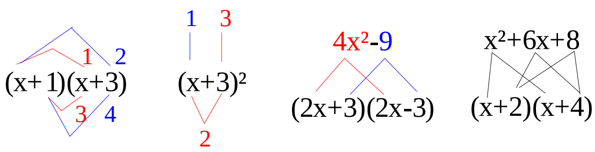
Pre-GCSE Material

To find the *square root* of a square number, **half** the powers when the number is written as a product of prime factors in index form. For example, $36 = 2^2 \times 3^2$, so $\sqrt{36} = 2^1 \times 3^1 = 6$. To **make** a number into a square number, write the number in terms of its *prime factors*, and then multiply the number so that all of the indexes are even. For example, to make $120 = 2^3 \times 3^1 \times 5^1$ into a square number, multiply by $2 \times 3 \times 5$ to make **all** of the powers even powers. The **Highest Common Factor** of two numbers is the product of the *common factors* of the two numbers when the two numbers are written in terms of their prime factors. **Mean**, **Median** and **Mode** are measures of *central tendency*.



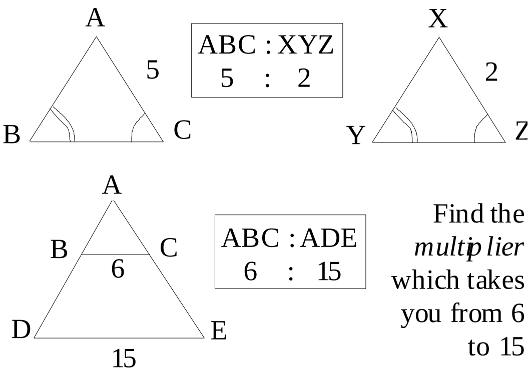
Multiply to make 8,
Add to make 6

Multiplying out and factorising. *Common Factors*: $x^2+2x = 0$; $x(x+2) = 0$; either $x = 0$, or $x+2 = 0$; so either $x = 0$, or $x = -2$. *Difference of two squares*: $x^2 = 4$; $x^2-4 = 0$; $(x-2)(x+2) = 0$; either $x-2 = 0$, or $x+2 = 0$; so either $x = 2$, or $x = -2$.

GCSE Material

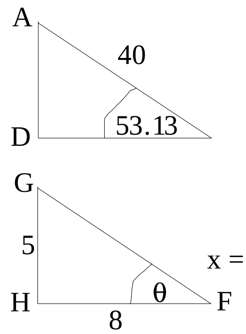
Indices and Fractional Indices

$$a^x \times a^y = a^{x+y}; \quad a^x/a^y = a^{x-y}; \quad (a^x)^y = a^{x \times y}; \quad 10^{-3} = {}^1/_{10^3}; \quad a^{-x} = 1/a^x; \quad a^0 = 1;$$
$$a^{½} = \sqrt{a}; \quad (¼)^{-½} = 1/(¼)^{½} = 1/\sqrt{¼} = {}^1/_{½} = 1 \times ({}^2/_1) = 2.$$



Standard Form: $a \times 10^n$, where $1 \leq a < 10$, and n is an integer. **Not** in standard form: 62.4×10^2; 0.3×10^1. **In** standard form: 3.4×10^2, 6.0×10^{12}. Equivalent fractions: ${}^4/_5$, ${}^{16}/_{20}$, ${}^{20}/_{25}$, ${}^{24}/_{30}$. *Mixed numbers* converted to *fractions*: $2½ = {}^5/_2$, $3¾ = {}^{15}/_4$, ${}^{32}/_3 = 10{}^2/_3$. Multiplying fractions: top×top, bottom×bottom — for example $¼ \times ½ = {}^1/_8$. Dividing: the same as multiplying, but turn the **second** fraction upside down. ¾ is the reciprocal of ${}^4/_3$, and ${}^1/_n$ is the reciprocal of n.

Volume of a **sphere** = ${}^4/_3\pi r^3$. Volume of **half a sphere** = ${}^2/_3\pi r^3$. Volume of a **cone** = ${}^1/_3\pi r^2$. Volume of a **cylinder** = $\pi r^2 h$. 'Ω' = estimate, e.g. 499 Ω 500. The volume of any *uniform cross section* = area of the cross section × length. **SIN** ${}^O/_H$; **COS** ${}^A/_H$; **TAN** ${}^O/_A$. *Pythagoras'* rule: $A^2 = B^2+C^2$ (A = *hypotenuse*). **Tessellate**: angles around a point *add up* to 360°.



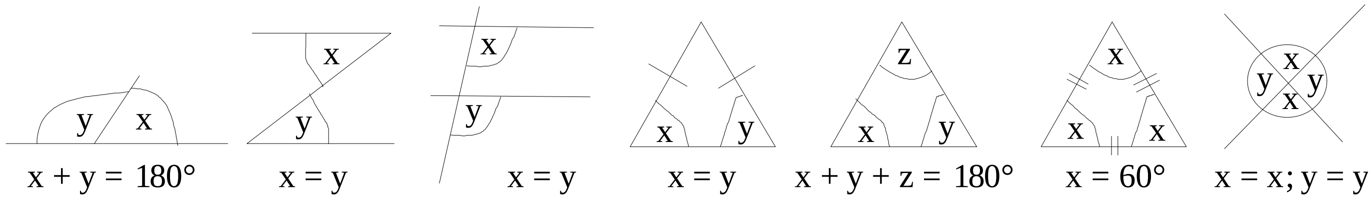
Sin 53.13 = AD/40;

40 × 0.7999 = AD

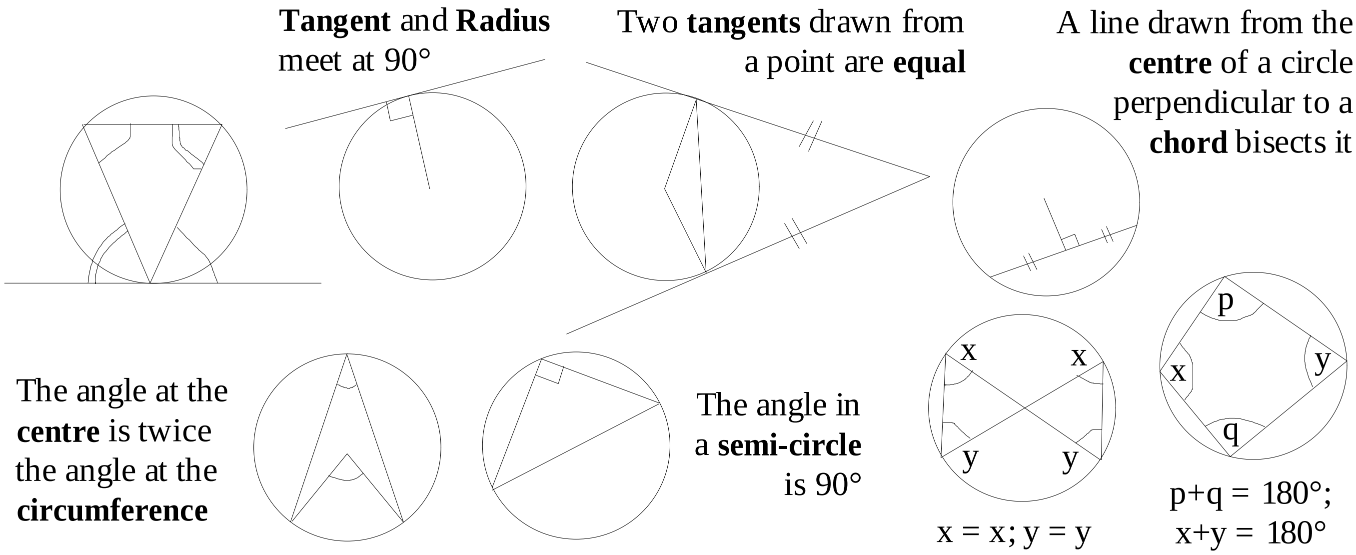
x = Inv Tan(5/8);

x = Inv Tan(0.625) = 32°

Complementary: add up to 90°. **Supplementary**: add up to 180°.
Acute: between 0° and 90°. **Obtuse**: between 90° and 180°. **Reflex**: over 180°.
Alternate: Z angles. **Corresponding**: F angles. **Vertically opposite**: X angles.



Interior angle = [(n-2)×180]/n. *Exterior* angle = $^{360}/_{n}$.
Pentagon: 5 axes of bilateral symmetry; rotational symmetry = 5.
In y = mx+c, m is the *gradient* while c is the *y-intercept*.
Special lines: The y-axis: x = 0. The x-axis: y = 0. The *gradient*, $^{V}/_{H}$, can be a fraction.



Inequalities and Inequations

> *greater* than; $\geq$ *greater* than or *equal* to; < *less* than; $\leq$ *less* than or *equal* to. Integers = {...-2, -1, 0, 1, 2,...}. Solve **inequalities** like equations. **Graphs**: ‘>’ or ‘<’: dotted lines; ‘$\geq$’ or ‘$\leq$’: solid lines. Test co-ordinates to see if some region must be *shaded* in. For example, test the point (3,2) in x+2y $\geq$ 4. So 3 + 2×2 $\geq$ 4.7; 4.7 $\geq$ 4, which is true, so the point **is** in the region to be shaded in. If you multiply or divide an inequality by a *negative* number, you must change the sign. **Quadratic**: x^2, x, n. **Linear**: x, n. The solution is where the graph *cuts the x-axis*. Scale factor: k. Area factor: k^2. Volume factor: k^3. Reduction factors are less than 0.

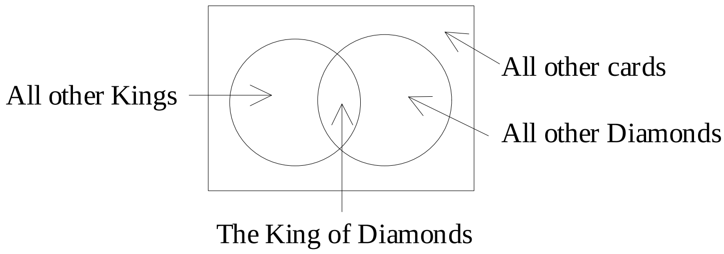
Decimal search method for the answers of an equation: substitute in *successive values* into the equation; pinpoint where the *sign changes*, and then repeat on a *smaller* scale until you reach some required degree of accuracy. For example, in $f(x) = x^2-7x+3$, f(0) = 3, and f(1) = -1 — so there is a solution between 0 and 1. Now look at a smaller scale (e.g. 0.1, 0.2, 0.3, ..., 0.9, 1) and look for the *change in sign* again, etc.

Simultaneous Equations

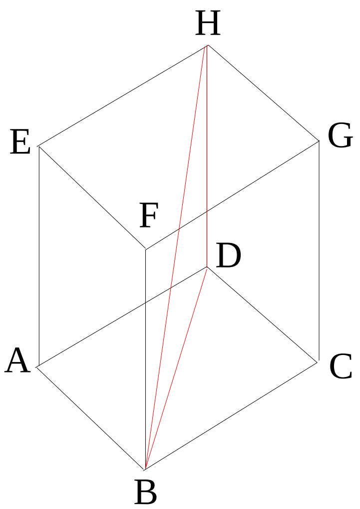
Method: multiply your two equations so that the coefficients of some variable are the **same**. Then add or take away (**same** signs: take away; **different** signs: add) so that you obtain a simple equation in one variable which you can solve. Example: 3x+2y = 19 (---(1)) and 2x+5y = 20 (---(2)). Multiply (1) by 5 and (2) by 2 to get 15x+10y = 95 (---(3)) and 4x+10y = 40 (---(4)). Now do (3) - (4) to obtain 11x = 55; x = 5. Now substitute for x in (2) to get y = 4.

Probability

AND = multiply; **OR** = add. 0 = *impossible*; 1 = *certain*. Use tree diagrams. **Mutually exclusive** events: one cannot happen if the other happens: P(A or B) = P(A) + P(B). **Non-mutually exclusive** events: one can happen if the other happens. **Independent** events (replace): P(A and B) = P(A)×P(B). **Dependent** events: not replace. Venn Diagrams: P(K or D) = P(K) + P(D) - P(K and D).



Trigonometry in Three Dimensions



Use *Pythagoras' rule* to get the length HB (the hypotenuse); use *trigonometry* (usually tan) to work out an angle e.g. angle CBH

In a question, make sure **which** angle you are dealing with; clarify whether you are looking to calculate an **angle** or a **length** of a side; and always remember the acronyn **"SOHCAHTOA"**
(**S**in = **O**pposite/**H**ypotenuse, **C**os = ...)

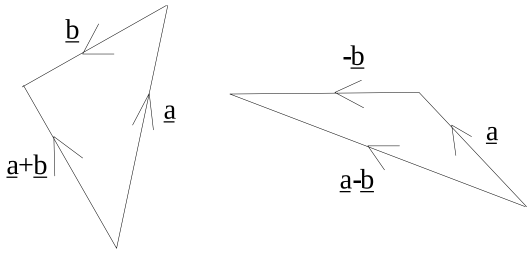
Sequences

Arithmetic (adding) sequence: the *first* term is called 'a'; and the value of the n^{th} *term* is a+(n-1)d, where d is the **difference** between successive elements in the sequence. **Geometric** (multiplying) sequence: the formula is a×d(n-1). Formula for the *triangle* sequence: $^{n(n+1)}/_{2}$. *Exponential growth*: e.g. y = 2x. Here, when x = 5, y = 2^5 = 2×2×2×2×2 = 32. *Exponential decay*: if you multiply something by 0.92, it is a **reduction** of 8%. Dividing by a number is the same as **multiplying** by its **reciprocal**. For example, 5÷¼ = 5×4 = 20. The formula for Pascal's triangle (for x^2) is [n(n-1)]÷2.

Algebraic Fractions

$({}^{U}/_{W})\times V = {}^{UV}/_{W}$. $[{}^{(a+b)}/_{c}]\times d = {}^{d(a+b)}/_{c}$. $b\times{}^{(c+d)}/_{(e+f)} = {}^{b(c+d)}/_{(e+f)}$. *Cancelling*: ${}^{4x^2y}/_{3ax^2} = {}^{4y}/_{3a}$. ${}^{12pq^2r}/_{3pqr^2} = {}^{4q}/_{r}$. The a's **cannot** be cancelled in ${}^{a+b}/_{ac}$ or ${}^{ax+b}/_{ac}$, but **can** be cancelled in ${}^{a(x+b)}/_{ac}$.

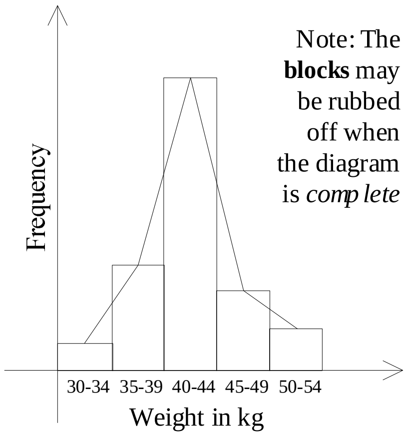
Vectors



Vectors represent **magnitude** and **direction**. They can be represented on paper by a *column* vector where the upper number is the distance **across** and the lower number is the distance **up**. The symbol for a vector is an underlined letter. A *linear* sum of vectors u and v is any vector in the form au+bv, where a and b are *numbers*. For example, if u = $[{}^{2}{}_{1}]$ and v = $[{}^{1}{}_{3}]$, then 2u+v = $2[{}^{2}{}_{1}]+[{}^{1}{}_{3}] = [{}^{4}{}_{2}]+[{}^{1}{}_{3}] = [{}^{5}{}_{5}]$. Question: if a = $[{}^{2}{}_{5}]$ and b = $[{}^{1}{}_{3}]$, express c = $[{}^{9}{}_{25}]$ as a *linear sum* of a and b. Answer: solve the **two simultaneous equations** 2a+1b = 9 and 5a+3b = 25 *for* a and b.

Examples of **functions**: x→x-1; x→2x; and x→4x+0.5. For functions, make *tables* for values and then draw graphs. For example, if you have the function t(x) = 2x-4, make a table saying what the values of t(x) are for x = -3, -2, ..., 2, 3; and then draw a graph **plotting** x against t(x) to see the function *graphically*. Draw *parallel line diagrams* of these functions and their inverses. When in e.g. ${}^{1}/_{x}$ you can't get a value, e.g. when x = 0, then the number isn't in the **domain** and you leave the space **blank**.

Percentage *increase*/*decrease* = $({}^{\text{increase}}/_{\text{original value}})\times 100$. **Accuracy**: ±½ a unit either side. For example, a side measured 35cm *to the nearest centimetre* can vary from 34.5cm to 35.5cm.



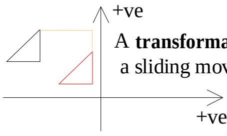
Quadratic = x^2, **Cubic** = x^3, **Quartic** = x^4. Locus: a path or region defined by certain conditions, e.g. *equidistant* — a perpendicular bisector. Median: put a list of numbers in order, and the *middle* number is the median. Mode: most *often* occurring number. Mean: average. $C = 2\pi r$ or $C = \pi D$; $A = \pi r^2$. **Correlation**: is there a pattern in the graph? If so, do the dots form a best fit line that slants **down** (*negative*) or **up** (*positive*). Pie chart = fraction×360°. Inter-quartile range: a measure of *how spread out* the measures are. Inter-quartile range = **Upper** quartile (Q3) - **Lower** quartile (Q1). (Note: Q2 is the *median*).

Relative Frequency

If, after 10 throws of a coin, **seven** throws yielded a head, then the *relative frequency of heads so far* is ${}^{7}/_{10} = 0.7$. If **another** 10 throws are carried out, and 5 heads are obtained, then the *relative frequency of heads so far* is now ${}^{12}/_{20} = 0.6$. Now draw a graph, plotting throws on the **bottom** and relative frequency **up**. Draw the line of best fit to find the *approximate estimated probability* of obtaining a head in any throw.

Rational (Ratio) numbers can be written as **fractions**. Types of Decimals. *Finite*: 0.75, 0.5, 0.125. *Recurring*: $^1/_3$, $^2/_3$, $^1/_9$. *Repeating*: $^1/_7 = 0.\dot{1}4285\dot{7}$. **Irrational** numbers are not finite, and do not *repeat* of *recur*. Examples: π, √2, ... Simplifying **surds**: $\sqrt{(3)}\times\sqrt{(6)}+\sqrt{(72)} = \sqrt{(18)}+\sqrt{(72)} = \sqrt{(9)}\times\sqrt{(2)}+\sqrt{(36)}\times\sqrt{(2)} = 3\sqrt{(2)}+6\sqrt{(2)} = 9\sqrt{(2)}$.

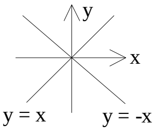
Transformations, Rotations and Reflections



A **transformation** is a sliding movement

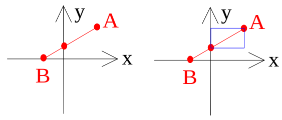
$\begin{pmatrix}2\\1\end{pmatrix}$ 2 units parallel to Ox; -1 unit parallel to Oy

A **reflection** is where a shape is reflected in a line which (on a set of *cartesian co-ordinates*) has a definite **equation**



Both images must be the same distance from the "mirror", which may be e.g. x = -1. Label the *second* image as A', B', C'; or A_1, B_1, C_1

Question: reflect the line from A(2,3) to B(-1, 0) around the point (0,1) by 180°.



Answer: think of a ***box*** around A and the point (0,1); rotate the box to get A's new position; do the same with B; and then *join up the two new points* to get the new line.

Congruent Figures: two figures with exactly the *same shape and size* are called congruent figures. Two figures with the *same shape* but *different size* are called **similar** figures. Remember that when stating the *number of rotational symmetry's*, include the one you **start** with. For example, the letter **Z** has **two** rotational symmetry's, not **one**.

Perimeter = (length + breadth)×2. **Area** = length × breadth. **Volume** = length × breadth × height. **Density** = Mass (kg) ÷ Volume. m.p.h. = m.p.m. × 60. *3-D co-ordinates* = (x, y, z). Do them in *this* order. **Quadratic Formula**: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Example: in x^2+6x+3, a = 1, b = 6 and c = 3. There are <u>two</u> answers to the quadratic formula — you split the answer up after reaching a certain stage in your calculations (the ± bit). In questions, look for *brackets* to multiply out of, *factorising* to be done, *diagrams* to be drawn, etc. **Remember** that $(-1)^2 = 1$, and that -(-2) = 2.

Simultaneous Equations: 1 Quadratic, 1 Linear

Consider $x^2+3y^2 = 21$ (---(1)) and x+y = 5 (---(2)). In this situation, express either x or y in terms of the **other** — here choose x. (It is easier to substitute for x^2 rather than for $3y^2$). From (2), x = 5-y. Substitute 5-y for x in (1), and then *solve* the equation.

The Sine and Cosine Rules

Consider that the **lengths** of a triangle are denoted by a, b and c; and that the **angles** opposite these lengths are denoted by A, B and C. The **Sine** rule is $^a/_{\text{Sin A}} = {}^b/_{\text{SinB}} = {}^c/_{\text{Sin C}}$; and the **Cosine** rule is $a^2 = b^2+c^2-2bc(\text{Cos A})$, or $\text{Cos A} = {}^{(b^2+c^2-a^2)}/_{2bc}$.

Choosing which formula to use: in an isosceles triangle, divide it into 2 right angled triangles and use a *right-angle triangle method*; if 2 angles and a side are given, use the *Sine rule* (finding the **third** angle first if necessary); if 2 sides and the included angle is given, the *Cosine rule* gives the side opposite the angle; and if 3 sides are given, use the *Cosine rule* to obtain the angles.

“**Centre of enlargement**”: when dealing with similar shapes, the centre of enlargement is the *light source* — join the corresponding dots with lines, carry the lines on — and they should meet at a *point* called the light source. A shape is **similar** if: (i) the *three angles are the same* in both shapes; (ii) we have *one pair of equal angles*, and the sides *containing these angles* are equal.

Histograms with **different class widths**: the *area* of the rectangle is proportional to the *frequency*. Height of a column (frequency density) = Frequency ÷ Class Width. Work out the frequency density to **correctly** plot Histograms with uneven class widths. Cumulative Frequency Curve: if you have e.g. $-30 \leq P < 50$, **start** at 30.

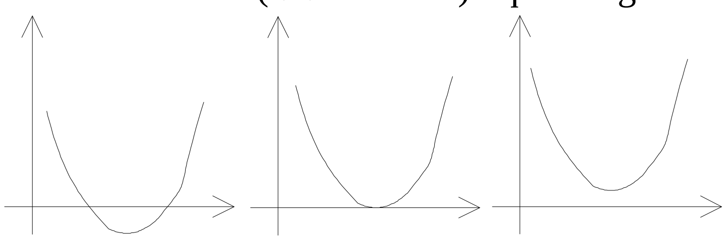
To get the *mean from a frequency table*, use the formula Mean ($\bar{x}$) = $^{\Sigma fx}/_{\Sigma f}$, where Σ = ‘sigma’. The **most used** measure of spread (dispersion) is given by the *following* formula: $\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+...+(x_n-\bar{x})^2}{n}}$. If the inter-quartile range is *lower*, the values are *less spread out*. **Number** of standard deviations = range ÷ standard deviation.

calls/min	**f**	**x**	**fx**	**$(x-\bar{x})^2$**	**$f(x-\bar{x})^2$**	**cum. f**
0-4	1	2	2	309	309	1
5-9	2	7	14	158	316.5	3
10-14	15	12	180	57.5	861.8	18
15-19	42	17	714	6.7	279.6	60
20-24	45	22	990	5.86	263.5	105
25-29	16	27	432	55.1	880.9	121
30-34	3	32	96	154.3	462.8	124
35-39	0	37	0	303.5	0	124
Σ	124		2428	1049.96	3374.1	

mean = 2428 ÷ 124 = 19.58	s.d. = √(3374.1 ÷ 124) = 5.2

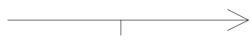
Many distributions, such as population heights, IQ’s, etc. follow a *normal distribution*, which has a **bell-shaped** graph. 67% of the values lie within ±1 s.d. of the mean. ±2 s.d.’s: 95%; ±3 s.d.’s: 99.7%. The normal distribution is *symmetrical* about the mean.

Completing the square: for a perfect square, (e.g. $(x+3)^2 = x^2+6x+9$), half the coefficient of x (i.e. 6÷2 = 3) squared gives the *constant* term. Note: in ax^2+bx+c, a is the *coefficient of x^2*; b is the *coefficient of x*; and c is the *constant term*. Make an expression into a perfect square by finding $(b÷2)^2$ and **adding** it onto both sides. Example: consider $x^2-2x+3 = 0$, or $x^2-2x = -2$. Here, $(b÷2)^2$ is 1, so add 1 to both sides, giving $x^2-2x+1 = -1$; $(x-1)^2 = -1$.



2 solutions, 2 roots 2 solutions, same root No roots

A body moves from A — 10m to the **right** and then 15m to the **left**. We have travelled 25m, but the *displacement from A* is 5m. Remember that when working with hours, use **decimals**. Velocity has *magnitude* and *direction*. If <u>either</u> of these change, then the velocity changes.



A

Speed is a scalar quantity while **velocity** is a vector quantity. The gradient gives the *velocity* in a displacement/time graph. Draw a tangent to the curve to find the gradient. In a *velocity/time* graph, the gradient gives the acceleration, while the area gives the distance travelled. To work out the **area** under the curve, split the graph up into sections; join the sections up with straight lines; and work out the *areas* of the segments and add them up.

Rationalising the denominators: multiply throughout with the irrational number on the bottom, remembering to *inverse* it when necessary to produce the difference of two squares. To write a *repeating/recurring* decimal as a fraction, multiply the number by **two** different powers of ten so that when you take the difference of the two numbers, you get a *whole number*.

For example, consider the *repeating number* $0.\dot{3}$. Multiply by 100 to get $100\times0.\dot{3} = 33.\dot{3}$. Multiply by 1 to get $1\times0.\dot{3} = 0.\dot{3}$. Now take the **second** number away from the **first** number to give $99\times0.\dot{3} = 33$, so that $0.\dot{3} = {}^{33}/_{99} = {}^{1}/_{3}$, as expected. Another example: consider the repeating number $0.\dot{0}3245\dot{1}$. Multiply by 1,000,000 to give the answer $32451.\dot{0}3245\dot{1}$. Multiply by 100 to give the answer $3.\dot{0}3245\dot{1}$. Note: we choose *these two multipliers* (1,000,000 and 100) so that the repeating parts are "underneath each other" when we come to do the subtraction. Therefore, performing the subtraction, $0.\dot{0}3245\dot{1}\times999{,}900 = 32448$, *so that* $0.\dot{0}3245\dot{1} = {}^{32448}/_{999900}$.

Question: give two **irrational** numbers between 3 and 4. Answer: π, and anything between √9 and √16, for example √10 or √13. Two numbers are **co-prime** if they have *no* common factors.

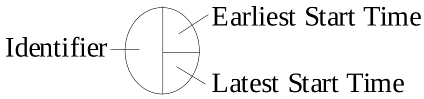
Sequences *converge* towards a limit or *diverge*. Rearrange the formula to give an **iterative** formula; put an approximate value in the formula; and iterate until both *starts* and *finishes* are the same. To get the starting value off other graphs, put in values for x, and find out where it **crosses** the x-axis — between a *positive* and *negative* value. In graphs of inequalities, the minimum/maximum cost obtainable occurs at the **corners** of shaded areas. Slide the objective function *towards the shaded area* — the first vertex you hit will be the optimal vertex.

Matrices are *regular arrays* arranged in rows and columns. You can **add** and **subtract** matrices only if they are of the **same** size. You can multiply matrices by a number, or by another compatible matrix. Order: $\begin{pmatrix}2&4&2\\4&2&6\end{pmatrix}$ is a 2×3 matrix. Addition: $\begin{pmatrix}1&2&4\\3&2&6\end{pmatrix} + \begin{pmatrix}2&4&3\\1&8&6\end{pmatrix} = \begin{pmatrix}3&6&7\\4&10&12\end{pmatrix}$. To *multiply* two matrices, there must be the **same** number of columns in the *left hand matrix* as there are rows in the *right hand matrix* — this is the '*domino*' effect. For $\begin{pmatrix}4&8\\1&2\end{pmatrix}\begin{pmatrix}-1&2&4\\3&5&3\end{pmatrix}$, we have orders 2×2 and 2×3. The inner numbers must be the **same** so we can *perform* the multiplication. The outer numbers give the **size of the product** matrix. (Here, a 2×3 matrix). Matrices can be used to *represent transformations* (**reflections**: x-axis, y-axis, y = x, y = -x, etc.; **rotations**: +90° (anticlockwise), -90° (clockwise), 180° (reflection in the origin); and **translations**).

Example: $\begin{pmatrix}4&8\\1&2\end{pmatrix}\begin{pmatrix}-1&2&4\\3&5&3\end{pmatrix} = \begin{pmatrix}(4\times-1)+(8\times3) & (4\times2)+(8\times5) & (4\times4)+(8\times3)\\(1\times-1)+(2\times2) & (1\times2)+(2\times5) & (1\times4)+(2\times3)\end{pmatrix} = \begin{pmatrix}20&48&40\\5&12&10\end{pmatrix}$.

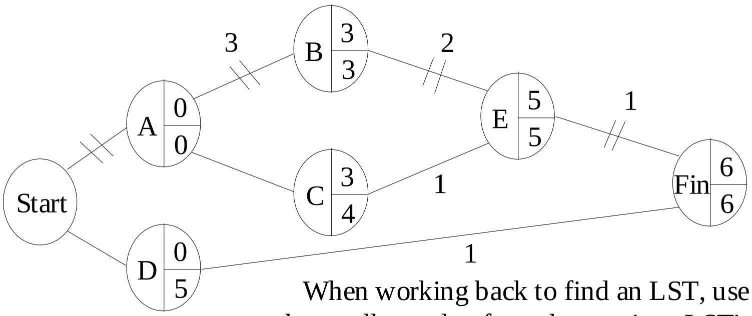
Variation (*Direct* and *Inverse*). If F varies as m_1 and m_2, and *inversely* as the square of d, then $F \propto (m_1m_2)\div d^2$. **Example**: if $y \propto (xz)\div t^2$, then $y = (kxz)\div t^2$, where k is a *constant*. If y = 100 when x = 25, z = 2 and t = 1, then $100 = (k\times25\times2)\div1^2 = 50k$; k = 2.

Critical Path Analysis

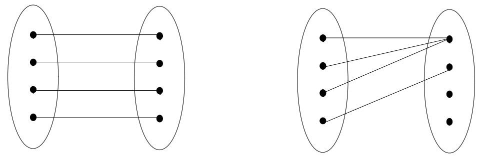


Arranging a Party

Task	*Preceded by*	*Time (Days)*
A Invite Friends	Nothing	3
B Buy Food	A	2
C Buy Drink	A	1
D Organise Music	Nothing	1
E Final Preparations	A, B, C	1



When working back to find an LST, use the smallest value from the previous LST's

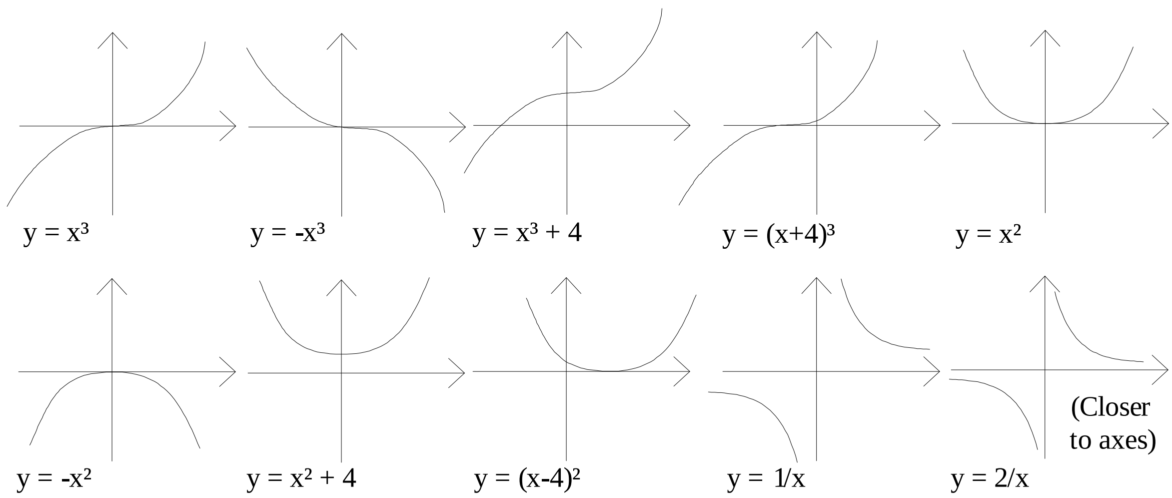


Mappings: all elements of the *first* set (the <u>domain</u>) must be used up. Each element may be mapped only **once** from the domain. In the diagrams shown on the right, the dots in the *left hand side oval* represent the domain; the *lines* represent the function; and the dots in the *right hand side oval* represent the range. If you have a graph of a function, and there is **more than one** value for a value of x (i.e. more than one f(x) for a particular x), then the function is *not* a valid function.

Area of a triangle = ½ab(Sin C). When sampling, obtain a *representative* sample. A **biased** sample is not a representative sample. For **random** sampling, *random tables* or the *random button on the calculator* are often used. In **stratified** sampling, the population is divided into *groups* — and random samples are taken <u>from each group</u>.

Maximum error allowance = maximum value - minimum value. In critical path analysis, the **longest** path is the critical path. Standard deviation is a *measure of spread* — a <u>smaller</u> value for the standard deviation means that the values are <u>less</u> spread out.

Related Functions



If $f(x) = x^2$, then $f(x)+3 = x^2+3$; $f(3x) = (3x)^2 = 9x^2$; $f(x+3) = (x+3)^2 = x^2+6+9$; $3f(x) = 3\times x^2 = 3x^2$; and $f(-x) = (-x)^2 = x^2$. **Rewrite** x^2-6x+7 as $(x+a)^2+b$ to find the co-ordinate of the vertex of the graph. Example: $x^2-6x+7 = (x^2-6x)+7 = [(x-2)^2-9]+7 = (x-3)^2-2$. This is the *same* as $y = x^2$ being translated by (3,-2) — the vertex point is (3, -2).