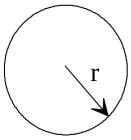
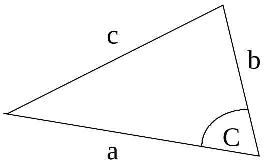
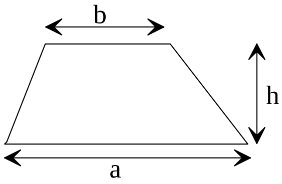
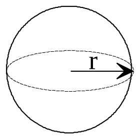
Pure Mathematics (Modules P1 and P2)

Shape and Space Formulae







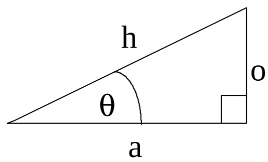


Area of a *Triangle:* A = $\frac{1}{2}$abSinC; or A = $\frac{1}{2}$ × base × height; or A = $\sqrt{s(s-a)(s-b)(s-c)}$, where s = $\frac{1}{2}$(a+b+c).
Area of a *Circle*: A = πr^2; Circumference: C = $2\pi r$.
Area of a *Trapezium*: A = $\frac{1}{2}$(a+b)h.
Area of a *Rhombus*: A = $\frac{1}{2}d_1d_2$.

Volume of a *Sphere*: V = $\frac{4}{3}\pi r^3$.
Volume of a *Cuboid*: V = Width × Breadth × Height.
Volume of a *Cone/Pyramid*: V = Base Area × Height.
Volume of any *Uniform Cross-section*: V = Area of cross-section × length.

Interior Angle of a *polygon* = $\frac{180(n-2)}{n}$; Exterior angle of a *polygon* = $\frac{360}{n}$.
Length of an *arc* = rθ, with θ in radians.
Area of a *sector*: A = $\frac{1}{2}r^2\theta$. Area of *Triangle*: A = $\frac{1}{2}r^2\text{Sin}\theta$.

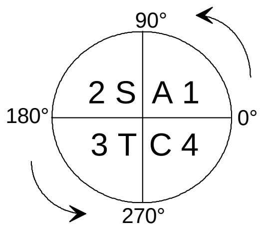
Trigonometry



In a *right-angled* triangle, normal trigonometric ratios apply, i.e.
Sinθ = $\frac{o}{h}$; Cosθ = $\frac{a}{h}$; Tanθ = $\frac{o}{a}$. Also, *Pythagoras'* rule applies: $h^2 = a^2 + o^2$.
In *non-right-angled* triangles, use the *Sine* and *Cosine* rules:
(*Sine*) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; (*Cosine*) $a^2 = b^2+c^2-2bc(\cos A)$, *or* $\cos A = \frac{b^2+c^2-a^2}{2bc}$.
With **acute** angles (up to 90°), Sinθ = Cos(90°-θ); Cosθ = Sin(90°-θ); and Tanθ = $\frac{\sin\theta}{\cos\theta}$.
With **obtuse** angles (90° to 180°), Sinθ = Sin(180°-θ); Cosθ = -Cos(180°-θ); and Tanθ = -Tan(180°-θ).

Trigonometric Identities

Angle	Sin	Cos	Tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞



cosecθ = $\frac{1}{\sin\theta}$; secθ = $\frac{1}{\cos\theta}$; and cotθ = $\frac{1}{\tan\theta}$.
$\sin^2\theta+\cos^2\theta = 1$; (and *from this)*
$\sec^2\theta = 1+\tan^2\theta$; and $\text{cosec}^2\theta = 1+\cot^2\theta$.
Sin(A ± B) = Sin*A*Cos*B* ± Cos*A*Sin*B*.
Cos(A ± B) = Cos*A*Cos*B* ∓ Sin*A*Sin*B*.
Tan(A ± B) = $\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.
Sin2A = 2sinAcosA; **Cos2A** = $\cos^2A-\sin^2A = 1-2\sin^2A = 2\cos^2A-1$; **Tan2A** = $\frac{2\tan A}{1-\tan^2A}$.

Tips: Always **factorise** (never *divide through)*; Look out for the <u>ambiguous</u> case (2 angles when using the **Sine** rule); in $\cos^2\theta = \frac{3}{4}$, $\cos\theta = +\sqrt{\frac{3}{4}}$ **or** $-\sqrt{\frac{3}{4}}$ i.e. we have 4 answers.

Number Manipulation

Find the square root of a square number: **half** the powers when written as a product of its prime factors. To make a square number, multiply a number so that all of its factors have *even* powers. Q: What is the Highest Common Factor of 60 and 80? A: 60 = <u>2</u> × <u>2</u> × 3 × <u>5</u>; and 80 = <u>2</u> × <u>2</u> × 2 × 2 × <u>5</u>; so that the HCF is **2 × 2 × 5 —** the Common Factors.

Rules of **Indices**: $a^x \times a^y = a^{x+y}$; $\frac{a^x}{a^y} = a^{x-y}$; $(a^x)^y = a^{x \times y}$; $a^{-x} = \frac{1}{a^x}$; $a^0 = 1$; and $a^{\frac{b}{c}} = \sqrt[c]{a^b}$.

Rules of **Logs**: $\log_c(ab) = \log_c a + \log_c b$; $\log_c(\frac{a}{b}) = \log_c a - \log_c y$; $\log_c a^n = n\log_c a$; and $\log_a a = 1$.

If $\log_m c = d$, then $m^d = c$. *Change* of base: Suppose that $y = \log_b a$, so that $a = b^y$. Now $\log_c a = \log_c b^y$, and $\log_c a = y\log_c b$. Therefore, $y = \frac{\log_c a}{\log_c b}$, i.e. $\log_b a = \frac{\log_c a}{\log_c b}$.

If c = a, then $\log_b a = \frac{\log_a a}{\log_a b}$, i.e. $\log_b a = \frac{1}{\log_a b}$.

Sequences. (1) *Arithmetic Progression* (AP). n^th term: [a+(n-1)d]. Sum: $S_n = \frac{n}{2}[2a + (n-1)d]$.

(2) *Geometric Progression* (GP). n^th term: ar^{n-1}. Sum: $S_n = \frac{a(1-r^n)}{(1-r)}$ when **r < 1,** and $S_n = \frac{a(r^n-1)}{(r-1)}$ when **r > 1**. When **|r| < 1**, the series is *convergent*, and the sum to infinity is $S_\infty = \frac{a}{1-r}$.

Equation Solving

Methods: Common factor; Difference of two squares; Factorising a Quadratic; Quadratic formula; Completing the Square; Simultaneous Equations; and Factorisation by Long Division.

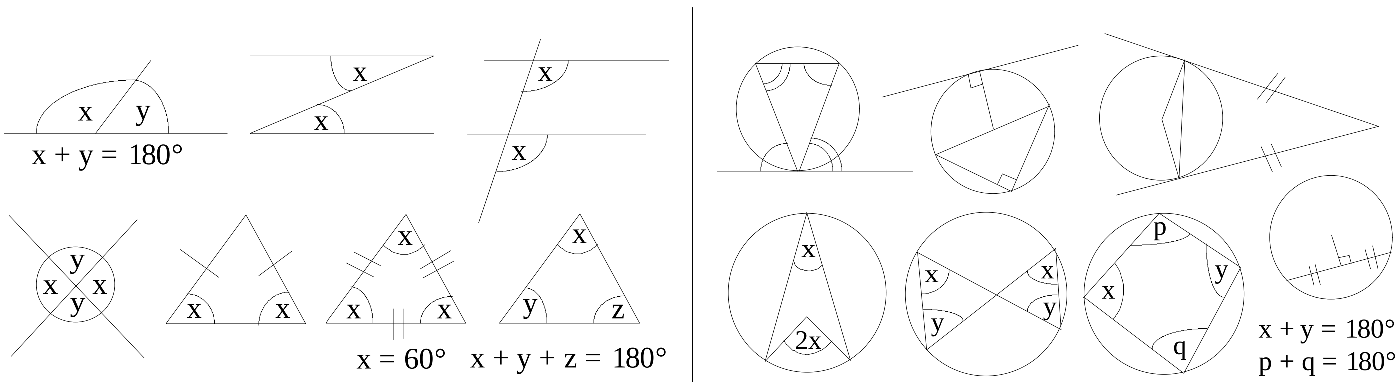
Techniques: $x^2+2x = x(x+2)$; $4x^2-9 = (2x+3)(2x-3)$; and $x^2+6x+8 = (x+2)(x+4)$. If $ax^2+bx+c = 0$, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. ($b^2$-4ac is the *discriminant).*	*Factor* theorem: If f(a) = 0, then (x-a) is a perfect factor of f(x). For example, in $f(x) = x^2+6x+8$, we have f(-2) = 0, so that we can do the following division: $$\begin{array}{r\|l} & x+4 \\ \hline x+2 & x^2+6x+8 \\ & x^2+2x \\ \hline & 4x+8 \\ & 4x+8 \end{array}$$	Completing the *Square*: $3x^2-2x-2 = 0$; $3(x^2-\frac{4}{3}x)-2 = 0$; $3((x-\frac{2}{3})^2-(\frac{2}{3})^2)-2 = 0$; $3(x-\frac{2}{3})^2 = \frac{10}{3}$; $(x-\frac{2}{3})^2 = \frac{10}{9}$; $x = \frac{2}{3} \pm \sqrt{\frac{10}{9}}$.

Functions: If f(x) = 2x+3, and g(x) = x^2, when performing gf(x), perform f(x) first, and then do g(result of f(x)). To find $f^{-1}(x)$, define y = f(x); so that y = 2x+3, and then *rearrange* for x, i.e. $f^{-1}(x) = \frac{x-3}{2}$. We have an **Odd** function when f(-x) produces -f(x). We have an **Even** function when f(x) is the same as f(-x). *Binomial Theorem*: $(1+x)^n = 1^n + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + x^n$. Or, $(1+x)^n = \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + ... + x^n$.

Co-ordinate Geometry

Given two points on a graph, **A(x_1,y_1)** and **B(x_2,y_2)**, we can find things from this information:
Length AB = $\sqrt{(x_2 - x_1) + (y_2 - y_1)}$; Midpoint AB = $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$; Gradient AB = $\frac{y_2-y_1}{x_2-x_1}$; and Equation AB = $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$. We can also use $y-y_1 = m(x-x_1)$, where **m** is the gradient.
The gradient of the *normal* is the <u>negative reciprocal</u> of the *tangent* gradient.

Angles; Circle Theorem



A circle with equation $x^2 + y^2 = r^2$ has centre **(0,0)** and radius **r**.
A circle with equation $(x-a)^2 + (y-b)^2 = r^2$ has centre **(a,b)** and radius **r**.
A circle with equation $x^2 + y^2 + 2fy + 2gx + c = 0$ has centre **(-g,-f)** and radius $\sqrt{f^2 - g^2 - c}$.
Differentiate *implicitly* to find the tangent. Intersection of a line in 2 places: we have a chord.
Intersection of a line in 1 place: we have a tangent.
Circle through 3 points: substitute into $x^2 + y^2 + 2fy + 2gx + c = 0$, and solve for f, g and c.
Intersecting circles: **externally**: $d = r_1+r_2$; **internally**: $d = r_1-r_2$; cut at **right angles**: $d^2 = r_1^2+r_2^2$.

Inequalities; Partial Fractions

When **multiplying/dividing through** by a negative number, *change* the sign. For $-7 \le 2x-3 \le 7$, solve by dividing into **2** inequations: $-7 \le 2x-3$, and $2x-3 \le 7$; and get a range.

	Denominator	Example		Expression Used
1	**Linear** Factors	$\frac{5}{(x-2)(x+3)}$	$\equiv$	$\frac{A}{x-2} + \frac{B}{x+3}$
2	**Quadratic** Factor (Doesn't factorise)	$\frac{2x+3}{(x-1)(x^2+4)}$	$\equiv$	$\frac{A}{x-1} + \frac{Bx+C}{x^2+4}$
3	**Repeated** Factor	$\frac{5x+3}{(x-2)(x+3)^2}$	$\equiv$	$\frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

Solve *modulus* inequations (e.g. $|x-2| \le 5$) by using 2 inequations as follows: $x-2 \le 5$, and $x-2 \ge -5$.

Solve **Partial Fraction** problems as **follows**:

To find the **coefficients** A, B, C, etc.,
either **substitute in** values for *x*,
or **equate** the coefficients of *x* on either side of the equation.

Differentiation; Integration

If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$. For f(x), *minimum / maximum / inflexion* points occur at f'(x) = 0. To check what these are, either check the values on either side of f'(x), e.g. +ve to -ve gives a **maximum** point; or *substitute* into y'' or $\frac{d^2y}{dx^2}$: y'' > 0 gives a **minimum** point; y'' < 0 gives a **maximum** point; and y'' = 0 gives an **inflexion** point.

Chain Rule: Defining something as u, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. Small changes: $\delta y \approx \frac{dy}{dx}\delta x$.

If $\frac{dy}{dx} = ax^n$, then y = $\frac{ax^{n+1}}{n+1} + c$, provided that n ≠ -1. Find the area under a curve between x = a and x = b as follows: use $\int_a^b f(x)dx$. Definite Integrals: $\int_0^3 x^2 - 3xdx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$.

If y = $[f(x)]^n$, then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$. **Integrate** f'(x)[f(x)] by *inspection*.

Product rule: If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. **Quotient** rule: If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

If y = $e^{f(x)}$, then $\frac{dy}{dx} = f'(x) \times e^{f(x)}$. If y = ln[f(x)], then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$. Integrate both by inspection.

When integrating *expressions* of the type $\frac{f'(x)}{f(x)}$, the **answer** you get is of the form ln|f(x)| + c.

f(x)	sin*x*	cos*x*	tan*x*	cot*x*	sec*x*	cosec*x*	$\sin^{-1}x$	$\cos^{-1}x$	$\tan^{-1}x$
f'(x)	cos*x*	-sin*x*	$\sec^2 x$	$-\text{cosec}^2 x$	sec*x*tan*x*	-cosec*x*cot*x*	$\frac{1}{\sqrt{(1-x^2)}}$	$\frac{-1}{\sqrt{(1-x^2)}}$	$\frac{1}{1+x^2}$

If y = sin2*x*, then y' = 2cos2*x*. If y = cos(x^2+1), then y' = -2*x*sin(x^2+1).
If y = $\sin^3 2x$, then y' = $3\sin^2 x\cos 2x \times 2$.
When *integrating*, inspect and change as necessary. Always work in **Radian** mode.
If y = $\sin^{-1}\frac{x}{a}$, then $\frac{dy}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$. If y = $\tan^{-1}\frac{x}{a}$, then $\frac{dy}{dx} = \frac{a}{a^2+x^2}$.

Integration by **Substitution**: $\int f(x)dx = \int f[g(u)]\frac{dx}{du}du$.
Integration by **Parts**: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Note: $\int a^x dx = \frac{a^x}{\ln a} + c$.
Write $\int \sec x dx$ as $\int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}dx = \ln|\sec x + \tan x| + c$.
Similarly, write $\frac{\text{cosec}\, x(\text{cosec}\, x - \cot x)}{(\text{cosec}\, x - \cot x)}$ for cosec*x*.

Parameters; Implicit Functions

For a **cartesian** equation, it is sometimes useful to use a *third* variable — and express *x* and *y* in terms of this third variable. x = f(t) and y = f(t) are parametric equations of the line, for example x = 2t-1 and y = $1-t^2$. *Rearrange* and *substitute* to get rid of the *t*. Use the **chain rule** to get $\frac{dy}{dx}$.

Take care with $\frac{d^2y}{dx^2}$: differentiate *with respect to x*: $\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \times \frac{dt}{dx}$.

Implicit functions do not define **clearly** y = f(x), for example $y^2-2yx = x^2$. Differentiate each *term* in turn. Example: $y^2+4x = 6x^2$. *Differentiating*, $2y\frac{dy}{dx} + 4 = 12x$; $\frac{dy}{dx} = \frac{12x-4}{2y}$.

Differential Equations

An **equation** of the form $f(y)\frac{dy}{dx} = g(x)$ can be solved by *separating the variables* and integrating both sides. Use initial conditions to get rid of constants, and '*accumulate*' constants, for example use ln*A*. Example: Solve $\frac{dy}{dx}$+5 = 6x. (Initial Condition: x = 1 when y = 2). *Separating* the variables, dy = 6x-5.dx. Integrating **both** sides gives y = $3x^2-5x+c$. Substituting in the *initial condition* gives c = 4, and therefore y = $3x^2-5x+4$.

Mechanics (Module M1)

A *scalar* quantity has **magnitude** only. Examples: mass, temperature, distance, time, speed. A *vector* quantity has both **magnitude** and **direction**. Examples: force, displacement, velocity. For *constant* speed or velocity, distance/displacement = speed/velocity × time. **Graphical** representation: in a *velocity-time* graph, the gradient gives the *acceleration* while the area under the graph gives the *distance* travelled. To get the area under the graph, use the **trapezium** rule, ½(a+b)d, where a and b are the *lengths* of the parallel sides, and d is the *height* of the trapezium.

The Equations of Motion

Let u = initial velocity; v = final velocity; a = acceleration; t = time; and S = displacement. The equations of motion are as follows: (1) $v = u+at$. (2) $S = (u+v)t/2$. (3) $S = ut+\frac{1}{2}at^2$. (4) $v^2 = u^2+2as$. For vertical motion under *gravity*, $g = 9.8m/s^2$ unless otherwise told. With questions, write down what you **know** and what you **don't know**, and then find the **equations** that give you the **unknowns**. When dealing with two objects e.g. thrown upwards within one second of each other, remember that 't' for the *second* object is (t-1). Conversions: $1km/h = {}^{1000}/_{3600}ms^{-1}$; and $1ms^{-1} = {}^{3600}/_{10000}km/h$.

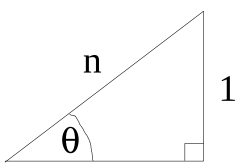
Forces and Acceleration

Conventions: light strings have *no weight*; smooth pulleys have *no friction*; and rough pulleys *do have friction*. **Forces** can be W (*Weight*); F (*Friction*); T (*Tension*); P (*Push/Thrust*); and R (*Normal* (90°) *Reaction*). Draw force diagrams when dealing with questions; consider **resultants**; and show **all** forces on a diagram.

Resolving Forces

When finding a *resultant*, use **trigonometry**; **pythagoras' rule**; and the **Sine** and **Cosine** rules. Always quote the magnitude *and* direction of a resultant. Resolve in mutually perpendicular directions. **Tip**: remember that "*cos contains the angle*". Try to leave answers in **Surd** form. If a system is in equilibrium, the sum of the forces in any direction is *vectorially zero*. Newton's **Second** Law: F = ma. Questions commonly involve the use of **pulleys** and **lifts**.

Motion on an Inclined Plane



An inclination of "*1 in n*" means a vertical rise of 1 unit for every 'n' units along the slope. This gives $\sin\theta = {}^{1}/_{n}$. Always consider the resultant force in F = ma. In questions, consider which **part** of the system of forces you are dealing with at any one time. When sliding does **not** take place on a slope, frictional forces act of the system. The *frictional force* is proportional to the *normal reaction*, so that $F \propto R$, or $F = \mu R$, where μ is the coefficient of friction.

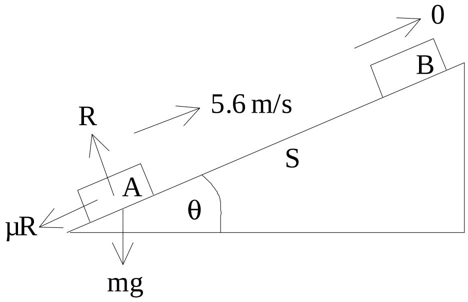
Work, Energy, Power

The *work done* by a constant force is defined as the **product** of the constant force and the distance through which the point of application moves. Along a frictionless horizontal surface, Work Done = F×S. On a slope of angle θ, the work done in moving a distance S up the slope is given by Fcosθ×S.

Kinetic Energy is the energy possessed by virtue of *motion*. K.E. = ½mv². **Potential Energy** is the energy possessed by virtue of *position*. P.E. = mgh. *Vertically*, Loss in P.E. = Gain in K.E. + Work done against resistances.

Power is the rate at which work is being done (the work done per second). 1 joule/second = 1 Watt (W). Typical questions involving power include measuring the *volume of water passing through a water pump*.

Worked Example: A plane is inclined at $\tan^{-1} {}^{4}/_{3}$. A mass is projected up the plane at 5.6m/s, and stops at a point B. The *coefficient of friction* of the plane is $\mu = {}^{4}/_{7}$. By energy considerations, find the distance AB.



Answer: Looking at the *diagram*, $\tan\theta = {}^{4}/_{3}$, $\sin\theta = {}^{4}/_{5}$, and $\cos\theta = {}^{3}/_{5}$. The Loss in K.E. is ½m(5.6)²-½m(0)² = 15.68*m* J. The Gain in P.E. is mgh = m×9.8×sSinθ = m(9.8)s(${}^{4}/_{5}$)= 7.84*ms* J. Resolving in the *perpendicular* direction to the plane, R = mgCosθ. Therefore, the *frictional* force μR is μmgCosθ = ${}^{4}/_{7}$m(9.8)(${}^{3}/_{5}$) = 3.36*m* N. The work done *against friction* is given by μR×s = 3.36*ms* J. **But**, Loss in K.E. = Gain in P.E. + Work done against friction; so 15.68m = 7.84m + 3.36ms; 15.68 = 11.2s; s = 1.4m. **Conclusion**: the distance AB is 1.4m.

Vehicles in Motion

If an *engine* is working at a rate of P watts, then P = work done per second = Force × Distance moved per second; P = FV; $F = {}^{P}/_{V}$. This is the **Tractive force**. Remember to work with SI units, e.g. use ms^{-1} instead of km/h, etc.

Momentum & Impulse

The **momentum** of a body is given by m×v. **Unit**: Newton Seconds. The **Impulse** of a constant force F acting for a time t is F×t. Impulse is also given by the *change in momentum*, mv-mu. If there is **no** external force acting on a system of bodies in a particular direction, the total momentum of the system in that direction remains *constant*.

Newton's Empirical Law

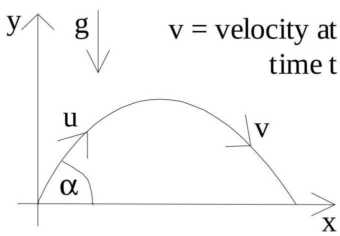
When two bodies collide directly, the relative velocity **after** collision bears a constant ratio to the velocity **before** collision, and is in the *opposite* direction. Therefore, $\frac{V_A - V_B}{U_A - U_B} = e$, or V_A-V_B = -e(U_A-U_B), where e is the *coefficient of restitution*, depending only upon the **material** of which the two bodies are made out of. e = 1: the bodies are *perfectly elastic*; e = 0: the bodies are *perfectly inelastic*. **Glass** balls: e = 0.9; **Lead** balls: e = 0.2. In collision questions, use the *conservation of momentum* and *Newton's Empirical Law* to give **two equations** which you can solve **simultaneously**. If a ball falls onto a *rough* surface, or hits normally a *fixed vertical wall*, and its velocity just **before** impact is 'u', then its velocity immediately **after** impact is 'eu'.

Impulsive Tensions in Strings

When a string *jerks*, equal and opposite **tensions** act suddenly on each end, so that equal and opposite **impulses** act on the object at each end. If one end of the string is fixed, then the attached end is *not affected* by the jerk, and the moveable object at the free end undergoes a <u>change in momentum</u> equal to the *impulsive tension*. If *neither* end is fixed; if *both* ends are attached to a moveable object; and if the two objects attached to the string are given *velocities in opposite directions* away from each other, then the string becomes taut, and equal and opposite **impulses** act on the objects, producing equal and opposite changes in **momentum**.

Projectiles

Definition: a particle given an initial velocity then moves freely under gravity (*negligible air resistance*). If the initial velocity is vertical, it moves in a <u>straight line</u>. If the initial velocity is not vertical, then the particle moves <u>parabolically</u>, and the horizontal and vertical motions are considered *separately*.



Vertical motion: subject to the force of *gravity*, g; and the *equations of motion* are used. **Horizontal** motion: the velocity is *constant*, and **no** force acts horizontally. Tips: to find the *greatest height*, remember it "stops" here so that v = 0. To find *the time of flight*, use $S = ut - \frac{1}{2}gt^2$, where S = 0 (it reaches the ground again). Obtain *horizontal distances* using S = Vt, and remember that the time to travel a distance is the **same** horizontally and vertically.

	Horizontal	**Vertical**
u	ucosα	usinα
a	0	-g
v	ucosα	usinα-gt
S	(ucosα)t	$(u\sin\alpha)t - \frac{1}{2}gt^2$

The **range**, R, is the total *horizontal* distance travelled. The equation of trajectory is given by $y = x\tan\alpha - gx^2\left(\frac{1+\tan^2\alpha}{2u^2}\right)$. It is a **quadratic equation** in tanα.

Light Strings and Springs

For **light springs**, it is assumed that the *tension is constant* throughout. A **spring** behaves like an elastic string but it may also be *compressed*. **Hooke's Law:** $T \propto x$, or $T = \frac{\lambda x}{l}$, where l = natural length, x = extension, and λ = the modulus of elasticity.

Work Done and/or Energy Stored for Elastic Springs/Strings

The *work done* in extending a spring from 0 to x is the area under the graph between S = 0 and S = x, and is given by $^{\lambda x^2}/_{2l}$. The work done *against tension* in increasing the extension of an elastic spring from a to b is given by $^{\lambda}/_{2l}(b^2-a^2)$. Since energy is the capability of *doing* work, the **energy stored** in an elastic spring/string is given by $E = {}^{\lambda x^2}/_{2l}$.

Rectilinear Motion

The *position* of a particle at any time is determined by it's x co-ordinate (the **displacement** from the origin). In a *displacement-time* graph, the **slope** at any time shows the instantaneous **velocity**. *Acceleration* is the rate of change of velocity w.r.t. time. It is also the *slope* of a velocity-time graph. The **opposite** of acceleration is deceleration/retardation.

Impulsive Tensions

An impulsive tension in a string is the result of a *jerk*, and acts in equal and opposite directions down the string. Two situations: (1) One end is fixed: the momentum changes in the direction of the **string**; (2) Both ends are attached to moveable objects: the impulse is the tension in the string, and acts in the **line** of the string. Total applied impulse: I = u(m+M). *Notation* used: I = *Impulse*; J = *Impulsive tension*; u = *velocity 'before'*; and v = *velocity 'after'*.

Equilibrium and Moments

A system of forces is in **equilibrium** if (a) the *resultant force* is zero; and (b) the *resultant moment* is zero. A force may be resolved into two **perpendicular** components. Important Note: *Anticlockwise* moments are taken as positive; and *clockwise* moments are taken as negative.

General Tips

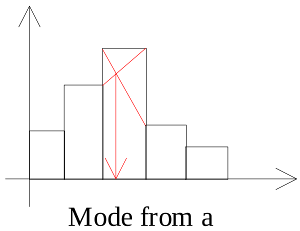
Leave answers **correct** to *two or three* decimal places, if not otherwise requested. Be careful with assumptions of *conservation of energy*, especially where a *variable* force is discussed — friction perhaps? Be careful also with run of the mill algebra — **double check** if time permits. Always draw *big* diagrams, *labelling* them correctly. **Read** through a question thoroughly before **attempting** it.

Statistics (Module S1)

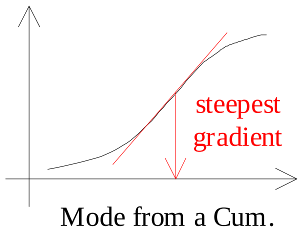
Chain of events: Obtain the **raw data** — **Sort** (frequency tables) — **Analyse** (Mean, Mode, etc.) — **Display** (Pie Chart, Histogram, etc.) — Draw **conclusions** (trends) — **Predict**. In Histograms with *uneven class widths*, height = frequency ÷ class width.

Measures of Central Tendency

Mode by calculation: an estimate from the *modal class*: the mode is given by $L+[\Delta_1/(\Delta_1+\Delta_2)]\times C$, where L is the *lower class boundary of the modal class*, Δ_1 is the *difference in frequency* between the modal class and the previous class, Δ_2 is the *difference* between the next class and the modal class, and C is the *width of the median class* from the cumulative frequency distribution (the **median** is the $\frac{1}{2}(n+1)^{th}$ value).



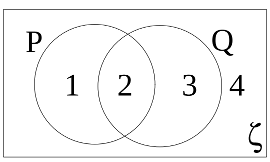
Mode from a Histogram



Mode from a Cum. Frequency Curve

Q_1 = the $\frac{1}{4}(n+1)^{th}$ value; Q_3 = the $\frac{3}{4}(n+1)^{th}$ value. The *Semi-interquartile range* is given by $\frac{1}{2}(Q_3-Q_1)$. The i^{th} percentile is given by $(^{i}/_{100})\times(n+1)^{th}$ value. If $\bar{x}$ is the arithmetic mean, then the **standard deviation** is given by the formula $S = \frac{\sqrt{\sum (x_n-\bar{x})^2}}{n}$, or $S = \sqrt{(\sum \frac{x^2}{n})} - \bar{x}^2$. Note: if each number is *increased by a constant*, c, the mean will **increase** by c — but the standard deviation will **stay the same**. If each number is *multiplied by* k, then **both** the mean and the standard deviation are **multiplied** by k as well.

Set Theory



1 = P∩Q'
2 = P∩Q
3 = Q∩P'
4 = (P∪Q)'

Definitions: ∈ = is a *member* of; ⊂ = is a *subset* of; ∩ = *intersection* of sets; ∪ = *union* of sets; ϕ or { } = the *empty* set; ζ = *everything*; n(a) or nA = the *number of elements* in A; and A' or Ā = the *complement* of A (everything outside A). Draw **Venn diagrams** to illustrate sets. An *example* is shown on the left.

Probability

The **sample space** contains *all* possible outcomes. If E is an event, then the probability of E is defined to be P(E) = n(E)÷n(S). **Results**: P(A') + P(A) = 1; P(A∪B) = P(A) + P(B) - P(A∩B); for *mutually exclusive* events, P(A∪B) = P(A) + P(B); *provided* that P(A) ≠ 0 and P(B) ≠ 0, we have P(A|B) = P(A∩B)÷P(B), or P(A|B)P(B) = P(B|A)P(A); if A and B are *independent*, we have P(A|B) = P(A) and P(A∩B) = P(A)P(B); for *exhaustive* events, we have P(A∪B) = 1; for *mutually exclusive* events, we have P(A∩B) = 0; and *Bayes' Theorem* is P(A|B) = P(B|A)P(A)÷P(B).

Permutations and Combinations

Order doesn't matter in a *combination*, but order matters in *permutations*. The number of arrangements of 'n' things consisting of 'p' things the same and 'q' things the same is given by $^{n!}/_{p!q!}$. **Definition**: $^nC_r = {}^{n!}/_{(n-r)!r!}$. Always consider whether you are dealing with *permutations* or *combinations*, and consider whether there are '*blocks*' of things in questions.

Discrete Random Variables (DRV's)

Discrete: only takes *particular values* (whole numbers). **Random**: *cannot* determine the outcome. **Variable**: can take *different* numerical values. If an outcome of a process can be stated as a number X, which can take *different values* $x_1, x_2, ..., x_n$ at random, then X is a DRV. The *probabilities* of different values must add up to **one**.

Expectation: E(X) = ΣP(X=x)×x. If g(x) is *any function* of the DRV X, then E(g(x)) = ΣP(X=x)×g(x). **Rules**: If a and b are constants, then E(a) = a, E(aX) = aE(X), and E(aX+b) = aE(X)+b. The **Variance** is given by $Var(X) = E(X^2) - [E(X)]^2$. **Rule**: $Var(aX+b) = a^2Var(X)$. *Cumulative Distribution Function* (CDF): $F(X) = \Sigma P(X=x_i)$ for $x_1, x_2, ..., x_i$. It *follows* that F(x) = P(X≤x). If X and Y are two **independent** random variables, then E(X+Y) = E(X) + E(Y), and Var(X+Y) = Var(X) + Var(Y).

Special DRV's

In a **Binomial Distribution**, there are *two* possible outcomes, and the events are random and independent. If *success* is denoted by 'p'; if *failure* is denoted by 'q' (so that q = 1-p); and if X denotes the random variable "*the number of successful outcomes in n independent trials*", then the probability density function (p.d.f.) is given by $P(X=x) = {}^nC_xq^{n-x}p^x$. If X follows a Binomial Distribution, we write X~Bin(n,p). Tips: P(X≥1) = 1 - P(X=0); in *surveys*, outcomes are independent of each other; *statistical tables* are "≤" tables, not "≥" tables; and P(X≤r | X~Bin(n,p)) = P(X≥n-r | X~Bin(n,1-p)). For a **Binomial Distribution**, E(X) = np, and Var(X) = npq.

The **Poisson Distribution** is generally used for *rare* events. $P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}$, where lambda is the mean, np. We write X~Po(λ) for a Poisson Distribution, where E(X) = λ and Var(X) = λ. If X and Y are *independent*, with X~Po(α) and Y~Po(β), then X+Y~Po(α+β).

Continuous Random Variables (CRV's)

A CRV is a *theoretical representation* of a **continuous** variable such as length, time or mass. It is specified by its p.d.f., written f(x), where $\int_a^b f(x)dx = 1$ (the p.d.f. is valid for the *range* a≤x≤b). Remember to use *geometry* to work out "*the area under the graph*" if possible. The **mean** of a CRV is given by $E(X) = \int_{\text{all } x} x\times f(x)dx$. The **Variance** of a CRV is given by $Var(X) = E(X^2) - E^2(X)$. The **Expected Value** of a **function** of x, g(x), is given by $E[g(x)] = \int_{\text{all } x} g(x)f(x)dx$.

If X is a CRV with p.d.f. f(x) *defined for* $a \le x \le b$, then the **Cumulative Distribution Function**, F(X), is defined by $F(X) = P(x \le t) = \int_a^t f(x)dx$. Be careful with *piecewise functions*. The **inter-quartile range**, as before, is $\frac{1}{2}(Q_3 - Q_1)$. Here, to find the *quartiles*, solve e.g. $\int f(x)dx = 0.5$ for x to get Q_2; = 0.25 to get Q_1; and = 0.75 to get Q_3. The **mode** occurs at the *highest value* of f(x). You can go "the *other* way", that is finding f(x) from F(x) by **differentiation**.

Special CRV's

The **Uniform Distribution** is represented by a *rectangle* on a graph, with constant value y = c in the range $a \le x \le b$. The p.d.f. is given by $f(x) = {}^{1}/_{(b-a)}$, with $E(X) = \frac{1}{2}(a+b)$, and $Var(X) = {}^{1}/_{12}(b-a)^2$.

The **Normal Distribution** is represented by $X \sim N(\mu, \sigma^2)$, where μ is the *expectation*, and σ^2 is the *variance*. Statistical **Tables** give the values for X~N(0,1). If you do *not* have this particular distribution, you can **convert** your normal distribution to this special distribution by using the *transformation* $Z = {}^{X-\mu}/_{\sigma}$, so that Z~N(0,1).

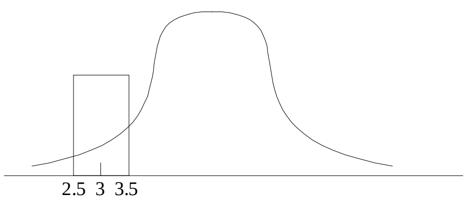
The Normal Distribution is represented by a *bell-shaped curve*, with the origin at the centre of the curve. Statistical tables give values for $P(X<n)$, so we must use '*tricks*' such as $P(x>n) = 1-P(X<n)$ to get e.g. $P(x>n)$. When you want to say that you've looked up a value in a **table**, use the character Φ, e.g. $P(Z<1.4) = \Phi(1.4) = 0.91924$. More complicated questions involve *working back*, e.g. given a **probability value**, find the **distribution**.

Approximations

In certain situations, we may *model* a distribution by **another** distribution. This is often useful when it is *difficult* to calculate the values for one distribution on a **calculator**.

	Distribution of X	**Restrictions** on the parameters	**Approximation**
1	X~Bin(n,p)	n large (say n > 50); p small (say p < 0.1)	X~Po(np)
2	X~Bin(n,p)	n > 10; p close to 0.5 or n > 30 (say); p moving away from 0.5	X~N(np,npq)
3	X~Po(λ)	λ > 20 (say)	X~N(λ,λ)

When *approximating* the Binomial Distribution by the Normal Distribution, we must apply a **continuity condition** — the Binomial graph is *discrete* and '*blocky*'. Consider the graph shown on the right. If we want $P(X \le 3)$, we need to find the probability of **all the block** as well as **all before it**, so $P(X \le 3)$ translates to $P(X<3.5)$ with the continuity correction. But if we require $P(X<3)$ to begin with, we **don't** count any of the block, so we use $P(X<2.5)$ when using the probability tables. Finally, if we require $P(X=3)$, we use $P(2.5<X<3.5)$. **Note**: when using a *Normal* approximation to a *Poisson* distribution, we must **also** here apply the continuity condition. Use *interpolation* for better accuracy.



Sum and Difference of Two Independent Normal Variables

If X and Y are *two independent normal variables* such that $X \sim N(\mu_1,\sigma_1^2)$, and $Y \sim N(\mu_2,\sigma_2^2)$, then $X+Y \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$, and $X-Y \sim N(\mu_1-\mu_2,\ \sigma_1^2+\sigma_2^2)$. Expectation of *more than two* independent normal variables: if $X_1, X_2, ..., X_n$ is **any** set of random variables, then $E(X_1+X_2+...+X_n) = E(X_1)+E(X_2)+...+E(X_n)$. If the random variables are *independent*, then $Var(X_1+X_2+...+X_n) = Var(X_1)+Var(X_2)+...+Var(X_n)$.

If $X_1, X_2, ..., X_n$ are n *independent* normal variables, such that $X_1 \sim N(\mu_1,\sigma_1^2)$, $X_2 \sim N(\mu_2,\sigma_2^2)$, ..., $X_n \sim N(\mu_n,\sigma_n^2)$, then $X_1+X_2+...+X_n \sim N(\mu_1+\mu_2+...+\mu_n,\ \sigma_1^2+\sigma_2^2+...+\sigma_n^2)$. Special Case: If $X_1, X_2, ..., X_n$ are *independent* observations of the **same** normal distribution, so that $X_f \sim N(\mu,\sigma^2)$ for f = 1,2,...,n, then $X_1+X_2+...+X_n \sim N(n\mu,\ n\sigma^2)$.

Derivations of Mean and Variance

(1) The Binomial Distribution: $P(X=x) = {}^nC_x p^x q^{n-x}$.

X	0	1	2	3	....	n
P(X=x)	${}^nC_0 p^0 q^{n-0}$ $= q^n$	${}^nC_1 p^1 q^{n-1}$ $= nq^{n-1}p$	${}^nC_2 p^2 q^{n-2}$ $= \frac{n(n-1)q^{n-2}p^2}{2!}$	${}^nC_3 p^3 q^{n-3}$ $= \frac{n(n-1)(n-2)p^3q^{n-3}}{3!}$		${}^nC_n p^n q^{n-n}$ $= p^n$

$E(X) = \Sigma_{\text{all } x}\ x \times p(X=x) = (0 \times q^n) + (1 \times nq^{n-1}p) + (2 \times (^{n(n-1)}/_{2!})q^{n-2}p^2) + ... + (n \times p^n) = np[q^{n-1} + (n-1)q^{n-2}p + (^{(n-1)(n-2)}/_{2!})q^{n-3}p^2 + ... + p^{n-1}]$. The purple bit is the *expansion* of $(q+p)^{n-1}$, so as $q+p = 1$, we have $(q+p)^{n-1} = 1^{n-1} = 1$. **Therefore**, $E(X) = np \times 1 = np$. **QED**.

To get the **Variance**, use $Var(X) = E(X^2)-[E(X)]^2$.

$$\begin{aligned}\text{Now } E(X^2) &= \Sigma_{\text{all } x}\ x^2 \times P(X=x) = (0^2 \times q^n) + (1^2 \times nq^{n-1}p) + (2^2 \times (^{n(n-1)}/_{2!})q^{n-2}p^2) + ... + (n^2 \times p^n)\\ &= (nq^{n-1}p) + ((^{4n(n-1)}/_{2!})q^{n-2}p^2) + ((^{9n(n-1)(n-2)}/_{3!})q^{n-3}p^3) + ... + (n^2p^n)\\ &= np[q^{n-1} + 2(n-1)q^{n-2}p + (^{3(n-1)(n-2)}/_{2!})q^{n-3}p^2 + ... + np^{n-1}]\\ &= np[q^{n-1} + 1(n-1)q^{n-2}p + (^{1(n-1)(n-2)}/_{2!})q^{n-3}p^2 + ... + 1p^{n-1}\\ &\qquad + 1(n-1)q^{n-2}p + (^{2(n-1)(n-2)}/_{2!})q^{n-3}p^2 + ... + (n-1)p^{n-1}].\end{aligned}$$

The **top** row is the expansion of $(q+p)^{n-1}$, so $E(X^2) = np[(q+p)^{n-1} + (n-1)p[q^{n-2} + (n-2)q^{n-3} + .. + p^{n-2}])$. Now the red bit is the *expansion* of $(q+p)^{n-2}$, which is **also** 1. Therefore, $E(X^2) = np[1+(n-1)p \times 1] = np(1 + np-p) = np + n^2p^2 - np^2 = np(1-p) + n^2p^2$. It follows that $Var(X) = E(X^2) - [E(X)]^2 = np(1-p) + n^2p^2 - n^2p^2 = np(1-p) = npq$. **QED**.

(2) The Poisson Distribution: $P(X=x) = e^{-\lambda}\frac{\lambda^x}{x!}$.

X	0	1	2	3	...	n
P(X=x)	$e^{-\lambda}$	$(e^{-\lambda})\lambda$	$(e^{-\lambda})\frac{\lambda^2}{2!}$	$(e^{-\lambda})\frac{\lambda^3}{3!}$		$(e^{-\lambda})\frac{\lambda^n}{n!}$

$E(X) = e^{-\lambda}\lambda[1 + \lambda + (\lambda^2/2!) + (\lambda^3/3!) + ... + (\lambda^n/n!)] = e^{-\lambda}\lambda e^{\lambda} = \lambda$. **QED**. Now $E(X^2) = e^{-\lambda}[0 + \lambda + (4\lambda^2/2!) + (9\lambda^3/3!) + ...] = e^{-\lambda}\lambda[1 + 2\lambda + (4\lambda^2/2!) + (4\lambda^3/3!) + ...] = \lambda e^{-\lambda}[(1 + \lambda + (\lambda^2/2!) + (\lambda^3/3!) + ...) + (\lambda + (2\lambda^2/2!) + (3\lambda^3/3!) + ...)] = \lambda e^{-\lambda}(e^{\lambda}+\lambda e^{\lambda}) = \lambda e^{-\lambda}e^{\lambda} + \lambda e^{-\lambda}\lambda e^{\lambda} = \lambda+\lambda^2$. **Therefore**, $Var(X) = E(X^2) - [E(X)]^2 = \lambda+\lambda^2-\lambda^2 = \lambda$. **QED**.