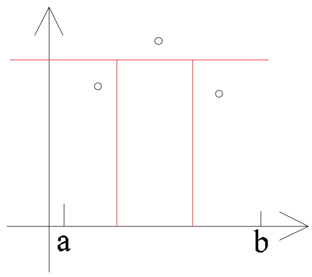
Continuous Functions (continued!). How do we know *functions* such as $^1/_x + \sqrt{(x^2-1)} + \sin(2x-1)$ are cts.? Start with *rules for combining continuous functions*. (1) Any **constant** function is cts. (2) Any **linear** function, $x \to mx$, is cts. (3) If f & g are cts, **then so are** $f+g: x \to f(x)+g(x)$, and $fg: x \to f(x)g(x)$. [$dom(f+g) = dom(f)+dom(g)$]. (4) $^1/_x$ is cts. (5) **Composite** rule: If f & g are cts, then so is $f\circ g: x \to f(g(x))$. [$dom(f\circ g) = \{s \in \mathbf{R}: g(x) \in dom(f)\}$. (6) If f is cts, and $a \in \mathbf{R}$, then $f|a$ is cts: this is *'f cut down' or 'restricted'*. [Domain: $a \cap dom(f)$].

Example: $f(x) = x^2$. f is not one-to-one, so no inverse. *However*, let $g = f|[0, \infty)$. Now g **is** one-to-one and so an inverse function exists, $\sqrt{x}$. By **definition**, $\sqrt{x} \geq 0$. Example: Let $f(x) = \sin(x)$. <u>Not</u> one-to-one. Let $g = f|[-^\pi/_2, ^\pi/_2]$. g <u>is</u> one-to-one, the *inverse* is $\sin^{-1}(x)$ or $\arcsin(x)$. $dom(\arcsin(x)) = [0,1]$, range $= [-^\pi/_2, ^\pi/_2]$. Similarly, cos is **restricted** to $[0,\pi]$ and tan to $[-^\pi/_2, ^\pi/_2]$.

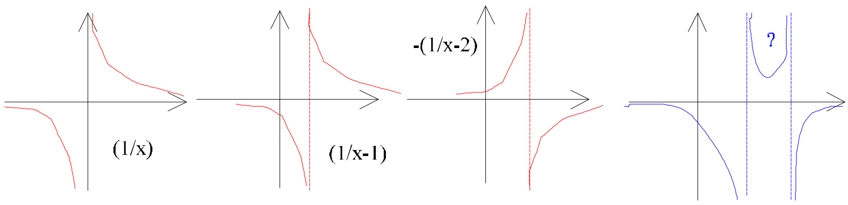
Remark on proofs: when we say f is continuous at a, we mean *intuitively* that the value f(a) can be "estimated" by f(x) for x near to a. To <u>prove</u> that $f\times g$ is cts, basic input is the estimate of $|uv-cd|$ in **comparison** with $|u-c|$, $|v-d|$. Another: $|\alpha-\beta| \leq |\alpha-\chi| + |\chi-\beta|$, or the **triangle** equality: $|\theta+\phi| \leq |\theta| + |\phi|$. Easy proof: **Square** both sides. Now $|uv-cd| = |uv-cv+cv-cd| \leq |uv-cv| + |cv-cd| = |v||u-c| + |c||v-d|$. So can **estimate the** LHS in terms of $|u-c|$, $|v-d|$.

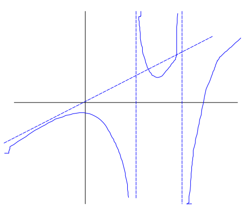
Consider $f(x) = {^1/_x} + \sqrt{(x^2-1)} + \sin(2x-1) + {^1/_{x^2+2}}$. Built up of **basic** functions. We know that the following are continuous: x, 2x, 2x-1, x^2 (by *product* rule), x^2-1, x^2+1, $^1/_{x^2+2}$ (by *composite* rule). **What** about sin(...) and $\sqrt{(...)}$? Rule: sin, cos, log, exp, etc. are *continuous*. Can't prove without definitions. Later will define e.g. as $\int^x \frac{dt}{t}$ and get basic <u>properties</u>. For $\sqrt{(...)}$, this is the *inverse* of $x^2|[0, \infty)$.

Inverse Rule: If f is a cts one-to-one function on an **interval**, then f^{-1} is cts. Note that one-to-one *implies* strictly monotonic. Reasoning: suppose that this is not the case. Then the **IVT** implies that f is not one-to-one. This is how one **shows** $\sqrt{x}$, $\sin^{-1}$, etc. are cts. Of course, the basic method of *building complex things* from basic things is standard.



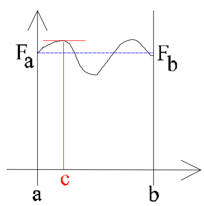
<u>Example</u>. $f(x) = {^1/_{x-1}} - {^1/_{x-2}} + x$. Method: Use a graphical **calculator or Maple** to understand the function. In the diagram, we don't *know* about the region marked with the blue question mark. To find out more, we need, $^d/_{dx}(...) = -{^1/_{(x-1)^2}} + {^1/_{(x-2)^2}} = -{^{(x-2)^2+(x-1)^2}/_{(x-1)^2(x-2)^2}} = {^{2x-3}/_{(...)(...)}}$. Because it has only **one** stationary point, it has to have the form *shown*. Finally, take the graph y = x for comparison, so roughly we have the graph shown on the **left**.





Basic facts on differentiation on an interval: Rolle's Theorem

Suppose that f is **continuous** on [a,b]; *differentiable* on (a,b), and that f(a) = f(b). Then there exists a **c** in (a,b) such that f'(c) = 0. This *looks* obvious. We now want the maths to ensure that this holds. A note that might be useful: Real numbers are specified by **infinite** decimals. (Take care with *infinite* processes!)

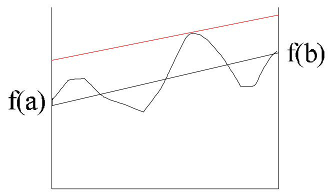


<u>Aside</u>: The "cover-up rule" gives ${}^{3x^2+2}/_{x(x-1)(x+2)} = {}^{(2/-2)}/_{x} + {}^{(5/3)}/_{x-1} + {}^{(14/2)}/_{x+2}$. How do we **get** this? Cover up x, put x = 0 in the rest: *gives* ${}^{2}/_{-2}$. Cover up (x-1), put x = 1 in the **rest**: gives ${}^{5}/_{3}$. Cover up (x-2), put x = 2 in the **rest**: gives ${}^{14}/_{2}$.

<u>10th March 1999</u>

<u>Rolles' Theorem</u>

The **<u>proof</u>** certainly holds true if f is constant. Suppose that f is *not* constant, and suppose that ∃ z ∈ (a,b) s.t. f(z) > f(a). As stated before, ∃ c ∈ (a,b) such that f(c) is a max value. Key point: the maximum is "*obtained*" Then we cannot have f'(c) > 0 or f'(c) < 0. So f'(c) = 0. Generally, we use a <u>consequence</u> of Rolle, called the **<u>Mean Value Theorem</u>** (MVT):



Suppose that (1) f is cts on [a,b]; (2) f is *differentiable* on (a,b); (3) f(a) ≠ f(b). Then ∃ c ∈ (a,b) s.t. $f'(c) = {}^{f(b)-f(a)}/_{b-a}$. This is a '**sheared** version of Rolle, so we deduce it from Rolle. *<u>Proof.</u>* Let φ(x) = f(x)+λ(x-a), where λ is chosen so *that* φ(a) = φ(b) (*). Now φ'(x) = f'(x)+λ. **Rolle** implies that there exists an x in (a,b) such that φ'(c) = 0 i.e f'(c) = -λ. But by (*), f(a)+λ(a-a) = f(b)+λ(b-a), so $-\lambda = {}^{f(b)-f(a)}/_{b-a}$.

General Mathematical Method: reduce a new problem to *one already solved*. But we often need a <u>fresh</u> look. Good example: *partial* differentiation. Consequences of the MVT. [Assumptions: **Suppose** f is cts on [a,b], differentiable on (a,b)]. (1) *Suppose* f'(x) = 0 for all x in (a,b). Then f is constant on [a,b]. **<u>Proof</u>**: <u>Let</u> d ∈ (a,b). By the MVT, there exists a c in (a, d] such **that** $f'(c) = {}^{f(d)-f(a)}/_{d-a}$. But f'(c) = 0, so we must have f(d) = f(a).

(2) **Suppose** f'(x) > 0 for all x ∈ (a,b). Then f is *strictly* increasing on [a,b]. **<u>Proof</u>**: Let a<x<x'<b. By the MVT, there **exists** a c in (x,x') such that $f'(c) = {}^{f(x')-f(x)}/_{x'-x}$. But f'(c) > 0 and x' > x, so that f(x') > f(x). (3) **Suppose** f'(x) < 0 for all x in (a,b). Then f is *strictly decreasing* on [a,b]. **<u>Proof</u>**: Similar to (2), or apply (2) to -f.

(4) Suppose that ∃ m,M ∈ **R** s.t. ∀ x ∈ (a,b), m ≤ f'(x) ≤ M (*). Then m(b-a) ≤ f(b)-f(a) ≤ M(b-a). **<u>Proof</u>**: Use $f'(c) = {}^{f(b)-f(a)}/_{b-a}$. LOGIC: **General** rule (*) applies to the c we *find later*.

<u>General Comments:</u> (1) All of these consequences are about functions differentiable on an **interval**. They do not apply to e.g. ${}^{1}/_{x}$, where ${}^{d}/_{dx}({}^{1}/_{x}) = -{}^{1}/_{x^2} < 0$. (2) They show that the maths **models** what we want. In the above, (1) implies that zero speed <u>implies</u> that *we don't move*. (2) implies that +ve velocity <u>implies</u> that *we move forward*. (3) implies that -ve velocity <u>implies</u> that *we move backward*. (4) implies that if the velocity *lies between* m & M, <u>then</u> the distance travelled in a time t **lies** between mt & Mt.

Some **Applications**. Inequalities. (1) Prove *that* for x > 0, sin(x) < x. <u>Solution</u>: Let f(x) = x-sin(x). Then f(0) = 0. f'(x) = 1-cos(x). f'(x) > 0 except at **isolated** points (we have not *defined* "isolated"). So f is strictly **increasing**. But f(0) = 0, so f(x) > 0 for x > 0. (2) Prove that cos(x) > $1-{}^{x^2}/_{2}$ for x ≠ 0. **Solution**: Let $f(x) = \cos(x)-(1-{}^{x^2}/_{2})$; so f'(x) = -sin(x)+x. By the *previous* example, f'(x) > 0 for x > 0. So f is <u>strictly</u> increasing for x > 0. Hence f(x) > f(0) = 0 for x > 0. Finally, f is **even,** so f(x) > 0 for x < 0. Trick: *Differentiate* until you come to something familiar, then climb back <u>up</u> again!

11th March 1999

Workshop

Q: Let $f(x) = 3x^2-6x$. Find a point c in [0,4] which **satisfies** the MVT for f on this interval. A: $f'(x) = 6x-6$. We require a point c where the **gradient** is 6 (Because $6 = {}^{f(b)-f(a)}/_{b-a} = {}^{24-0}/_4 = 6$). So $6 = 6c-6$; $c = 2$.

Q: Prove that for $0<x<{}^{\pi}/_2$, (i) $\tan(x) > x$, (ii) $\tan(x) > x+{}^{x^3}/_3$. What would you *expect* for a further inequality? A: **Let** $f(x) = \tan(x)-x$. Then $f'(x) = \sec^2-1 \geq 0$, with $f'(x) = 0$ only at *isolated* points. Also, $f(0) = 0$. Since f is strictly <u>increasing</u> on $[0,{}^{\pi}/_2]$, we have $f(x) > 0$ for x in $(0,{}^{\pi}/_2)$. (ii) Let $g(x) = \tan(x) - (x+{}^{x^3}/_3)$. Then $g'(x) = \sec^2(x)-1-x^2$. By (i), $g'(x) > 0$ for $0<x<{}^{\pi}/_2$. So g is strictly *increasing* on $(0,{}^{\pi}/_2)$. And as $g(0) = 0$, we have $\tan(x) > x+{}^{x^3}/_3$. This **suggests** that the third inequality should be $\tan(x) > x+{}^{x^3}/_3+{}^{x\wedge 5}/_5$, and so on.

Q: Let $f(x) = x^3+3x+1$. Prove that f is strictly increasing. Hence show that f has exactly **one** root in [-1,1]. A: $f'(x) = 3x^2+3 > 0$. So f is strictly **increasing**. Now $f(-1) = -2$, $f(1) = 5$. By the IVT, f has a root in [-1,1]. Since f is strictly *increasing*, it only has <u>one</u> root in this interval.

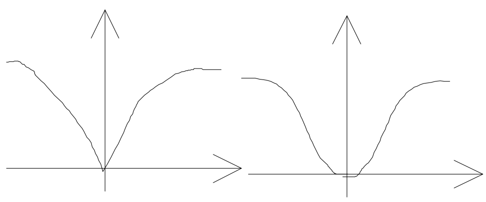
Q: Let $f(x) = {}^4\sqrt{x}$. Prove that f' is strictly *decreasing*. Use a consequence of the MVT on the interval [16, 19] to <u>prove</u> that ${}^1/_{20} < {}^4\sqrt{(19)}-2 < {}^1/_8$. A: $f'(x) = ¼x^{-¾}$ and $f''(x) = -{}^3/_4.{}^1/_4.x^{-1/4} < 0$ for $x > 0$. So $f'(x)$ is **decreasing**. So on [16, 19], $¾.19^{-¾} < {}^4\sqrt{(19)}-2 < {}^3/_{32} < {}^4/_{32} = {}^1/_8$. Also, $¾.19^{-¾} > ½$ iff $15 > 19^{¾}$ iff $15^4 > 19^3$ iff $50625 > 6859$. So the result **follows**.

Q: Let $f(x) = x^6+5x^3+8$. Find the *number* of points x in the interval [-1,1] which satisfy the MVT for the <u>function</u> f on the interval [-1,1]. A: This leads to **solving** $g(x) = 0$ on (-1, 1], where $g(x) = f'(x) = 6x^5+15x^2-5$. But $g(-1) = 4$, $g(1) = 16$. By *differentiation*, g has a <u>minimum</u> at 0 only, $g(0) = -5$. So there exists **2** solutions.

16th March 1999

Main Application: Taylor's Theorem (G1M52). We will be concerned with *approximation*. General problem: given a function in one form, how do you "calculate" it or approximate it by "<u>convenient</u>" means? This depends on the circumstances. Note: We can approximate an expression where the answer is a real number to a specified accuracy, for example fsolv(sin(x)-(xcos(x))). But what about approximating functions? By what do we approximate them?

In Taylor's theorem, we approximate by *polynomials*. Warning: this cannot always be done. Easy example: $\sqrt{(|x|)}$, or $f(x) = e^{-\pi/x^2}$ when $x \neq 0$, and 0 when $x = 0$. Now it is too hard to prove that $f^{(n)}(0) = 0$ for all $n \geq 0$. This function is very flat at 0. People use other *approximations* — Fourier (sums of cos(nx) & sin(nx)) and Puisaux (sums of fractional series).



Taylor's Theorem. Suppose the function f has *derivatives* of all orders on an interval I. Suppose that c is in I. Then when a+x is in I, $f(a+x) = f(a) + f'(a)x + {}^{f''(a)}/_{2!}x^2 + \ldots + (f^{(n)}(a)x^n)/n! + R_n(x)$ [*], where $R_n(x) = (f^{(n+1)}(c)x^{n+1})/(n+1)!$ [**] for some x *between* a and x.

Remarks:

(1) Proof by **differentiating** that the n! arises because $(d^n/dx^n)x^n = n!$. Proof by *induction*: $^d/_{dx}(x) = 1$. But $^d/_{dx}(x^n) = nx^{n-1}$. The result follows by *induction* since $n! = n(n-1)!$.

(2) Looking at [*], we have a **tautology**. The important point is the form of $R_n(x)$ given by [**], which includes the *unknown* number c. In some cases, this allows us to estimate $R_n(x)$. $R_n(x)$ is often called the "*Remainder Term*". There are various estimates of $R_n(x)$.

(3) If you can prove for a *specific* f, a and x that $R_n(x)$ tends to 0 as n tends to ∞, then you are asserting that f(a+x) is getting *better* & *better* approximated by the polynomials $f(a)+f'(a)x+...$ Then you **calculate** only f(a), f'(a), ..., $f^{(n)}(a)$, otherwise work with *polynomials*.

Examples where it is easy to calculate $f^{(n)}$

Suppose that $f(x) = e^x$. Assume we *know* that $f'(x) = e^x$. Then **immediately we get that** $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$. Note: $f^{(0)}(x) = f(x)$ by definition. So the *Taylor* series for e^x about a = 0 is $a + x +(x^2/2!) + (x^3/3!) + ... + (x^n/n!) + ...$ But we still need to prove that $R_n(x)$ tends to 0 as n tends to ∞.

(2) $f(x) = \sin(x)$; $f(0) = 0$. $f'(x) = \cos(x)$, $f'(0) = 1$. $f''(x) = -\sin(x)$, $f''(0) = 0$. $f'''(x) = -\cos(x)$, $f'''(0) = -1$. Then it **repeats** periodically. So the *Taylor* series for sin(x) is $x-(x^3/3!)+(x^5/5!)-(x^7/7!)+...+[(-1)^n x^{(2n+1)}]/(2n+1)!$. The last term is all right for n = 0, hence all right for all n. *Similarly* f(x) = cos(x) gives a Taylor series of $1 - (x^2/2!) + (x^4/4!) - (x^6/6!) + ... + [(-1)^n x^{2n}]/(2n)!$. Again the last *term* is all right for n = 0 (+ve), so all right for all n. (4) $f(x) = e^{-1/x^2}$ when $x \neq 0$; 0 when x = 0. So the Taylor *series* for f(x) is $0 = 0+0x+0x^2+...$ This series gives zero information on f.

Proof of Theorem

Assume that x > 0 so that a < a+x. **Define** $p_n(x) = f(a)+f'(a)x+...+[f^{(n)}(a)x^n]/n!$. *Observation*: $p_n^{(r)}(a) = p^{(n)}(a)$ for 0<n<a. Explanation: $p_n'(x) = f'(a) + f''(a)x + [f'''(a)x^2/2!] + ... + [f^{(n)}(a)x^{n-1}]/(n-1)!$, etc. *Observation 2*: The same holds for $\phi(x) = p_n(x)+kx^{n+1}$ where k is a constant. We need a fixed b, variable x to have a function. **Choose** b later appropriately.

Next Step: choose k so that $f(b) = p_n(b-a)+k(b-a)^{n+1}$. (!) Consider the function $f(x) = f(a+x)-\phi(x)$. (!!) By (!), f(0) = 0, f(b-a) = 0. **Also** $f^{(r)}(0) = 0$. Now apply Rolle *successively*. Since f(0) = 0 and f(b-a) = 0, there exists a c_1 in (0, b-a) such that $f'(c_1) = 0$. Since $f'(0) = 0$ and $f'(c_1) = 0$, there exists a c_2 in $(0,c_1)$ such that $f'(c_2) = 0$.

Continuing in this way, there exists a c_{n+1} in (0,b-a) such that $f^{(n+1)}(c_{n+1}) = 0$. But $f^{(n+1)}(c_{n+1}) = f^{(n+1)}(a+c_{n+1})-k(n+1)!$. Remember we **had** $f(x) = f(a+x)-\phi(x)$ and $\phi(x) = p_n(x)+kx^{n+1}$. Get $k = f^{(n+1)}(a+c_{n+1})/(n+1)!$. Let $c = a+c_{n+1}$. Main **plan** of proof: Like the MVT proof but *done* n or (n+1) times. Form of remainder: $R_n(x) = f^{(n+1)}(c)x^{n+1}/(n+1)!$. Now consider the estimate of $|R_n(x)|$ in *examples*.

The *error* term is $R_n(x)$. Differentiate (n+1) times and apply Rolle n times (with some fiddling) to get $R_n(x) = [f^{(n+1)}(c)x^{n+1}]/(n+1)!$ for some c between a and (a+x). The *dependence* of c on x can be annoying! This is the Cauchy form of the remainder. Later: integral form (easier?).

Example 1: $f(x) = e^x$ about a = 0. Previous *calculations* show that $f^{(n)}(0) = 1$, so the Taylor series is $1+x+{}^{x^2}/_{2!}+{}^{x^3}/_{3!}+...+(x^n/n!)+...$ Also, the **radius** of convergence is *infinite*: converges for all x. Not the **same** as saying it converges **to** e^x. Need to consider $R_n(x) = [f^{(n+1)}(c)x^{n+1}]/(n+1)! = e^c x^{n+1}/(n+1)!$ for $0<c<x$ if $x > 0$. For fixed x, *we* have $e^c < e^x$. But $x^{n+1}/(n+1)!$ tends to 0 as n tends to ∞. So $e^c x^{n+1}/(n+1)!$ tends to 0 as n tends to ∞. So the Taylor series **converges** to e^x. Consequence: e is irrational.

Proof: Suppose that $e = {}^a/_b$ in its lowest terms (a,b natural numbers). Let $\alpha = b!(e-(1+1+{}^1/_{2!}+{}^1/_{3!}+...+{}^1/_{b!}))$. Then $\alpha > 0$ and α is a natural number. (since $e = {}^a/_b$). But $\alpha = {}^1/_{b+1} + {}^1/_{(b+1)(b+2)} + {}^1/_{(b+1)(b+2)(b+3)} + ... < {}^1/_{b+1}+{}^1/_{(b+1)^2}+{}^1/_{(b+1)^3}+....+(1/(b+1)^r)+... = ({}^1/_{b+1})({}^1/_{(1-(1/b+1))})$ = *(geometric series)* $= ({}^1/_{b+1})({}^{b+1}/_{(b+1-1)}) = {}^1/_b < 1$ so a is not a *natural* number. We have a contradiction! With **increasing** levels of difficulty, it can be proved that (1) e is *transcendental* i.e. not algebraic — not the root of a polynomial equation with integer coefficients; (2) π^2 is irrational; (3) π is *transcendental*.

We can calculate errors e.g. how many steps of the **Taylor** series are needed to calculate e^2 to within 10^{-6}? The Error Term is $R_n(2) = e^c x^{n+1}/(n+1)!$ for $0<c<2$; it is $< e^2 2^{n+1}/(n+1)! < 8.2^{n+1}/(n+1)!$. We want $8.2^{n+1}/(n+1)! < 10^{-6}$. This is all right if $2^{n+1}/(n+1)! < 10^{-7}$. Now use a calculator to try n = 6,7,8,...

Example 2: f(x) = sin(x). The **error term** is $|R_n(x)| = |f^{(n+1)}(c)||x|^{n+1}/(n+1)!$. Of course $f^{(n+1)}(c) = \pm\sin(c)$ or $\pm\cos(c)$. In any case, $|f^{(n+1)}(c)| \leq 1$. For fixed x, $|x|^{n+1}/(n+1)!$ tends to 0 as n *tends* to ∞. Taylor series for sin(x) converges to sin(x). Do polynomial approximations to e.g. sin(x) using **Maple** procedures. A polynomial is dominated by the term of **highest** degree. The polynomial approximation will, for fixed n, tend to ±∞ as x tends to ∞. As n increases, the part that is "well approximated" gets *larger*.

Notes: (1) Similarly for cos(x). (2) Because sin(x), cos(x) have **alternating** power series, then errors in approximation can be found from the *alternating series theorem* — concerned with $\Sigma_{n\geq 1} (-1)^n a_n$, where (i) $a_n \geq 0$; (ii) $a_n < a_{n-1}$, (iii) a_n *tends* to 0 as n *tends* to ∞. Then $\Sigma_{n\geq 1} (-1)^n a_n$ converges, and if S_n is the partial sum $\Sigma^n_{i=1}(-1)^i a_i$, then $|S_\infty - S_n| \leq a_{n+1}$.

Example 3: log(1+x) = f(x). f(0) = log(1) = 0. $f'(0) = {}^1/_{1+0} = 1$. $f''(x) = -{}^1/_{(1+x)^2}$, $f''(0) = -1$; ..., $f^{(n)}(x) = [(-1)^{n-1}(n-1)!]/(1+x)^n$. $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. So the **Taylor** series is $x-{}^{x^2}/_2+{}^{x^3}/_3-...+[(-1)^{n-1}x^n]/n$.

General Point: We are assuming that we know about sin(x) and cos(x); and how to *differentiate* them. We then deduce the Taylor series. How do we know about sin and cos? Two expositions have been used in our work: (1) **Power series method**. If $\Sigma_{n\geq 0} a_n x^n$ is a power series with radius of convergence R>0, then we can *define* $f(x) = \Sigma_{n\geq 0} a_n x^n$ for $|x|<R$. This defines a *function*. Theorems on power series show that $f^{(i)}(x)$ exist and $a_n = f^{(n)}(0)/n!$. We still have to *prove that* cos(a+b) = cos(a)cos(b)-sin(a)sin(b), etc.; this involves manipulating Power series.

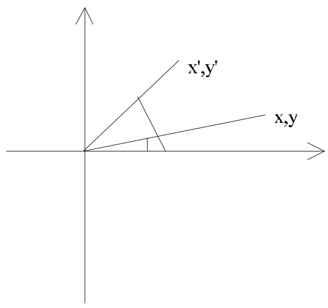
23rd March 1999

We have already looked at the Taylor expansions of e^x, cos(x), sin(x), e^x: the radius of *convergence* is ∞ (converges for all x). The error in taking n terms for sin and cos is found by the Alternating Series Estimation Theorem. Now $\cosh(x) = ½(e^x+e^{-x}) = 1 + (x^2/2!) + (x^4/4!) + ... + (x^{2n}/(2n)!)$. And $\sinh(x) = x + (x^3/3!) + (x^5/5!) + ... + (x^{2n+1}/(2n+1)!)$. It looks like *cos, cosh, sin* and *sinh* are related!

There is a strange **formal** relationship based on the power series: cos(x) = cosh(ix), isin(x) = sinh(ix). This allows the production of *hyperbolic* identities corresponding to the trigonometric identities e.g. cos(x+y) = cosxcosy-sinxsiny leads to cosh(x+y) = coshxcoshy+sinhxsinhy. We can prove this: LHS = $½(e^{x+y}+e^{-(x+y)})$. RHS: $¼\{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})\}$ = ... = LHS.

Question: How do we know that $e^xe^y = e^{x+y}$? We could try $(1+x+(x^2/2!)+...)(1+y+(y^2/2!)+...) = (1+(x+y)+[(x+y)^2/2!]+...)$. The justification requires **complex analysis**. Now sin(x+y) = cosxsiny + sinxcosy, leading to sinh(x+y) = coshxsinhy+sinhxcoshy. This can be proved from the *definitions* in terms of e^x and e^y.

These power series lead to the following **formula:** e^{ix} = cosx+isinx (Real x). **Famous** formula: $e^{i\pi} = -1$. For *complex* numbers, multiplication by $e^{i\alpha}$ corresponds to *rotation* through an angle α *counter clockwise*. (x', y') = ((cosα)x-(sinα)y, (sinα)x+(cosα)y) is represented by the *matrix* $\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$. Now $\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$. *Trigonometric* formulae for cos(α+β), sin(α+β) are **related** to rotations in Euclidean space. Q: Are cosh(α+β) and sinh(α+β) related to **hyperbolic** geometry?



Binomial Series

Recall: if n is a **natural** number (non zero) then $(1+x)^n = 1+nx+{}^nC_2x^2+...+{}^nC_{n-1}x^n$, where ${}^nC_r = \binom{n}{r} = {}^{n!}/_{(n-r)!r!}$. There are various **proofs** of the Binomial Theorem. (1) *Induction*, used with the expression $(1+x)^{n+1} = (1+x)^n(1+x)$. (2) Use $(1+x)^n = (1+x)(1+x)(1+x)...(1+x)$ *(n (1+x)'s).* Coefficient of x^r = number of ways of **choosing** r things from n **things**: ${}^nC_r$.

(3) Apply *Taylor's* Theorem. Let $f(x) = (1+x)^n = a_n + ... + [a^{(r)}(0)x^r]/r! + ...$ **But** $f^{(r)}(x) = n(r-1)(n-1+r).(1+x)^{n-r}$. An amazing *generalisation* (due to Newton) allows the case of when n is not a Natural number. So **look** at $(1+x)^\alpha$. Let $f(x) = (1+x)^\alpha = 1 + ... + [f^{(r)}(0)x^r]/r!$. We get $f^{(r)}(0) = \alpha(\alpha-1)(\alpha-r+1)$.

Now try to extend to new cases. *Can't say* that if α = √2, then $\binom{\sqrt{2}}{r}$ means "the number of ways of **choosing** r things from √2 things" — this is **nonsense**! Compare to $10^{\sqrt{2}}$ i.e. 10 **multiplied** by itself √2 times — also *nonsense*! In fractals, there are "*fractional dimensions*". Koch snowflake assigned dimensions: $\log_e4/\log_e3$ (4 from the number of **pieces**; 3 from the **reduction** factor). It is not hard to prove that the *binomial series* has a radius of convergence of 1 (The ratio test). It is harder to prove that for |x| < 1, the binomial **series** goes to $(1+x)^\alpha$.

24th March 1999

Power Series, Manipulations; Convergence & Taylor Series

Let us recall the **difference** between a *polynomial* and a polynomial *function* e.g. $1+2x+3x^2$ and $x \mapsto 1+2x+3x^2$. LHS: x is a "symbol". RHS: x is a bound variable. **What** are bound variables? Look at the statements "$\forall\ x \in \mathbf{R}$, $x^2 \geq 0$" and "$\forall\ y \in \mathbf{R}$, $y^2 \geq 0$". Logically, these statements are **equivalent**. In a formula, a bound variable such as x can be replaced by any other bound variable not *occurring* elsewhere.

Polynomials: $1+2x+3x^2$ is not the same as $1+y+3y^2$. x and y are often called "variables". Key point — we can *recognise* x whenever it occurs. All these logical points have been crucial to the development of maths; especially for computer implementation. Important aspect: we know how to **add** and **multiply**.

We *assume* that $x^m x^n = x^{m+n}$ (m,n are natural numbers). $x^0 = 1$ (which is real). Then we add polynomials by adding **coefficients**: $(1+2x+3x^2) + (1-3x-x^2) = 2-x+2x^2$. We multiply by using distributivity: $(a+b)c = ac+bc$. We emphasise the *general* laws in order to make analogies. There are some analogies between addition & multiplication in commutativity e.g. $a+b = b+a$, $ab = ba$. This leads to **abstract** algebra — looking at general laws to exploit analogies. *We are surprised* in maths when the same idea pops up in different situations e.g. $e^{i\pi} = -1$. The π comes from **circles**; but is also important in **population statistics**. Polynomials with *coefficients* in **R** or **Z** or **C** or **Q** or... form a *Ring*.

When we write a polynomial as e.g. $1+2x+3x^3$, we ask: what is x? In effect, it is a place marker. The rule $x^m x^n = x^{m+n}$ is convenient for specifying *multiplication*. The ring of all polynomials over **R** is written **R**[x]. In this setup, there is no problem in having an infinite number of powers of x. Formal **power** series over **R** in x is $\sum_{n=0}^{\infty} a_n x^n$. ($a_n$ Real).

This corresponds to or is a nice *notation* for an infinite sequence $(a_0, a_1, a_2, a_3, \ldots)$. x is a notation for (0,1,0,0,0,...) But we **define** addition by $\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n+b_n)x^n$. Multiplication is given by the definition $x^n = (0,0,\ldots,0,1,0,\ldots)$ (the 1 in the n^{th} place). This is consistent with the *definition* $x^m x^n = x^{m+n}$. Then multiply by $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$. The n in the *first* bracket is bound to the *first* bracket The n in the second bracket can be **replaced** by e.g. m. So = $\sum_{n=0}^{\infty}(\sum_{p+q=n} a_p b_q)x^n$. Example: $a_0b_0 + (a_1b_0+a_0b_1)x + (a_2b_0+a_1b_1+a_0b_2)x^2 + \ldots$

The formal power series over **R** in x forms a ring **R**[[x]]. All the *rules* of a ring apply. Also, we can differentiate and integrate formal power series: define $Dx^n = nx^{n-1}$ ($n \geq 1$); $Dx^0 = 0$. So $\int x^n = (x^{n+1})/(n+1)$. This *extends* to give $D(\sum_{n=0}^{\infty} a_n x^n) = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $\int \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n x^{n+1})/(n+1)$. So far — no convergence questions! Note: even if we write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we cannot (yet) write f(a) for Real a. Except for $f(0) = a_0$, all we can say is that $a_n = f^{(n)}(0)/n!$, and that $f^{(n)}(x)$ is a **formal** power series $= D^n f(n)$, so $f^{(n)}(0)$ is *defined*.

Can we take $f(x)^{-1}$? Claim: $(\sum_{n=0}^{\infty} a_n x^n)^{-1}$ exists iff $a_0 \neq 0$. **Explanation:** looking for $f(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x)g(x) = 1 = (1+0x+0x^2+\ldots)$.

When you look at the **coefficients** on the LHS, and equate with the RHS, you get $a_0b_0 = 1$. So $b_0 = a_0^{-1}$. This is all right as $a_0 \neq 0$. $a_0b_1+a_1b_0 = 0$; $b_1 = -a_1b_0/a_0 = -a_1(b_0)^2$. Now $a_0b_2+a_1b_1+a_2b_0 = 0$. But $a_1b_1+a_2b_0$ is already known, so we *can get* b_2 from previous information. In general, if $b_0, b_1, \ldots, b_{n-1}$ are known, **then** $a_nb_0+a_{n-1}b_1+\ldots+a_1b_{n-1}+a_0b_n = 0$ *determines* b_n.

Formal **power** series define functions from x to f(x). *Defined* for Real x such that $\sum_{n=0}^{\infty} a_nx^n$ converges and is then that limit. Now $\sum_{n=0}^{\infty} a_nx^n$ has a radius of *convergence* R where $0 \le R \le \infty$. R = 0 means convergence *only* for x = 0. R = ∞ means convergence for all x. $0<R<\infty$ means convergence for $|x| < R$ and **possible convergence** for x = R or x = -R.

Example: $\Sigma_{n\ge1} (x^n/n)$ has **radius** of convergence 1. Also converges for x = -1, but not x = 1. The function has domain [-1,1). **Emphasis**: if f(x) is defined by a power series of +ve radius of convergence, then the Taylor series T(x) of f(x) is the power series and *converges* to f for all $|x|<R$. Could define $(1+x)^\alpha$ for Real α by its *Taylor* series. Is this a **good** thing?

25th March 1999

Tutorial

Q: Find the **Taylor** approximation to degree 9 (with *remainder* of degree 10) for f(x) = 3cosx+4sinx, by calculating the derivatives $f^{(n)}(0)$. Write down an *expression* for the remainder term. A: f(x) = 3cosx+4sinx, f(0) = 3. f'(x) = -3sinx+4cosx; f'(0) = 4. f''(x) = -3cos-4sinx; f''(0) = -3. f'''(x) = 3sinx-4cosx, f'''(0) = -4. f(4)(x) = 3cosx+4sinx. It then **repeat**s.

So the *Taylor* series of degree 9 is $3 + 4x - (3x^2/2!) - (4x^3/3!) + (3x^4/4!) + (4x^5/5!) - (3x^6/6!) - (4x^7/7!) + (3x^8/8!) + (4x^9/9!)$. The *remainder term is* $R_9(x) = [f^{(10)}(c)/(10)!]x^{10} = [(-3\cos c-4\sin c)/10!]x^{10}$.

Q: Let f be as above. Write f(x) in the **form** $r\cos(x+\alpha)$. Hence give a form for the remainder in *Taylor's* theorem, from which you can show that the error in using this Taylor approximation at x = 1 is *less* than 2×10^{-6}. A: Now $r\cos(x+\alpha) = r(\cos x\cos\alpha-\sin x\sin\alpha)$. So *comparing* with f(x) = 3cosx+4sinx; $r\cos\alpha = 3$, $r\sin\alpha = -4$. Squaring, $r^2(1) = 9+16$, so $r = 5$. Hence $f(x) = 5(^3/_5\cos x+^4/_5\sin x) = 5(\cos(x+\alpha))$ where $\tan\alpha = -^4/_3$. Hence $R_9(x) = [-5\cos(c+\alpha)/10!]x^{10}$. We require $|R_9(1)| < 10^{-6}$ i.e. $^5/_{10!} < 10^{-6}$. But $^5/_{10!} = 0.13779\times10^{-5} < 0.2\times10^{-6}$.

Q: Write down the **Alternating Series Estimation** Theorem, but including all the conditions of validity of the Theorem given in the *previous* Theorem 8. Applying this to the Taylor series for cosx, what is the smallest n that this theorem gives so that the **series** for x = 1 is accurate to within 10^{-6}?

A: $\cos(1) = 1-^1/_{2!}+^1/_{4!}-^1/_{6!}+^1/_{8!}-^1/_{10!}+\ldots$ The alternating series theorem tells us that the *error* in taking the sum to $\pm^1/_{(2n)!}$ is less than $^1/_{(2n+1)!}$. But $^1/_{8!} = 0.000025$; $^1/_{10!} = 0.276\times10^{-6}$. So $1-^1/_{2!}+^1/_{4!}-^1/_{6!}+^1/_{8!}$ will **do**.

20th April 1999

Q: What do we mean by x^a when *x and a are Real?* A nice answer is obtained by using **Riemann** Integration. The old problem was determining the area under a curve. For this, we start from knowing the area of a rectangle. We then deduce the areas of **parallelograms** (base×height), **triangles** (½×base×height) and hence **polygons** by cutting them up into triangles. The area of a circle was tackled by Archimedes by approximation: he *enscribed* & *inscribed* polygons of increasing number of sides into a circle.

Modern account — **Riemann** — approximate by sums of areas of strips. We need a new notion: sup (suprenum) = least upper bound, inf (infinum) = greatest lower bound. Compare A = [0,1] and B = [0,1). For A, max A = greatest/*maximum* element of A, which here is 1. However, for the second interval, 1 is not the maximum element because it is not in the interval. Need a name for it: **sup** B.

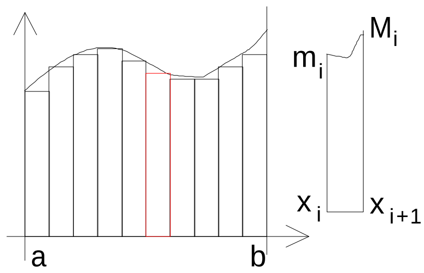
More definitions: suppose X is a subset of **R**. We say X is bounded above if there exists an M in **R** such that $x \in X \Rightarrow x \leqslant M$. Then M is called an upper bound for X. Similarly, X is bounded *below* if there exists an m in **R** such that $x \in X \Rightarrow x \geqslant m$. Then m is called a *lower* bound for X. X is bounded if X is bounded above **and** below. Q: Are the following bounded above and/or below?: [1,2]; [1,3); (-∞, 1]; {x ∈ **R**, x < 2}; the integers.

Aside: The proof that ∃ an infinite number of prime numbers. Suppose that p is real and prime. Consider N = (2×3×5×...×p)+1, the product of the prime numbers up to p *plus* one. Let q be a prime factor of N. q cannot be any of the numbers 2, 3, 5, ..., p since they all leave remainder 1. So q > p and q must therefore be **prime**.

1 is an **upper** bound for the interval B = [0,1). But nothing *smaller* than 1 is an upper bound of B. So 1 is the least upper bound of B, the sup. Similarly, if C = (0,1], then 0 is the **g**reatest **l**ower **b**ound of C = inf. *Example*: Let A = {$1-{}^1/_n$, $n \in$ Œ\{0} }. Then 1 = sup A, 0 = inf A = min A. max A *does not exist*. **Extra** information: 1 = $\lim_{n\to\infty}(1-\frac{1}{n})$.

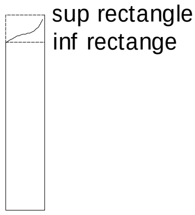
Basic property of **R** (*completeness* axiom): every nonempty subset X ∈ **R** which is bounded above has a least upper bound = sup. This distinguishes **R** from **Q**. Example: Let A = {x ∈ **Q**, $x^2<2$}. A is *nonempty*, bounded above, and 0 ∈ A. But in **R**, sup A = √2, which is not *rational*. Facts on *continuous functions:* if f is a real function with domain a closed bounded interval, say [a,b], then (1) **the range** of f is bounded (i.e. f is bounded); (2) f **attains** its bounds i.e. there is a c ∈ [a,b] s.t. f(c) = sup(range of f) = sup(f[a,b]). There also exists a **minimum**.

Integration of continuous functions on a closed, bounded interval (Riemann)



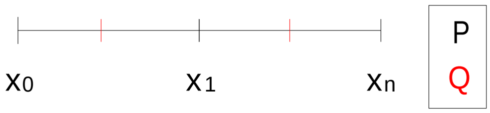
A **partition** of [a,b] is a set $\{x_1,x_2,x_3,..,x_n\} \subseteq$ **R** such that x = a < x_1 < x_2 < ... < x_{n-1} < x_n = b. **Now** let m_i = inf {f(x), x ∈ [x x_{i+1}]}. And let M_i = sup{f(x), x ∈ [x_i, x_{i+1}]}. So we have L(P,f) = $\sum_{i=0}^{n-1} m_i(x_{i+1}-x_i)$ = the sum of the *areas* of lower rectangles (like the red rectangle in the picture). Note: area is signed.

mesh P = $\max_{i=0,\dots,n}\{x_{i+1}-x_i\}$. Let the lower sum for (P, f) be L(P, f) = $\Sigma^{n-1}_{i=0}$ inf{f(x): x∈[x_i, x_{i+1}]}x(x_{i+1}-x_i). Notice: we need to know that f is **bounded** for thi to make sense, i.e. Riemann *lower* and *upper* sums are defined only for boundec functions. For example, f(x) = $^1/_{x^2}$ (x > 0) is not bounded at x = 0: the **upper** sum i: not defined. Similarly, U(P, f) = $\Sigma^{n-1}_{i=0}$ sup{f(x): x∈[x_i, x_{i+1}]}x(x_{i+1}-x_i).

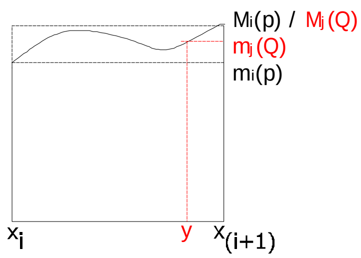


For X = [0,1), the set of X's upper bounds is [1, ∞) — but 1 is *not* in X. Clearly L(P,f) ≤ U(P,f) since inf X ≤ sup X for any set X. It is *convenient* to say that inf X = -∞ if X is not the empty set and not bounded below. Similarly, sup X = +∞ if X is not empty and bounded above. Warning: ±∞ are not real numbers — we cannot have $^\infty/_\infty$ or ∞-∞.

<u>Refinement</u>: if Q and P are partitions of [a,b], then Q refines P if Q ⊇ P (add **more** points). Basic Fact: if Q is a refinement of P, then [L(P,f) ≤ L(Q,f) and U(P,f) ≥ U(Q,f)]* Immediate *consequence*: if p_1 and p_2 are partitions of [a,b], then L(p_1, f) ≤ U(p_2, f), or, any lower sum ≤ any upper sum.

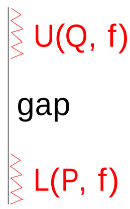


Proof: Let P = p_1, Q = p_1 ∪ p_2. So Q is a refinement of p_1 and p_2 (the *common* refinement). By (*), L(p_1,f) = L(P,f) ≤ L(Q,f) ≤ U(Q,f) ≤ U(p_2,f). *Subdivision* (forming partitions) is a basic method in mathematics. Proof of (*): given that Q refines P, looking at the <u>diagram</u> we see that M_i(p) = sup{f(x): x∈[x_i, x_{i+1}]} and m_i(p) = inf{f(x): x∈[x_i, x_{i+1}]}.

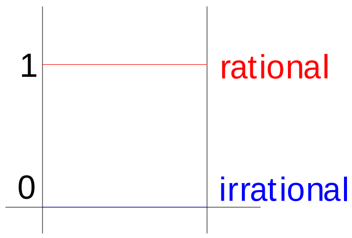


Let us **suppose** that Q has one more point 'y' in [x_i, x_{i+1}] than P has. So [x_i, x_{i+1}] = [x_i,y]∪[y,x_{i+1}]. Then sup {f(x): x∈[x_i,y]} ≤ sup {f(x): x∈[x_i, x_{i+1}]}. Similarly for [y, x_{i+1}]. Now sup{f(x): x∈[x_i,y]}(y-x_i) + sup {f(x): x∈[y, x_{i+1}]}(x_{i+1}-y) ≤ M_i(p)(y-x_i) + M_i(p)(x_{i+1}-y) = M_i(p)(x_{i+1}-x_i). Similarly for *more* points and for *lower* sums.

Recall that given a bounded f on the interval [a,b], then for **partitions** P and Q of [a,b], L(P,f) ≤ U(Q,f). These are numbers! If sup{L(P,f): P = partition on [a,b]} = inf {U(Q, f): Q = partition on [a,b]} i.e. we have **<u>no</u>** gap, then we say that f is R-integrable and that the *common* value is called the Riemann integral, $\int^b_a$ f.



The **main** point is how the idea of a *definite* integral arises from approximating areas. Example: Let $\underline{\int}^b_a$ f = sup{L(P, f), P = ...}. And let $\overline{\int}^b_a$ f = inf {U(Q, f), Q = ...}. Here is an *example* where these two are **not** the same. Let f(x) on [0,1] be *defined* by f(x) = 0 for x∈Q and 0 for x∉Q (0≤x≤1). For any partition P of [0,1], L(P,f) = 0 and U(P,f) = 1. So $\underline{\int}^1_0$ f = 0 and $\overline{\int}^1_0$ f = 1. Silly example? Better ones come from *Fourier* series. Next time: If f is **cts** on [a,b], then f is **R-integrable** on [a,b].



Worksheet: Riemann Integration

Suppose that X is a subset of **R**. U(X) = set of *upper* bounds of X. L(X) = set of *lower* bounds of X. X *is bounded above* iff U(X) is not empty. X *is bounded below* iff L(X) is not empty. So for X = $(-\infty, 9]$, U(X) = $[9, \infty)$ and L(X) is **empty**. So 9 = sup X = l.u.b. of X.

Q: Write down the **sup** and **inf** in **R** (if they exist) of the *following* sets: (i) $(-\infty, 9]$; (ii) $(-2,\infty)$; (iii) $(-1, 1] \cup (2,3]$; (iv) **Q** $\cap$ $(0, \sqrt{3}]$; (v) (**R** \ **Q**) $\cap$ $[2, \infty)$; (vi) $\{1-n^{-1}: n\in$ **N** $\setminus \{0\}\}$. A: (Note: these are **unchecked** — some might be incorrect!) (i) see above. (ii) U(X) is *empty*; L(X) = $(-\infty, -2)$. -2 = inf X. (iii) U(X) = $[3, \infty)$. L(X) = $(-\infty, -1)$. -1 = *inf* X; 3 = *sup* X. (iv) 0 = *inf* X. **No** *sup*. (v) U(X) is empty. L(X) = $(-\infty, 2]$. No *sup*. (vi) U(X) = $[1, \infty)$. L(X) = $(-\infty, 0)$. 0 = *inf* X. 1 = *sup* X. See the **solution** sheet for more detail.

Assignment 2

Q: Find a & b s.t. x = acosht, y = bsinht gives a **parametirc** form for the branch of the *hyperbola* $x^2/9 - y^2/4 = 1$ with x<0. Write down the **equation** of the normal to this hyperbola at the point with parameter t. Find a *parametric* form for the envelope of the normals to the curve. Hence find an equation in x & y for this envelope.

A: The **branch** of the hyperbola $x^2/9 - y^2/4 = 1$ with x<0 has **parametric** equations x = -3cosht, y = 2sinht . Thus $\dot{x}$ = -3sinht, $\dot{y}$ = 2cosht. The **normal** at the point with parameter t has equation 2cosht(y-2sinht) = 3sinht(x+3cosht) i.e. 3xsinht-2ycosht - 13sinhtcosht = 0. This implies that F(x,y,t) = 3xtanht - 2y - 13sinht = 0 (---(1)).

So $\partial f/\partial t$ = $3x\text{sech}^2 t - 13\cosh t = 0$ (---(2)). From (2), we **obtain** $x = {}^{13}/_3 \cosh^3 t$. Substituting in (1) gives $2y = -13\sinh t + 3x\tanh t = -13\sinh t + 13\sinh t.\cosh^2 t = 13\sinh t(\cosh^2 t - 1) = -13\sinh^3 t$. Thus $y = (-{}^{13}/_2)\sinh^3 t$. An equation in x, y for this is (because $\cosh^2 t - \sinh^2 t = 1$) $({}^{3x}/_{13})^{2/3} - ({}^{2y}/_{13})^{2/3} = 1$.

Q: Let f be the **real** function $f(x) = 2-5x^2+2x^5$. Find the *intervals* in which f is increasing and decreasing, and sketch the graph of f. Locate the **zeros** of f between successive integers, stating clearly which *theorems* you are using. Show that for $0<\delta<1$, $f(1)<f(1+\delta)<-1+47\delta^2$ and $f(1)<f(1-\delta)<-1+25\delta^2$. **Hence** find an r>0 *such that* $|x-1|<r$ implies $|f(x)+1|<10^{-6}$.

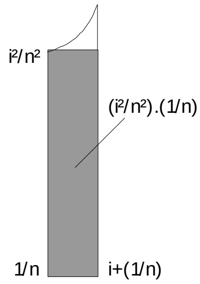
A: $f'(x) = -10x+10x^4 = -10x(-1+x^3) = -10x(x-1)(1+x+x^2)$. **Sign** this i.e. draw a graph of where f'(x) is *+ve and -ve*. Thus f'(x) > 0 (i.e. f is increasing, for x>1 and x<0) on $(-\infty, 0] \cup [1, \infty)$, and f is **decreasing** on [0,1]. Also f(-1) = -5, f(1) = -1, f(0) = 2, f(2) = 46, f(-2) = -82 as a *check*. The Intermediate Value Theorem implies that there are zeros in (-1,0), (0,1) and (1,2). Rolle's Theorem implies that these are the **only** zeros, since further zeros would give *more* roots of f'(x) = 0.

Now for $0<\delta<1$, $f(1+\delta) = 2-5(1+\delta)^2+2(1+\delta)^5 = ... = -1+15\delta^2+20\delta^3+10\delta^4+2\delta^5 < -1+15\delta^2+20\delta^2+2\delta^2$ since $\delta^n<\delta^2$ = $-1+47\delta^2$. And $f(1-\delta) = -1+15\delta^2-10\delta^3+10\delta^4-2\delta^5 <$ (from the above) $-1+15\delta^2+0+10\delta^2+0 = -1+25\delta^2$. These equations **show** that $|x-1|<r$ implies $|f(x)+1|<47r^2$. So for $47r^2<10^{-6}$, it is sufficient to take $r^2<10^{-8}$ i.e. $r<10^{-4}$.

Q: **Sketch** the curve $f(x) = -1+x+{}^{1}/_{(x+2)}-{}^{1}/_{(x-1)}$ by building it up from the **basic** curves -1, x, ${}^{1}/_{x}$ using the *operators* +, -. Sketch the curves given by |f(x)|, √[f(x)], paying care to the form of the **slope**. A: Sketch ${}^{1}/_{x}$ and shift to give $-{}^{1}/_{x}$, $-{}^{1}/_{(x-1)}$ and ${}^{1}/_{x+2}$. Then **superimpose** and **add**.

27th April 1999

Example of Riemann sum



Simple example to illustrate: $f(x) = x^2$ on [0,1]. Take the *partition* $z_0 = 0$, ${}^{1}/_{n}, \ldots, {}^{n-1}/_{n}, {}^{n}/_{n} = 1$. Then $L(P_n, f) = \Sigma^{n-1}_{i=0} {}^{i^2}/_{n^3} = {}^{1}/_{n^3}\Sigma^{n-1}_{i=0} i^2$. Now $\Sigma^{n-1}_{i=0} i^2 = {}^{i(i+1)(2i+1)}/$ (Note on *this*: $f(n)-f(n-1) = n^2$. Aside: $\Sigma^{n}_{i=0} i^3 = ({}^{n(n+1)}/_{2})^2$. There exists an *algorithn* for indefinite **integration** and one for **summing** series). So ${}^{1}/_{n^3}\Sigma^{n-1}_{i=0} i^2 =$ ${}^{1}/_{n^3}{}^{(n+1)n(2n+1)}/_{6} = {}^{(1+(1/n)(2+(1/n))}/_{6} = {}^{1}/_{3}$ as n tends to ∞.

General facts of Riemann integration (no proofs)

Existence. If f is bounded on [a,b] and cts on all except *possibly* a finite number of points, then f is **R-integrable** on [a,b]. So we allow for some discontinuities on [a,b].

(1) **Inequality**. If f and g are *R-integrable* on [a,b], and $f(x) \leq g(x)$ on [a,b] (except possibly at a *finite* number of points), then $\int^b_a f \leq \int^b_a g$.

(2) **Absolute Value.** If mod f and f are R-integrable on [a,b], then $|\int^b_a f| \leq \int^b_a |f|$. Explanation: $\int^b_a$ allows for sign i.e. the *area* under the x-axis is -ve. Related to $|x+y| \leq |x|+|y|$.

(3) Linearity. If f, g are **R-integrable** on [a,b] and λ is in **R**, then $\int^b_a (\lambda f+g) = \lambda\int^b_a f + \int^b_a g$. **Reminder** of meaning: f & g are functions on [a,b] with values in **R**. Values of the function λf+g are *defined* by $(\lambda f+g)(x) = \lambda f(x)+g(x)$. (adding and scalar multiplying values). We emphasise this as a *general* procedure in Mathematics — applying an old idea in a more general situation so that (1) you can apply **old** tricks and (2) you can get more for the **same** amount of work. We are also treating functions as *objects*. Initially 4.3+1, then λx+y?

Other examples: $1+r+r^2+\ldots+r^{n-1} = (1-r^n)/(1-r)$ (**Geometric** Series). This tends to ${}^{1}/_{1-r}$ if $|r| < 1$. H1 course: same result but in more *general* situations to get results on fractals: convergence of sets of points.

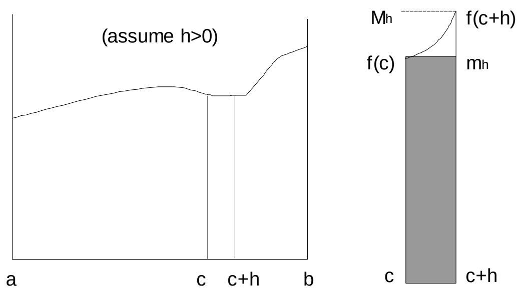
(4) **Additivity**. If f is R-integrable on [a,b] and $a \leq c \leq b$, then f is R-integrable on $[a,c] \cup [c,b]$ and $\int^b_a f = \int^c_a f + \int^b_c f$.

Fundamental Theorem of Calculus

(The main discovery of **Newton** and **Liepniz:** the connection between Area and Tangents). Suppose that f is *bounded* and R-integrable on [a,b], and that f is cts at $c \in (a,b)$. Let $f(x) = \int^x_a f(t)dt$. Then $f'(c) = f(c)$, i.e. f'(c) exists and has this *value*. Point to bear in mind: if f is not cts at c, then f'(c) might or might not exist.

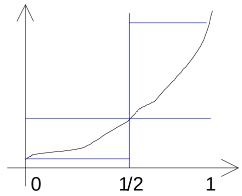
Examples. $f(x) = 2$ when $x = ½$ and $f(x) = 1$ when $x \neq ½$. Then $f'(½)$ *exists* but is not equal to $f(½)$. Example: $f(x) = 2x$ **when** $x \geq ½$ and x when $x < ½$. In this case $f'(x)$ has slope 1 up until $x = ½$ and slope 2 afterwards. Here $f'(½)$ does not exist — there is a corner.

Explaining the Fundamental Theorem (when f is cts at c). By *additivity*, $f(c+h)-f(c) = \int_c^{c+h} f(t)dt$. Let m_h and M_h be the inf and sup of the values of f. By the **inequality** rule $\int_c^{c+h} m_h \leq \int_c^{c+h} f(t)dt \leq \int_c^{c+h} M_h$ which can be **rewritten** as $m_h h \leq f(c+h)-f(c) \leq M_h h$, or $m_h \leq {}^{f(c+h)-f(c)}/_h \leq M_h$. Let h tend to 0. By the **continuity** of c, we can say that m_h tends to f(c) and M_h also tends to f(c). So f'(c) **exists** and is equal to f(c). *Similarly* for h<0.



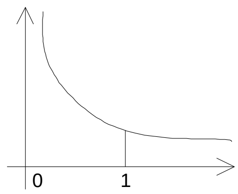
28th April 1999

If f is R-integrable on [a,b], and $m \leq f(x) \leq M$, then for *all* $x \in [a,b]$, we have $m(b-a) \leq \int_a^b f \leq M(b-a)$. Easy: uses the **inequality** $\int_a^b m \leq \int_a^b f \leq \int_a^b M$. But because m and M are **constants**, $\int_a^b m = m(b-a)$ and $\int_a^b M = M(b-a)$. The example *shown* is $\int_0^1 f\,dx$. Sometimes, f(x) is not something you can apply a **standard** antiderivative to, e.g. $\sqrt{(x^3+1)}$, $1/\sqrt{(x^4+1)}$, ${}^{\sin x}/_x$, etc. Maple: If there exists an *elementary* integral, it gives it, otherwise it **returns** the input.



Explain how the basic *properties* of log & exp are established by defining log as an integral. Original problem: we know what is meant by 5.3^{10}: 5.3 multiplied by itself 10 times. We can get an idea of what is meant by $5.3^{1/n}$ ($n \in \mathbf{N}\setminus\{0\}$) by using the fact that $x \to x^n$ is *continuous*. So the IVT shows that for any y>0, there exists a !x (unique x) such that $x^n = y$. This x is the *n^{th} root of y*, written as ${}^n\sqrt{y}$. Then we can define $y^{m/n} = (y^{1/n})^m$.

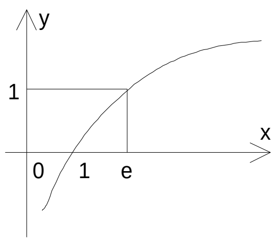
What is $y^{\sqrt{2}}$? This can be explained by **approximating** $\sqrt{2}$ by rationals and so approximating $y^{\sqrt{2}}$. Establishing properties is boring. Instead we define log by integration and easily establish its *properties*. Functions can be defined by integration, e.g. $y = {}^1/_x$. Input: cts on $(0, \infty)$; ${}^1/_x > 0$. Strictly decreasing and so we **define** $\log x = \int_1^x {}^1/_t\,dt$.



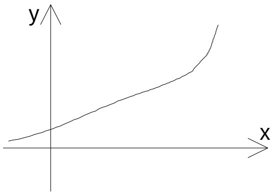
Consequences.

(1) ${}^d/_{dx}(\log x) = {}^1/_x$ (x>0). This is done by the fundamental *theorem* of calculus. (2) log x is strictly increasing. logx > 0 for x > 1, and logx < 0 for 0<x<1. log1 = 0. (3) If x,y > 0 then $\log(xy) = \log(x)+\log(y)$. PROOF: $\log(xy) = \int_1^{xy} {}^{dt}/_t = \int_1^x {}^{dt}/_t + \int_x^{xy} {}^{dt}/_t$. Substitute t = xs. *Then* $\int_x^{xy} {}^{dt}/_t = \int_1^y {}^{xds}/_{xs}$. **Then** $\log(xy) = \int_1^x {}^{dt}/_t + \int_1^y {}^{ds}/_s = \log x + \log y$. QED.

(4) logx is not **bounded** above. PROOF log 2 > 0. Also $\log 2^n = n\log 2$, by (3) and induction. But nlog 2 tends to ∞ as n tends to ∞ since log2>0. (5) logx is not **bounded** below. PROOF: $\log(x.{}^1/_x) = \log 1 = 0$. So $\log({}^1/_x) = -\log x$. So logx tends to -∞ as x tends to 0 (x>0). Now we get the *familiar* graph as shown.



Dom(log) = (0, ∞). Range(log) = (-∞, ∞) = **R** (*). *Proof* of (*) uses the IVT — we know log is unbounded and continuous. Now we define exp to be the inverse function to log so that exp(logx) = x and log(expy) = y. Then Dom(exp) = Range(log) = **R**. And Range(exp) = Dom(log) = (0, ∞). *Graph* is as shown.



Basic properties of exp, with exp(x) = e^x where e is defined as *loge = 1*. (1) (expx)(expy) = exp(x+y). PROOF: log((expx)(expy)) = log(expx)+log(expy) = x+y = log(exp(x+y)). Hence the result. Note: $\log x_1 = \log x_2$ *implies* $x_1 = x_2$. (2) $^d/_{dx}$(expx) = expx. The PROOF follows from the fact that log(expx) = x. So $^d/_{dx}$(log(expx)) = $^1/_{expx}$ $^d/_{dx}$(expx). This is a famous *equation* — the rate of growth is the current value. Important in Biology e.g. where infinite food supply **populations** grow exponentially.

Let a > 0. Define a^x = exp(xloga) [$e^{x\log a}$]. We know loga for a > 0, so a^x is defined for all x in **R**. So $2^{\sqrt{2}}$ does not mean 2 *multiplied* with itself √2 times. Lesson: a concept may have several definitions. Other examples: the Gamma function where Γ(x+1) = (x+1)Γ(x). If x is natural, then Γ(x) = x!. Also, fractional **dimension** and fractional **differentiation**.

29th April 1999

Worksheet: Riemann Integration

Q: Write down the **greatest** element, **least** element, sup and inf of the following sets where these exist. (i) (-∞, 9]. Greatest element sup = 9. (ii) (-2, ∞). inf = 2. (iii) (-1, 1] ∪ (2,3]. *Greatest* element sup = 3, inf = -1. (iv) **Q** ∩ (0, √3]. inf = 0. $\sup_{\mathbf{R}}$ = √3, $\sup_{\mathbf{Q}}$ = does not exist. (v) (**R** \ **Q**) ∩ [2, ∞). inf = 2. (vi) {$1-n^{-1}$: n∈**N**\{0}}. Least element inf = 0, *sup* = 1.

4th May 1999

Basic Properties of a^x

Define a^x = exp(xlogn). (1) a^1 = a because exp(loga) = a. (2) e is *defined* as loge = 1. So e^x = exp(xloge) = exp(x). (3) a^{x+y} = exp((x+y)loga) = exp(xloga).exp(xlogb) = $a^x a^y$. (4) $\log a^x$ = xloga. So $(a^x)^y$ = exp(ylog(a^x)) = exp(yxloga) = a^{xy}. (3) & (4) are called the "*exponential*" laws. For natural **numbers** they obviously hold, but we use a^x generally as well.

(5) (*) $^d/_{dx}(a^x)$ = $^d/_{dx}(e^{x\log a})$ = (loga)a^x. (6) If a^x = y we also write x = $\log_a y$. So $\log_e y$ = logy (*natural* logarithm). Logs are used to **change** × into +. Makes calculation *faster*. (7) Change to x^a. $^d/_{dx}(x^a)$ = $^d/_{dx}(e^{a\log x})$ = $^a/_x e^{a\log x}$ = $x^a(^a/_x)$ = ax^{a-1} (**Real** a).

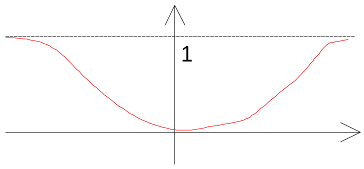
The behaviour of these functions as x tends to ∞ and zero. We have already said that log x tends to ∞ as x **tends** to ∞ (log increasing, unbounded above on (0, ∞)). But ±∞ are not **real** numbers. Similarly, logx tends to -∞ as x tends to 0+. We now show how logx tends to +∞ "more slowly" than any power of x, that is if α>0, then $\lim_{x\to+\infty} \log x/x^\alpha = 0$.

Proof: Choose β > 0, β < α. Suppose that x > 1. Then logx = $\int_1^x {}^{dt}/_t < \int_1^x dt/t^{1-\beta} = \int_1^x t^{\beta-1}dt$ = $x^{\beta-1}/\beta < x^\beta/\beta$. So $\log x/x^\alpha < x^{\beta-\alpha}/\beta$. But β < α so β-α < 0. So $x^{\beta-\alpha}$ tends to 0 as x tends to ∞. But 0 < $\log x/x^\alpha$ for x > 1. So $\log x/x^\alpha$ tends to 0 as x tends to ∞.

Next, put x = $^1/_y$. So as x tends to +∞, y *tends* to 0+. Then $x^{-\alpha}\log x = -y^{\alpha}\log y$. So $y^{\alpha}\log y$ tends to 0 as y tends to 0+. And logy tends to -∞ as y tends to 0+. This happens more slowly than any power of y. We can **relate** this to e^x: "e^x tends to ∞ as x tends to 0 more *rapidly* than any other power of x".

Proof, **Explanation**: start with $\log x/x^{\alpha}$ tends to 0 as x tends to +∞. Put x = e^y. Get $y/e^{\alpha y}$ tends to 0 as y tends to +∞. Hence $(y/e^{\alpha y})^{1/\alpha}$ tends to 0 as y **tends** to +∞. i.e. $y^{1/\alpha}/e^{\alpha}$ tends to 0 as y tends to +∞. (**<u>Note</u>**: we can write "x→y" instead of "x tends to y").

Let y = e^{-1/x^2} for **non** zero x and y = 0 for x = 0. Note: f(x) is **even**: f(x) = f(-x). Also, $-^1/_{x^2} < 0$, so $e^{-1/x^2} < 1$. But as x tends to infinity, e^{-1/x^2} tends to 1. By **symmetry**, expect f'(0) = 0. In fact, $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{1/h^2}}{h} = 0$. **What** is f''(0)? $f''(0) = \lim_{h\to 0}\frac{g'(h)-f'(0)}{h} = \lim_{h\to 0}\frac{f'(h)}{h}$. For h≠0, $f'(h) = {}^d/_{dh}(e^{-1/h^2}) = e^{-1/h^2}(2h^{-4})$. So $f''(0) = \lim_{h>0} e^{-1/h^2}/2h^4 = 0$.



Similarly, $f'''(0) = \lim_{h\to 0} {}^{f''(h)}/_h$; $f''(h) = {}^d/_{dh}(2e^{-1/h^2}h^{-3}) = e^{-1/h^2}({}^{\text{polynomial in h}}/_{\text{power of h}})$. Using the **same** idea, f'''(0) = 0. We can now set up a *proof* that $f^{(n)}(x) = e^{-1/x^2}\times({}^{\text{poly in x}}/_{\text{power of x}})$ when x≠0; and = 0 for x=0. **Contrary** to the picture above, f is <u>very flat</u> at 0. The *Taylor* expansion of f is 0, and this gives no information on f except at 0. Note: <u>Multiply</u> functions together as a *smoothing* function.

<u>Assignment 3</u>

Q: Give 2 *proofs* that when x > 0, cosx < $1-(x^2/2)+(x^4/4!)$, — (a) by **differentiation**; (b) by applying the **Alternating Series Estimation Theorem**. A: (a) Let f(x) = $1-(x^2/2!)+(x^4/4!)-\cos x$. Then $f'(x) = -x+{}^{x^3}/_{3!}+\sin x$, f'(0) = 0; $f''(x) = -1+{}^{x^2}/_{2!}+\cos x$, f''(0) = 0; f'''(x) = x-sinx; f'''(0) = 0. We have **already** proved that x > sinx *for* x > 0. So f''(x) is strictly *increasing* for x > 0, and f''(x) > 0. Hence f'(x) is strictly <u>increasing</u> for x $\geq$ 0, and f'(x) > 0. Hence f(x) is **strictly** increasing for x $\geq$ 0, and so f(x) > 0 *for* x > 0.

(b) The *Alternating Series Estimation Theorem* says that if (a_n) is a sequence of **real** numbers such that (1) $a_n \geq 0$ for all n; (ii) $a_n \geq a_{n+1}$ for all n; and (iii) a_n tends to *zero, (all +ve, decreasing, tend to 0)*, then the series $S_k = \Sigma^k_{n=0}(-1)^n a_n$ is convergent with *limit* S, say; and $|S-S_k| \leq a_{k+1}$. (This is the "*error estimate*"). We would like to apply this to the **series** for cosx, with $a_n = x^{2n}/(2n)!$. Unfortunately, we can **apply** the theorem only to the case where (a_n) is decreasing (This is an trick question!). Now for x > 0, $a_n \geq a_{n+1}$ iff $x^{2n}/(2n)! \geq x^{2n+2}/(2n+2)!$ iff $(2n+2)(2n+1) \geq x^2$. If n = 2, this *gives* $30 \geq x^2$.

Q: Use the MVT to *prove* that $^1/_{10} < \sqrt{(83)}-9 < {}^1/_9$. A: Let f(x) = √x. Then $f'(x) = {}^1/_{2\sqrt{x}}$. Since √x is an *increasing* function, $^1/_{2\sqrt{x}}$ is a **decreasing** function (Alternatively, $f''(x) = -¼x^{-3/2}$, which is < 0). Now the MVT implies that if m ≤ f'(x) ≤ M on [a,b], then m(b-a) ≤ f(b)-f(a) ≤ M(b-a). (*). **Take** a = 81, b = 83. Then we can take M = f'(81) = $^1/_{18}$; m = f'(83) = $^1/_{2\sqrt{83}}$. So from (*), we *deduce that* $^2/_{2\sqrt{83}} \leq \sqrt{83}-\sqrt{81} \leq {}^2/_{18}$. Since $^1/_{\sqrt{83}} > {}^1/_{\sqrt{100}} = {}^1/_{10}$, the result *follows*.

Q: Find the **Taylor** approximation to degree 9 (with remainder of degree 10) for $f(x) = 3\cos x+4\sin x$, by calculating the derivatives $f^{(n)}(0)$. Write down an *expression* for the remainder term. A: $f(x) = 3\cos x+4\sin x$. $f(0) = 3$. $f'(x) = -3\sin x+4\cos x$. $f'(0) = 4$. $f''(x) = -3\cos x-4\sin x$. $f''(0) = -3$. $f'''(x) = 3\sin x-4\cos x$. $f'''(0) = -4$. Then we *repeat*.

Thus the **Taylor** series to degree 9 is $3 + 4x - (3x^2/2!) - (4x^3/3!) + (3x^4/4!) - (4x^5/5!) - (3x^6/6!) - (4x^7/7!) + (3x^8/8!) + (4x^9/9!)$. The **remainder** term is $R_9(x) = (f^{(10)}(c)/10!)x^{10}$ where $f^{(10)}(x) = -3\cos c-4\sin c$.

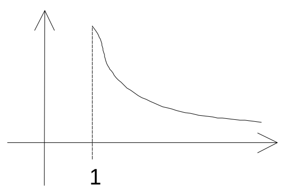
Q: Let f be as in the above question. Write f(x) in the form $r\cos(x+\alpha)$. Hence give a form for the remainder in **Taylor's** theorem, from which you can show that the *error* in using this Taylor approximation at x = 1 is $< 2\times10^{-6}$. A: $f(x) = 5(^3/_5\cos x+^4/_5\sin x) = 5\cos(x+\alpha)$ where $\cos\alpha = {}^3/_5$ and $\sin\alpha = -{}^4/_5$. This *shows* that $|f^{(10)}(c)| \le 5$. So $|R_9(1)| \le {}^5/_{10!} = 1.38\times10^{-6} < 2\times10^{-6}$.

5th May 1999

Famous Formula: $e^x = \lim_{n\to\infty}(1+{}^x/_n)^n = \lim_{n\to\infty}(1+{}^x/_n)^{-n}$. *Proof:* ${}^d/_{dt}(\log(1+xt)) = ({}^1/_{1+xt})x$ (*). From the *definition* of ${}^d/_{dt}$, (take t = c), $\lim_{h\to\infty} {}^{\log(1+xh)-\log(1+x.0)}/_h = x$, i.e. $\lim_{h\to\infty} {}^{\log(1+xh)}/_h = x$. Now let $h = {}^1/_k$, and let k tend to ∞. Get $\lim_{k\to\infty} k\log(1+{}^x/_k) = x$. Take the **exp** of this, so $\lim_{k\to\infty}\exp(\log((1+{}^x/_k)^k)) = \exp x$. And $\lim_{k\to\infty}(1+{}^x/_k)^k = \exp x$. Here, k tends to ∞ for real k. Restrict k to *integral* values; $\lim_{n\to\infty}(1+{}^x/_n)^n = e^x$. In (*), we are *using* (a) the **definition** of log, FTC; (b) the *differentiation* of a composite, i.e. ${}^{dz}/_{dt} = {}^{dz}/_{dy}.{}^{dy}/_{dt}$. After that, it is just limit manipulation.

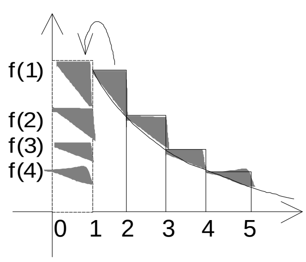
Integral test for series

Hypothesis: Let f(x) be a decreasing real function with domain including $[1, \infty)$, s.t. f(x) tends to 0 as x tends to ∞. **Conclusion**: then $\sum^{n-1}_{i=1} f(i) - \int^n_1 f(x)dx$ is convergent (cvt). Hence $\sum^{\infty}_{i=1} f(i)$ is cvt iff $\int^{\infty}_1 f(x)dx$ is cvt. Let us look at an *application* before looking at the proof.



Application: $\sum^{n-1}_{i=1} ({}^1/_i)-\log n$ tends to a finite limit, called **Euler's** constant, γ. So $\sum^{\infty}_{i=1} {}^1/_i$ diverges (v. slowly) like log n. How many *terms* do you need to get $\sum^{n}_{i=0} {}^1/_i > 100$? A: Roughly e^{100}. Famous question: Is γ **rational**? A: Presumably not! — γ is approx. 0.577.

Proof: $\sum^{n-1}_{i=1} (f(i))$ = Riemann **upper** sum, $U(p_n, f)$ = total *area* of the outside boxes. So $0 \le \sum^{n-1}_{i=1} f(i) - \int^n_1 f(x)dx$. ($0 \le$ ... because the sum = $U(p_n, f)$). So $\sum^{n-1}_{i=1} f(i) - \int^n_1 f(x)dx = \sum^{n-1}_{i=1} f(i) - \sum^{n-1}_{i=1}\int^{i+1}_{i=1} f(x)dx \le$ (as f is decreasing) $\sum^{n-1}_{i=1} f(i) - \sum^{n-1}_{i=1} \int^{i+1}_{i} f(1+i)dx = \sum^{n-1}_{i=1} (f(i)-f(1+i)) = f(1)-f(n) \le f(1)$ (as $f(n) > 0$). This calculation corresponds in the **picture** to the sum of the shaded bits $\le$ dotted rectangle. Further, $\sum^{n-1}_{i=1} f(i) - \int^n_1 f(x)dx = \phi(n)$ is an **increasing** sequence, so $\phi(1) \le \phi(2) \le ... \le \phi(n) \le f(1)$.



Let $l = \sup\{\phi(i): i\in \mathbf{N}\}$. Then $\phi(n)$ tends to l as n *tends* to ∞, with $l \le f(1)$. So we have proved more **information** than we stated in the theorem. Roll off more *applications*: (1) $\sum^{\infty}_{i=1} (1/n^s)$ (with $s\in\mathbf{R}$) = S(s), the Riemann **zeta** function. used in number theory. Aside: if s = 2, then $S(2) = 1+{}^1/_{2^2}+{}^1/_{3^2}+... = {}^{\pi^2}/_6$.

Let us examine the **convergence** of S(s): convergent for $s > 1$ but *dvt* for $s \le 1$. The proof uses the integral test: $\int_1^n 1/x^s\,dx$. We get **2** cases: (i) log n ($s = 1$); (ii) n^{1-s}-(1/1-s) ($s \ne 1$). If $s < 1$, n^{1-s} tends to ∞ as n tends to ∞. If $s > 1$, n^{1-s} tends to 0 as n tends to ∞. So $\Sigma^{\infty}_1 1/n^s$ is cvt iff $s > 1$. (This is *difficult* to prove for **other** tests e.g. the ratio test).

(2) $\Sigma^{\infty}_{i=1} 1/[n(\log n)^s]$. Consider $\int_1^n [1/x(\log x)^s]dx = [(\log x)^{1-s}/(1-s)]^n_1 = (\log n)^{1-s}/(1-s)$. This is cvt iff $s > 1$. **Mention** the Ratio test: Take $\lim_{n\to\infty} (n+1)^s/n^s = \lim_{n\to\infty} (1+(1/n))^s = 1$. *Note:* the ratio test, $\lim_{n\to\infty} a_n/a_{n+1} = l$, says that a sequence is cvt if $l < 1$, dvt if $l > 1$, and is **unknown** if $l = 1$.

6th May 1999

Tutorial: Integral Test

Q: Prove that the **series** $\Sigma^{\infty}_{n=1} {}^1/_{n^2+1}$ is cvt, and that the *sum* is $< \frac{1}{2}+\frac{1}{2}\pi$. A: Analyse $\int_1^{\infty} {}^1/_{n^2+1}dn = [\tan^{-1}(n)]^{\infty}_1 = \tan^{-1}(\infty)-{}^{\pi}/_4 = {}^{\pi}/_2-{}^{\pi}/_4 = {}^{\pi}/_4$. So $\Sigma^{\infty}_{n=1} {}^1/_{n^2+1}-{}^{\pi}/_4 < \frac{1}{2}$. (*Where* the ½ comes from f(1)). So $\Sigma^{\infty}_{n=1} {}^1/_{n^2+1} < {}^{\pi}/_4+{}^1/_2$ as *required*.

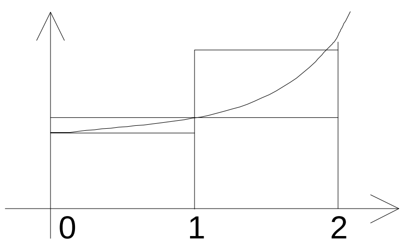
11th May 1999

Assignment 4

Q: Find the *greatest element, least element, sup* and *inf* of these sets, where they exist.

Set	Greatest element	Least element	sup	inf
$(-\infty, 8)$			8	
$[3, \infty)$		3		3
$(-1, 1) \cup (2,3)$		-1	3	-1
$\mathbf{R} \cap [0, \sqrt{5}]$		0	$\sqrt{5}$	0
$(\mathbf{R}\backslash\mathbf{Q}) \cap (-2, \pi]$	π		π	-2
$\{2-{}^3/_n: n \in \mathbf{N}\}$		-1	2	-1

Q: Prove that the *function* $f(x) = 1+(x^3+1)^{1/2}$ is increasing on the *interval* [0, 2]. By dividing this interval into two *equal* parts, show that $I = \int_0^1 f(x)dx$ *satisfies* $4.414 \le I \le 6.415$. Does one obtain **better** upper & lower bounds by dividing this interval into *unequal* intervals [0, 0.5], [0.5, 1.5], [1.5, 2]?



A: Prove that the function is *increasing* by proving that f’(x) is +ve on the interval. Then divide the interval into 2 equal parts, [0,1] and [1,2]. The integral is approximated by calculating the area of the rectangles under the graph. The lower sum is $(1\times f(0))+(1\times f(1)) = 4.414$ (Rounding **DOWN**). The upper sum is $(1\times f(1))+(1\times f(2)) = 6.415$ (Rounding **UP**). Therefore, I is situated between the *lower* and *upper* sums, i.e. $4.414 \le I \le 6.415$. Now do the same procedure but with the *different* rectangles to get the better bounds $4.6064 \le I \le 6.1220$.

General notes on this type of question: here the graph was increasing, so we calculated e.g. lower sum = (width of b-a)×f(a). If it WAS **decreasing,** we would calculate (width of b-a)×f(b) to get the lower sum. So with functions, split them up into domains where they are increasing or decreasing i.e. find the *stationary* points. Also, if the graph goes below the x-axis, the method changes again.

Exam Paper: May 1999

SECTION 1 (Compulsory)

(1) (a) Express the parametric curve $x = 3 \cosh 3t$, $y = 2 \sinh 3t$ in cartesian coordinates and give its name. **[3 marks]**

Write down without simplification the equations of the tangent and of the normal at the point of the curve with parameter t. **[4 marks]**

(b) Let f be the real function given by $f(x) = x^2-4x+3$. Sketch the function, and determine the image by f of the sets $(-\infty, 2]$, $(1, 3]$, $[0, 3)$. **[6 marks]**

(c) For each of the following sets, state whether it is bounded above or bounded below; in each case determine the maximum, minimum, sup, inf of each set, when they exist.

(i) $(2, 8]$;
(ii) $(-\infty, -\sqrt{5})$;
(iii) $\{-1-1/n : n \in$ Œ and $n \neq 0\}$. **[7 marks]**

SECTION 2 (Answer 2 out of 4 questions)

(2) (a) Find the arc length of the curve $x = 2t^2$, $y = 2t$ from $t = 0$ to $t = 1$. **[3 marks]**

(b) Let f be the real function given by $f(x) = x^4-2x^2+\frac{1}{2}$. Determine the intervals in which f is (i) increasing, (ii) decreasing, (iii) concave up and (iv) concave down. Locate the zeros of f at or between successive integers, and sketch the graph of f. **[7 marks]**

(c) Find an n such that the error in using the Taylor series for sin x up to the term involving x^{2n+1} for $x = 1$ in less than 10^{-6}. **[5 marks]**

(3) Let $a > 0$. Give a name for the curve C(t) with parametric form $(at^2, 2at)$. Find the equation of the normal at the point with parameter t of C(t). Hence find the parametric form for the evolute E(t) of C(t), i.e. the envelope of the normals of C. **[5 marks]**

Find also the radius of curvature of the curve C(t) at the point with parameter t and verify that this coincides with the distance from C(t) to E(t). [You may assume the formula for the radius of curvature $\rho = (\dot{x}^2+\dot{y}^2)^{(3/2)}/|\dot{y}\ddot{x}-\dot{x}\ddot{y}|$. Sketch this evolute. Your sketch should include the original curve C and some of its normals. **[4 marks]**

(4) (a) Prove that the function $f(x) = 2+(\cos x)^{1/2}$ is decreasing on the interval $[0, \pi/2]$. By dividing this interval into two equal parts show that $I = \int_0^{\pi/2} f(x)dx$ satisfies $3.8020 \leq I \leq 4.5874$. Does one obtain better upper and lower bounds by dividing this interval into unequal intervals $[0, 3\pi/8]$, $[3\pi/8, \pi/2]$?
[Use the rule $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$]. **[8 marks]**

(b) Let f be the real function given by $f(x) = 1+2x-x^2+x^3$. Prove that for $0 < \delta < 1$ we have $3-4\delta < f(1-\delta) < f(1) < f(1+\delta) < 3+5\delta$. Hence find a number $\delta > 0$ such that $|x-1| < \delta$ implies $|f(x)-3| < 10^{-6}$. **[7 marks]**

(5) (a) Find the general term of the series expansions, in terms of x, of $(1+2x)^{-1}$ and of $(3-x)^{-1}$. For which x are these series convergent? **[4 marks]**

By partial fractions, or otherwise, prove that the general term of the series expansion of $(1+2x)^{-1}(3-x)^{-1}$ is $\frac{1}{7}((-1)^n 2^{n+1}+3^{-n-1})x^n$. For which x is this series convergent? **[5 marks]**

(b) Let $f(x) = x(x-1)(x-2)$. Find a point c in [1, 3] which satisfies the Mean Value Theorem for f on this interval, and illustrate with a sketch graph. **[6 marks]**

<u>(Questions done: 1, 2, 4)</u>